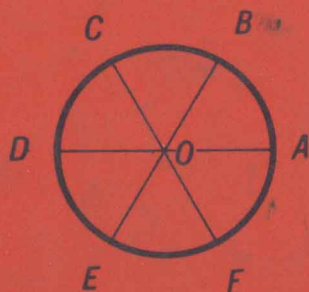


PROBLEM BOOK:

*Algebra
and
Elementary
Functions*

A. KUTEPOV
and
A. RUBANOV

MIR PUBLISHERS
MOSCOW



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by LEONID LEVANT

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MOSCOW

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ЗАДАЧНИК ПО АЛГЕБРЕ И ЭЛЕМЕНТАРНЫМ ФУНКЦИЯМ

Издательство «Высшая школа»

FOREWORD TO THE FIRST RUSSIAN EDITION

This problem book is intended for secondary school students. In this problem book, the authors have selected and methodologically arranged problems and exercises on algebra and elementary functions which encompass all the standard secondary school mathematics program.

Moreover, this problem book contains two review chapters which help the instructor to organize work for the students with the goal in mind of preparing them for examinations. In each paragraph, the problems and exercises are arranged in an ascending order of complexity.

The most difficult problems are noted with an asterisk. These problems can be used for group work or for homework.

In this work, the authors accepted much valuable advice from their review editors R. A. Kalnin and I. M. Smychkovich, and the authors would like to express their deep thanks to them.

The Authors

FOREWORD TO THE SECOND RUSSIAN EDITION

Because of requests from mathematics teachers, the number of problems and exercises in almost all of the chapters have been increased. In addition, there are three new chapters in the problem book: *Limits*, *The Derivative and its Application*, and *Review Problems Covering the Entire Course*. This problem book is very handy for students and teachers in those technical high schools where the elements of higher mathematics are not taught in a special course.

For the second edition, the authors studied all of the remarks and suggestions which were sent to them. Moreover, the authors would like to thank those individuals who took the time to send in their remarks and ideas. The authors, however, would like to express their special gratitude to their review editor A. T. Rogov who with great attention read the proofs and gave us a number of very valuable suggestions.

The Authors

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CHAPTER I

Review Exercises

1. SHORT-CUT MULTIPLICATION AND DIVISION WITH THE AID OF FORMULAS

Perform the indicated operations using the formulas for short-cut multiplication:

1. (Orally). (1) $(2b^3 - 1)^2$; (2) $(0.5 + 2a^5)^2$; (3) 52^2 ; (4) 1001^2 ;
(5) 198×202 ; (6) $84^2 - 16^2$; (7) $\left(2\frac{3}{4}\right)^2 - \left(1\frac{1}{4}\right)^2$;
(8) $\sqrt{313^2 - 312^2}$.
2. (1) $(0.5x^2 - 0.2y)^3$; (2) $(-2a - 3b)^2$;
(3) $(x^n + 4)^3$; (4) $(x - 2)(x^2 + 2x + 4)$.
3. (1) $(1 + 2u)(1 - 2u)(1 - 4u^2)$;
(2) $(2mn + 1)(4m^2n^2 - 2mn + 1)(8m^3n^3 - 1)$;
(3) $(4^{2m} + 4^{3n})(4^{2m} - 4^{3n})(4^{4m} - 4^{6n})$;
(4) $(1 + x - y)(1 + x + y)(1 + 2x + x^2 + y^2)$.
4. (1) $(16 - 25z^4) : (4 + 5z^2)$; (2) $\left(\frac{1}{8}x^3 + \frac{1}{27}y^3\right) : \left(\frac{1}{2}x + \frac{1}{3}y\right)$;
(3) $(a^{2x} - b^{2x}) : (a^x + b^x)$;
(4) $(3^{3x} + 3^{3a}) : (3^x + 3^a)$.

5. (1) $[(2a+3b)^2-1] : (2a+3b+1);$

(2) $(2^{4m}-2^{4n}) : (2^{2m}-2^{2n});$ (3) $(1-x^{6k}) : (1-x^{2k});$

(4) $[(m-n)^3-p^3] : (m-n-p).$

6. Prove the validity of the following identities:

(1) $(a-b)^2 = (b-a)^2;$ (2) $(-a-b)^2 = (a+b)^2;$

(3) $(a-b)^3 = -(b-a)^3;$ (4) $(-a-b)^3 = -(a+b)^3.$

7. (1) Find the sum of the squared difference between two numbers and their quadruple product.

(2) Find the difference between the squared sum of two numbers and their quadruple product.

8. (1) Prove the identity $(10a+5)^2 = 100a(a+1)+25$ and (a) check its validity for $a = 8$; (b) formulate the rule for an oral squaring of numbers whose last digit is 5.

