Kôsaku Yosida

Functional Analysis

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Preface to the First Edition

The present book is based on lectures given by the author at the University of Tokyo during the past ten years. It is intended as a textbook to be studied by students on their own or to be used in a course on Functional Analysis, i.e., the general theory of linear operators in function spaces together with salient features of its application to diverse fields of modern and classical analysis.

Necessary prerequisites for the reading of this book are summarized, with or without proof, in Chapter 0 under titles: Set Theory, Topological Spaces, Measure Spaces and Linear Spaces. Then, starting with the chapter on Semi-norms, a general theory of Banach and Hilbert spaces is presented in connection with the theory of generalized functions of S. L. SOBOLEV and L. SCHWARTZ. While the book is primarily addressed to graduate students, it is hoped it might prove useful to research mathematicians, both pure and applied. The reader may pass, e.g., from Chapter IX (Analytical Theory of Semi-groups) directly to Chapter XIII (Ergodic Theory and Diffusion Theory) and to Chapter XIV (Integration of the Equation of Evolution). Such materials as "Weak Topologies and Duality in Locally Convex Spaces" and "Nuclear Spaces" are presented in the form of the appendices to Chapter V and Chapter X, respectively. These might be skipped for the first reading by those who are interested rather in the application of linear operators.

In the preparation of the present book, the author has received valuable advice and criticism from many friends. Especially, Mrs. K. HILLE has kindly read through the manuscript as well as the galley and page proofs. Without her painstaking help, this book could not have been printed in the present style in the language which was not spoken to the author in the cradle. The author owes very much to his old friends, Professor E. HILLE and Professor S. KAKUTANI of Yale University and Professor R. S. PHILLIPS of Stanford University for the chance to stay in their universities in 1962, which enabled him to polish the greater part of the manuscript of this book, availing himself of their valuable advice. Professor S. ITO and Dr. H. KOMATSU of the University of Tokyo kindly assisted the author in reading various parts

of the galley proof, correcting errors and improving the presentation. To all of them, the author expresses his warmest gratitude.

Thanks are also due to Professor F. K. SCHMIDT of Heidelberg University and to Professor T. KATO of the University of California at Berkeley who constantly encouraged the author to write up the present book. Finally, the author wishes to express his appreciation to Springer-Verlag for their most efficient handling of the publication of this book.

Tokyo, September 1964

KÔSAKU YOSIDA

Preface to the Second Edition

In the preparation of this edition, the author is indebted to Mr. FLORET of Heidelberg who kindly did the task of enlarging the Index to make the book more useful. The errors in the second printing are corrected thanks to the remarks of many friends. In order to make the book more up-to-date, Section 4 of Chapter XIV has been rewritten entirely for this new edition.

Tokyo, September 1967

KÔSAKU YOSIDA

Preface to the Third Edition

A new Section (9. Abstract Potential Operators and Semi-groups) pertaining to G. HUNT's theory of potentials is inserted in Chapter XIII of this edition. The errors in the second edition are corrected thanks to kind remarks of many friends, especially of Mr. KLAUS-DIETER BIERSTEDT.

Kyoto, April 1971

KÔSAKU YOSIDA

Preface to the Fourth Edition

Two new Sections "6. Non-linear Evolution Equations 1 (The Kōmura-Kato Approach)" and "7. Non-linear Evolution Equations 2 (The Approach Through The Crandall-Liggett Convergence Theorem)" are added to the last Chapter XIV of this edition. The author is grateful to Professor Y. Kōmura for his careful reading of the manuscript.

Tokyo, April 1974

KÔSAKU YOSIDA

Preface to the Fifth Edition

Taking advantage of this opportunity, supplementary notes are added at the end of this new edition and additional references to books have been entered in the bibliography. The notes are divided into two categories. The first category comprises two topics: the one is concerned with the time reversibility of Markov processes with invariant measures, and the other is concerned with the uniqueness of the solution of time dependent linear evolution equations. The second category comprises those lists of recently published books dealing respectively with Sobolev Spaces, Trace Operators or Generalized Boundary Values, Distributions and Hyperfunctions, Contraction Operators in Hilbert Spaces, Choquet's Refinement of the Krein-Milman Theorem and Linear as well as Nonlinear Evolution Equations.

A number of minor errors and a serious one on page 459 in the fourth edition have been corrected. The author wishes to thank many friends who kindly brought these errors to his attention.

Kamakura, August 1977 KÔSAKU YOSIDA

Contents

0. Preliminaries

It is the purpose of this chapter to explain certain notions and theorems used throughout the present book. These are related to *Set Theory*, *Topological Spaces*, *Measure Spaces* and *Linear Spaces*.