(2) Calculate orally: 35^2 ; 95^2 ; 105^2 ; 145^2 ; 195^2 .

9. (1) Prove the identity $\left(a+\frac{1}{2}\right)^2 = a(a+1)+\frac{1}{4}$ and (a) check its validity for $a = 4$, $a = 5$; (b) formulate the rule for an oral squaring of mixed numbers whose fractional part equals $\frac{1}{2}$.

(2) Calculate orally: $\left(7\frac{1}{2}\right)^2$; $\left(8\frac{1}{2}\right)^2$; $\left(60\frac{1}{2}\right)^2$; $\left(100\frac{1}{2}\right)^2$.

10. Complete the following expressions to perfect squares:

(1) $x^2+4x+?$; (2) $36x^2-?+49y^2$; (3) $m^4-3m^2+?$;

(4) $4a^2+5a+?$; (5) $?-2a+\frac{1}{4}$; (6) $p^2-0.5p+?$.

11. Single the perfect squares of the sum or difference of two numbers out of the following expressions:

(1) x^2+2x+4 ; (2) $a^2-6a+13$; (3) $4a^4+5a^2-2$;

(4) $9x^2-8x+2$; (5) $m^2-0.2m+1$; (6) x^2+px+q .

12. Carry out the following operations:

(1) $(2x+y^2)^2+(2x-y^2)^2$;

(2) $\frac{\left(\frac{1}{x}+\frac{1}{y}\right)^2-\left(\frac{1}{x}-\frac{1}{y}\right)^2-xy}{2-xy}$;

- (3) $\frac{(2a+3)^3 - (2a-3)^3}{(3a+4)^2 + 3a^2 - 24a - 7}$;
- (4) $\frac{(2x+3)^2 + 2(4x^2-9) + (2x-3)^2}{(2x+3)^2 - 2(4x^2-9) + (2x-3)^2}$.

13. Prove the following identities:

- (1) $(4x^2 + 4ax + a^2) : (2x + a) - (2x + a)^3 : (4x^2 + 4ax + a^2) = 0$;
- (2) $\left(\frac{x-3}{2}\right)^3 - \left(\frac{x+3}{2}\right)^3 = -\frac{9(x^2+3)}{4}$;
- (3) $\left(\frac{x-3}{3}\right)^3 + \left(\frac{x+3}{3}\right)^3 = \frac{2x(x^2+27)}{27}$.

14. Simplify the following expressions and find their numerical values:

- (1) $9m^2 + 24mn + 16n^2$ for $m = 4$, $n = -2$;
- (2) $27a^3 - 54a^2b + 36ab^2 - 8b^3$ for $a = 3$, $b = -\frac{1}{2}$;
- (3) $\frac{a^6 + 8}{a^4 - 2a^2 + 4}$ for $a = -4$;
- (4) $\frac{4x^2 + 12xy + 9y^2 - 4}{2x + 3y + 2}$ for $x = 4$, $y = -3$.

2. FACTORIZATION

Factorize the following:

- 15.** (1) $2ab^3 + 6a^2b^2 + 8a^4b^4$; (2) $3x(x-3) - x + 3$;
 (3) $m(m-1) + (1-m)^2$; (4) $4a^2(b^2-1) + 4b^2(1-b^2)$.
- 16.** (1) $(2p+3q)(p-q) + (2p-3q)(q-p)$;
 (2) $(a^2-b^2)(a^2+ab+b^2) - a^3+b^3$;
 (3) $3x^3(x^2-x+1) + 2x^2(x-x^2-1) + x(-x^2+1-1)$.
- 17.** (1) $2^x + 2^{2x}$; (2) $a^{3m} + 2a^{2m} + a^m$;
 (3) $3^x + 3^{x+1}$; (4) $p^{2x+1} - 2p^{2x} + p^{2x-1}$.
- 18.** (1) In the polynomial $ax^2 + bx + c$ factor out a ;
 (2) in the polynomial $3a^2b - 5ab^2 - b^3$ factor out $3a^2b$.

Factorize the following:

19. (1) $a^3 - 5a^2 - 4a + 20$; (2) $x^3 + x^2 + x + 1$;

(3) $a^3 + 3 + 3a^2 + a$.

20. (1) $2ax + ay - az - 2bx - by + bz$;

(2) $4ax + 3xy^2 - 3ay^2 - 4x^2$;

(3) $4a^2x - 6x - 24a^2y + 36y$.

21. Factorize the following polynomials replacing the middle term by the sum or the difference of two terms:

(1) $2a^2 + 10a + 12$; (2) $a^4 + a^2 - 20$;

(3) $x^2 - 7xy + 12y^2$; (4) $x^3 - 3x + 2$.

Factorize:

22. (1) $64a^2 - 25b^4$; (2) $a^2 + 4ab + 4b^2 - 1$;

(3) $4(a+b)^2 - 9(a-b)^2$.

23. (1) $a^4 - a^2(b^2 + 1) + b^2$; (2) $a^5 - 4a^3 + 8a^2 - 32$;

(3) $a^2 + ax - 2x^2 + a + 2x$; (4*) $a^4 + 4$;

(5*) $x^4 - x^2 + 16$; (6*) $x^4 + 6x^2 + 25$.

24. Perform the following operations (orally):

(1) $394 \times 28 + 606 \times 28$; (2) $7.35 \times 2.5 - 1.35 \times 2.5$;

(3) $14 \frac{3}{8} \times 1 \frac{1}{4} - 4 \frac{3}{8} \times 1 \frac{1}{4}$;

(4) $24 \times 2.78 + 35 \times 2.78 + 41 \times 2.78$.

25. Reduce the given fractions:

(1) $\frac{x^2 + 2x - 8}{(x-1)^2 - 1}$; (2) $\frac{(x-1)^3 - 8}{x-3}$;

(3) $\frac{9 - 12x + 4x^2}{2x - 3}$.

26. (1) Prove that the polynomial $x^5 - x^4 - 2x^3 + 2x^2 + x - 1$ is divisible by (a) $x^2 - 1$; (b) $(x+1)^2$; (c) $(x-1)^3$.

(2) Prove that the fraction $\frac{(2x^2 + 2xy)(x-1)}{x^2 - x + xy - y}$ takes on even values for any integral x .

27. Will the value of the following fractions change if x and y are doubled:

$$(1) \frac{2x+3y}{3y-x}; \quad (2) \frac{2x^2-3y^2}{x+y}; \quad (3) \frac{2x^2-3y^2}{x^2+y^2}.$$

28. Solve the given equations:

$$(1) y+40+(y+3)^2 = (2y+7)^2;$$

$$(2) (a+2)(a-2)(a^2+4) = 25a^2 - 16.$$

29. (1) Prove that if a is an integer, then $a^3 - a$ is exactly divisible by 6.

(2) Knowing that $a^3 - a$, where a is an integer, is divisible by 6, prove that the expressions $a^3 - 7a$, $a^3 + 11a$, and $5a^3 + 13a - 30$ are also divisible by 6.

3. SIMPLIFYING AND EVALUATING ALGEBRAIC EXPRESSIONS

30. For what values of x do the following fractions have no sense?

$$(1) \frac{2x+1}{x-1}; \quad (2) \frac{3x^2-2}{x^2-1}; \quad (3) \frac{x}{x^3-1};$$

$$(4) \frac{x^2}{x^3+1}; \quad (5) \frac{2x-1}{3x+1}; \quad (6) \frac{3}{3x^2+5}.$$

Simplify the following expressions:

$$31. (1) \left(\frac{3}{9-m^2} + \frac{2m-1}{m-3} - \frac{m^2-4}{m^2+6m+9} \times \frac{m+3}{m-2} \right) : \frac{m}{m-3};$$

$$(2) \left(\frac{6a^2-2b^2-ab}{b^2-4a^2} + \frac{3a+1}{5a-1} \times \frac{5a^2-a}{6a^2-3ab-b+2a} + 1 \right) \frac{a-b}{2b}.$$

$$32. \frac{\frac{5z^2}{10z-8} - 9(z+2) \left(\frac{1}{14z+28} + \frac{1}{35z-28} \right) + \frac{2}{5z-4}}{\frac{18z-6}{1-7z} \left(\frac{z+3}{3-9z} - \frac{4z-12}{54z^2-6} + \frac{3z+1}{6z+2} \right)}.$$

$$33. (1) \left(\frac{3x+2}{2x+3} - \frac{4x-1}{2x+3} - \frac{2x^2+3x}{4x^2+12x+9} \right) : \frac{3-2x}{2x+3};$$

$$(2) \left(\frac{1}{a^2-9} : \frac{x-a}{3a^2+9a} - \frac{3a}{9-3x-3a+ax} \right) \times \frac{x^3-27}{3a}.$$

34. Prove the following identities:

$$(1) \left(\frac{y}{x} - \frac{x}{y}\right) : \left(2 - \frac{x}{y} - \frac{y}{x}\right) : \left(\frac{y}{x} + 1\right) = \frac{x}{x-y};$$

$$(2) \left[\left(\frac{3a}{a^3-b^3} \times \frac{a^2+ab+b^2}{a+b} - \frac{3}{b-a}\right) : \frac{2a+b}{a^2+2ab+b^2}\right] \frac{3}{a+b} = \frac{9}{a-b}.$$