1. Set Theory

Sets. $x \in X$ means that x is a *member* or *element* of the set X; $x \bar{\in} X$ means that x is not a member of the set X. We denote the set consisting of all x possessing the property P by $\{x; P\}$. Thus $\{y; y = x\}$ is the set $\{x\}$ consisting of a single element x. The *void set* is the set with no members, and will be denoted by $\emptyset$. If every element of a set X is also an element of a set Y, then X is said to be a *subset* of Y and this fact will be denoted by $X \subseteq Y$, or $Y \supseteq X$. If $\mathfrak{X}$ is a set whose elements are sets X, then the set of all x such that $x \in X$ for some $X \in \mathfrak{X}$ is called the *union* of sets X in $\mathfrak{X}$; this union will be denoted by $\bigcup_{X \in \mathfrak{X}} X$. The *intersection* of the sets X in $\mathfrak{X}$ is the set of all x which are elements of every $X \in \mathfrak{X}$; this intersection will be denoted by $\bigcap_{X \in \mathfrak{X}} X$. Two sets are *disjoint* if their intersection is void. A family of sets is disjoint if every pair of distinct sets in the family is disjoint. If a sequence $\{X_n\}_{n=1,2,\ldots}$ of sets is a disjoint family, then the union $\bigcup_{n=1}^{\infty} X_n$ may be written in the form of a sum $\sum_{n=1}^{\infty} X_n$.

Mappings. The term *mapping*, *function* and *transformation* will be used synonymously. The symbol $f: X \to Y$ will mean that f is a single-valued function whose *domain* is X and whose *range* is contained in Y; for every $x \in X$, the function f assigns a uniquely determined element $f(x) = y \in Y$. For two mappings $f: X \to Y$ and $g: Y \to Z$, we can define their *composite mapping* $gf: X \to Z$ by $(gf)(x) = g(f(x))$. The symbol $f(M)$ denotes the set $\{f(x); x \in M\}$ and $f(M)$ is called the *image* of M under the mapping f. The symbol $f^{-1}(N)$ denotes the set $\{x; f(x) \in N\}$ and $f^{-1}(N)$ is called the *inverse image* of N under the mapping f. It is clear that

$$Y_1 = f(f^{-1}(Y_1)) \text{ for all } Y_1 \subseteq f(X), \text{ and } X_1 \subseteq f^{-1}(f(X_1)) \text{ for all } X_1 \subseteq X.$$

If $f: X \to Y$, and for each $y \in f(X)$ there is only one $x \in X$ with $f(x) = y$, then f is said to have an *inverse* (mapping) or to be *one-to-one*. The inverse mapping then has the domain $f(X)$ and range X; it is defined by the equation $x = f^{-1}(y) = f^{-1}(\{y\})$.

The domain and the range of a mapping f will be denoted by $D(f)$ and $R(f)$, respectively. Thus, if f has an inverse then

$$f^{-1}(f(x)) = x \text{ for all } x \in D(f), \text{ and } f(f^{-1}(y)) = y \text{ for all } y \in R(f).$$

The function f is said to map X *onto* Y if $f(X) = Y$ and *into* Y if $f(X) \subseteq Y$. The function f is said to be an *extension* of the function g and g a *restriction* of f if $D(f)$ contains $D(g)$, and $f(x) = g(x)$ for all x in $D(g)$.

Zorn's Lemma

Definition. Let P be a set of elements $a, b, \ldots$ Suppose there is a binary relation defined between certain pairs (a, b) of elements of P, expressed by $a \prec b$, with the properties:

$$\begin{cases} a \prec a, \\ \text{if } a \prec b \text{ and } b \prec a, \text{ then } a = b, \\ \text{if } a \prec b \text{ and } b \prec c, \text{ then } a \prec c \text{ (transitivity)}. \end{cases}$$

Then P is said to be *partially ordered* (or *semi-ordered*) by the relation $\prec$.

Examples. If P is the set of all subsets of a given set X, then the set inclusion relation ($A \subseteq B$) gives a partial ordering of P. The set of all complex numbers $z = x + iy$, $w = u + iv, \ldots$ is partially ordered by defining $z \prec w$ to mean $x \leq u$ and $y \leq v$.

Definition. Let P be a partially ordered set with elements $a, b, \ldots$ If $a \prec c$ and $b \prec c$, we call c an *upper bound* for a and b. If furthermore $c \prec d$ whenever d is an upper bound for a and b, we call c the *least upper bound* or the *supremum* of a and b, and write $c = \sup(a, b)$ or $a \vee b$. This element of P is unique if it exists. In a similar way we define the *greatest lower bound* or the *infimum* of a and b, and denote it by $\inf(a, b)$ or $a \wedge b$. If $a \vee b$ and $a \wedge b$ exist for every pair (a, b) in a partially ordered set P, P is called a *lattice*. (格)

Example. The totality of subsets M of a fixed set B is a lattice by the partial ordering $M_1 \prec M_2$ defined by the set inclusion relation $M_1 \subseteq M_2$.

Definition. A partially ordered set P is said to be *linearly ordered* (or *totally ordered*) if for every pair (a, b) in P, either $a \prec b$ or $b \prec a$ holds. A subset of a partially ordered set is itself partially ordered by the relation which partially orders P; the subset might turn out to be linearly ordered by this relation. If P is partially ordered and S is a subset of P, an $m \in P$ is called an upper bound of S if $s \prec m$ for every $s \in S$. An $m \in P$ is said to be *maximal* if $p \in P$ and $m \prec p$ together imply $m = p$.

Zorn's Lemma. Let P be a non-empty partially ordered set with the property that every linearly ordered subset of P has an upper bound in P. Then P contains at least one maximal element.

It is known that Zorn's lemma is equivalent to Zermelo's axiom of choice in set theory.

2. Topological Spaces

Open Sets and Closed Sets

Definition. A system τ of subsets of a set X defines a *topology* in X if τ contains the void set, the set X itself, the union of every one of its subsystems, and the intersection of every one of its finite subsystems. The sets in τ are called the *open sets* of the *topological space* (X, τ); we shall often omit τ and refer to X as a topological space. Unless otherwise stated, we shall assume that a topological space X satisfies *Hausdorff's axiom of separation*:

For every pair (x_1, x_2) of distinct points x_1, x_2 of X, there exist disjoint open sets G_1, G_2 such that $x_1 \in G_1$, $x_2 \in G_2$.

A *neighbourhood* of the point x of X is a set containing an open set which contains x. A neighbourhood of the subset M of X is a set which is a neighbourhood of every point of M. A point x of X is an *accumulation point* or *limit point* of a subset M of X if every neighbourhood of x contains at least one point $m \in M$ different from x.

Definition. Any subset M of a topological space X becomes a topological space by calling "open" the subsets of M which are of the form $M \cap G$ where G's are open sets of X. The induced topology of M is called the *relative topology* of M as a subset of the topological space X.

Definition. A set M of a topological space X is *closed* if it contains all its accumulation points. It is easy to see that M is closed iff[1] its *complement* $M^C = X - M$ is open. Here $A - B$ denotes the totality of points $x \in A$ not contained in B. If $M \subseteq X$, the intersection of all closed subsets of X which contain M is called the *closure* of M and will be denoted by M^a (the superscript "a" stands for the first letter of the German: abgeschlossene Hülle).

Clearly M^a is closed and $M \subseteq M^a$; it is easy to see that $M = M^a$ iff M is closed.

Metric Spaces

Definition. If X, Y are sets, we denote by $X \times Y$ the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$; $X \times Y$ will be called the *Cartesian product* of X and Y. X is called a *metric space* if there is defined a func-

[1] iff is the abbreviation for "if and only if".

tion d with domain $X \times X$ and range in the real number field R^1 such that

$$\begin{cases} d(x_1, x_2) \geqq 0 \text{ and } d(x_1, x_2) = 0 \text{ iff } x_1 = x_2, \\ d(x_1, x_2) = d(x_2, x_1), \\ d(x_1, x_3) \leqq d(x_1, x_2) + d(x_2, x_3) \text{ (the triangle inequality).} \end{cases}$$

d is called the *metric* or the *distance function* of X. With each point x_0 in a metric space X and each positive number r, we associate the set $S(x_0; r) = \{x \in X; d(x, x_0) < r\}$ and call it the *open sphere* with centre x_0 and radius r. Let us call "open" the set M of a metric space X iff, for every point $x_0 \in M$, M contains a sphere with centre x_0. Then the totality of such "open" sets satisfies the axiom of open sets in the definition of the topological space.

Hence a metric space X is a topological space. It is easy to see that a point x_0 of X is an accumulation point of M iff, to every $\varepsilon > 0$, there exists at least one point $m \neq x_0$ of M such that $d(m, x_0) < \varepsilon$. The n-dimensional Euclidean space R^n is a metric space by

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}, \text{ where } x = (x_1, \ldots, x_n) \text{ and } y = (y_1, \ldots, y_n).$$

Continuous Mappings

Definition. Let $f: X \to Y$ be a mapping defined on a topological space X into a topological space Y. f is called *continuous at a point* $x_0 \in X$ if to every neighbourhood U of $f(x_0)$ there corresponds a neighbourhood V of x_0 such that $f(V) \subseteq U$. The mapping f is said to be *continuous* if it is continuous at every point of its domain $D(f) = X$.

Theorem. Let X, Y be topological spaces and f a mapping defined on X into Y. Then f is continuous iff the inverse image under f of every open set of Y is an open set of X.

Proof. If f is continuous and U an open set of Y, then $V = f^{-1}(U)$ is a neighbourhood of every point $x_0 \in X$ such that $f(x_0) \in U$, that is, V is a neighbourhood of every point x_0 of V. Thus V is an open set of X. Let, conversely, for every open set $U \ni f(x_0)$ of Y, the set $V = f^{-1}(U)$ be an open set of X. Then, by the definition, f is continuous at $x_0 \in X$.

Compactness

Definition. A system of sets G_α, $\alpha \in A$, is called a *covering* of the set X if X is contained as a subset of the union $\bigcup_{\alpha \in A} G_\alpha$. A subset M of a topological space X is called *compact* if every system of open sets of X which covers M contains a finite subsystem also covering M.

In view of the preceding theorem, *a continuous image of a compact set is also compact.*

Proposition 1. Compact subsets of a topological space are necessarily closed.

Proof. Let there be an accumulation point x_0 of a compact set M of a topological space X such that $x_0 \bar{\in} M$. By Hausdorff's axiom of separation, there exist, for any point $m \in M$, disjoint open sets G_{m,x_0} and $G_{x_0,m}$ of X such that $m \in G_{m,x_0}$, $x_0 \in G_{x_0,m}$. The system $\{G_{m,x_0};\, m \in M\}$ surely covers M. By the compactness of M, there exists a finite subsystem $\{G_{m_i,x_0};\; i = 1, 2, \ldots, n\}$ which covers M. Then $\bigcap_{i=1}^{n} G_{x_0,m_i}$ does not intersect M. But, since x_0 is an accumulation point of M, the open set $\bigcap_{i=1}^{n} G_{x_0,m_i} \ni x_0$ must contain a point $m \in M$ distinct from x_0. This is a contradiction, and M must be closed.

Proposition 2. A closed subset M_1 of a compact set M of a topological space X is compact.

Proof. Let $\{G_\alpha\}$ be any system of open sets of X which covers M_1. M_1 being closed, $M_1^C = X - M_1$ is an open set of X. Since $M_1 \subseteq M$, the system of open sets $\{G_\alpha\}$ plus M_1^C covers M, and since M is compact, a properly chosen finite subsystem $\{G_{\alpha_i};\, i = 1, 2, \ldots, n\}$ plus M_1^C surely covers M. Thus $\{G_{\alpha_i};\, i = 1, 2, \ldots, n\}$ covers M_1.

Definition. A subset of a topological space is called *relatively compact* if its closure is compact. A topological space is said to be *locally compact* if each point of the space has a compact neighbourhood.

Theorem. Any locally compact space X can be embedded in another compact space Y, having just one more point than X, in such a way that the relative topology of X as a subset of Y is just the original topology of X. This Y is called a *one point compactification* of X.

Proof. Let y be any element distinct from the points of X. Let $\{U\}$ be the class of all open sets in X such that $U^C = X - U$ is compact. We remark that X itself $\in \{U\}$. Let Y be the set consisting of the points of X and the point y. A set in Y will be called open if either (i) it does not contain y and is open as a subset of X, or (ii) it does contain y and its intersection with X is a member of $\{U\}$. It is easy to see that Y thus obtained is a topological space, and that the relative topology of X coincides with its original topology.

Suppose $\{V\}$ be a family of open sets which covers Y. Then there must be some member of $\{V\}$ of the form $U_0 \cup \{y\}$, where $U_0 \in \{U\}$. By the definition of $\{U\}$, U_0^C is compact as a subset of X. It is covered by the system of sets $V \cap X$ with $V \in \{V\}$. Thus some finite subsystem: $V_1 \cap X, V_2 \cap X, \ldots, V_n \cap X$ covers U_0^C. Consequently, $V_1, V_2, \ldots, V_n$ and $U_0 \cup \{y\}$ cover Y, proving that Y is compact.

Tychonov's Theorem

Definition. Corresponding to each α of an index set A, let there be given a topological space X_α. The Cartesian product $\prod_{\alpha \in A} X_\alpha$ is, by defini-