35. Compute orally:

$$(1) \frac{a}{a+1} \quad \text{if} \quad a = -\frac{3}{4}, \quad a = 3;$$

$$(2) \frac{7y}{(x-y)^2} \quad \text{if} \quad x = -5, \quad y = 2.$$

36. Simplify the following expressions and find their numerical values:

$$(1) \left[\left(\frac{1}{x} + \frac{1}{y}\right) : \frac{x+y}{2} + \frac{1}{x^2} + \frac{1}{y^2}\right] \frac{xy}{(x+y)^2} \quad \text{for} \quad x = -\frac{1}{2}, \quad y = -2;$$

$$(2) \left(\frac{a+1}{a-1} - \frac{a-1}{a+1}\right) \left(\frac{1}{2} - \frac{a}{4} - \frac{1}{4a}\right) \quad \text{for} \quad a = -3\frac{3}{4}.$$

37. Carry out the operation of division and check on the results by evaluating the dividend, the divisor and the quotient for the given values of the letters:

$$(x^4 + x^3y + 7xy^3 - 3y^4) : (3y^2 + x^2 - xy) \quad \text{for} \quad x = 4, \quad y = -4.$$

38. How many times is the number 3^{33} greater than 3^{32} ?

39. A batch of a kg of berries has been stored at a supply depot. The first analysis showed that the berries contain 99% of water. With time the water content reduced and has become equal to 98%. What is now the weight of the berries?

40. Find x if

$$\frac{(17.125 + 19.38 : x)0.2 + 3\frac{1}{12} : 2\frac{1}{18}}{\left(5\frac{17}{32} - 4\frac{11}{27} : 2 + 2\frac{1}{4} \times 1\frac{3}{8}\right) : 27.74 + \frac{7}{9}} = 6.48.$$

4. CARRYING OUT COMPUTATIONS ON A SLIDE RULE

41. Read the following numbers considering them as a number series and rounding them off to three significant digits:

60.1, 0.00601, 17.54, 632.562, 74.033, 540, 483.52.

42. Set the following numbers on the main number scale with the aid of the sighting line of the slide:

- (1) 1, 0.5, 2000, 16.250, 85, 0.0064, 10, 0.111;
- (2) 46.2, 85.23, 280, 10.2, 0.8426, 318.5, 513;
- (3) 1043, 46 452, 51 300, 2.07, 102.4, 0.00487.

43. Find the number of significant digits in the following numbers:

- (1) 3, 48, 735, 5007, 27 000, 0.125, 2.13, 156.7;
- (2) 14.8, 1002.3, 83745.7, 0.137, 0.0150, 0.00034.

44. Determine the number if its digits are 1, 3, and 8; the number of its significant digits being +3, +1, +6, 0, -2, -5.

Multiply the given numbers using the basic scale of the slide rule:

- 45.** (1) 2×4 , 30×800 , 0.6×0.7 , 14.5×4.9 , 5.25×0.8 ;
(2) 29.6×75 , 0.015×46 , 1.54×326 , 2.08×465 ;
(3) 0.0064×0.00056 , 6430×0.00017 , 9820×0.000542 .

- 46.** (1) 44×3.5 , 23.8×3.1 , 202×4.95 , 0.874×119 ;
(2) 0.0312×0.183 , 0.202×30.3 , 0.022×51.3 , 419×0.0368 .

47. (1) $0.285 \times 31.2 \times 42$; (2) $89.2 \times 0.63 \times 1.5$; (3) $42.4 \times 0.78 \times 0.0075$; (4) $435 \times 0.538 \times 0.207 \times 323$; (5) $18.4 \times 0.0248 \times 0.926 \times 41\,600 \times 0.0576$.

48. Find 10%, 15%, 25.3%, 44%, 56%, 68.5%, 92%, 112% of the numbers 480, 248, and 1120.

49. (1) Compute the weight of a pine board 5.75 m long, 6.5 cm thick, 34 cm wide. The density of pine is 580 kg/m^3 .

(2) Compute the volume of a room, whose dimensions are $8.4 \times 5.8 \times 3.2 \text{ m}$.

50. (1) Compile a table of wages if the former rates 62.3, 74.8, 86.5 roubles are increased by 18%.

(2) Compile a table for converting inches into millimetres (1 inch = 25.4 mm).

(3) Convert the following from degrees Centigrade to degrees Kelvin, 10° , 18.5° , 19.8° , 180° , 39.5° .

51. Divide the given numbers using the basic scale of the slide rule: