TOPOLOGY
A First Course

JAMES R. MUNKRES

Preface

This book is intended as a text for a one- or two-semester introduction to topology, at the senior or first-year graduate level.

The subject of topology is of interest in its own right, and it also serves to lay the foundations for future study in analysis, in geometry, and in algebraic topology. There is no universal agreement among mathematicians as to what a first course in topology should include; there are many topics that are appropriate to such a course, and not all are equally relevant to these differing purposes. In the choice of material to be treated, I have tried to strike a balance among the various points of view.

Prerequisites. There are no formal subject matter prerequisites for studying most of this book. I do not even assume the reader knows much set theory. Having said that, I must hasten to add that unless the reader has studied a bit of analysis or "rigorous calculus," he will be missing much of the motivation for the concepts introduced in the first part of the book. Things will go more smoothly if he already has had some experience with continuous functions, open and closed sets, metric spaces, and the like, although none of these is actually assumed. In Chapter 8, we do assume familiarity with the elements of group theory.

Most students in a topology course have, in my experience, some knowledge of the foundations of mathematics. But the amount varies a great deal from one student to another. Therefore I begin with a fairly thorough chapter on set theory and logic. It starts at an elementary level, and works up to a level that might be described as "semi-sophisticated." It treats those topics (and only those) which will be needed later in the book. Most students will already be familiar with the material of the first few sections, but many of them will find their *expertise* disappearing somewhere about the middle of the chapter. How much time and effort the instructor will need to spend on this chapter will thus depend largely on the mathematical sophistication and experience of his students. Ability to do the exercises fairly readily (and correctly!) should serve as a reasonable criterion for determining whether the student's mastery of set theory is sufficient for him to begin the study of topology.

How the book is organized. When this book is used for a one-semester course, some choices will have to be made concerning what material to cover. I have attempted to organize the book as flexibly as possible, so as to enable the instructor to follow his own preferences in this matter.

Part I of the book, consisting of the first four chapters, deals with that body of material which in my opinion should be included in any introductory topology course worthy of the name. This may be considered the "irreducible core" of the subject, treating as it does topological spaces, connectedness, compactness (through compactness of finite products), and the countability and separation axioms (through the Urysohn metrization theorem). Certain sections are marked with an asterisk; these do not form part of the basic core and may be omitted or postponed with no loss of continuity.

Part II of the book consists of four chapters which are entirely independent of one another. They depend only on the material of Part I; the instructor may take them up in any order he chooses. Furthermore, if he wishes to cover only a portion of one of these later chapters, he can consult the introduction to that chapter, where there appears a diagram showing the relations of dependence among the sections of the chapter. The instructor who wishes, for instance, to conclude his course with a proof of the Jordan curve theorem can determine from this diagram which of the earlier sections of Chapter 8 are essential, and which peripheral, to his purpose.

Some of the material of the later chapters depends on one or more of the asterisked sections in Part I. Each such dependence is indicated in a footnote at the beginning of the asterisked section, and again in the introduction to the chapter in question. Some of the exercises also depend on earlier asterisked sections, but in such cases the dependence is obvious.

Possible course outlines. Most instructors who use this text for a one-semester course will wish to cover the "core" material of Part I, along with the Tychonoff theorem (§5-1). Many will cover additional topics as well. One might, for instance, treat some of the asterisked sections of Part I. (I usually do local compactness, at least.) Or he may choose one or more topics from Part II. Possibilities include: the Stone-Čech compactification (§5-3), metrization theorems (Chapter 6), the Peano curve (§7-2), one or both versions of Ascoli's theorem (§7-3 and §7-6), dimension theory (§7-9), the fundamental group and applications (§8-1–§8-10), or the Jordan curve theorem (§8-13). I have in different semesters followed each of these options.

For the instructor who wishes to emphasize algebraic topology, one possible course outline would consist of Chapters 1 to 3 followed by Chapter 8 in its entirety. Omitting Chapter 4 will cause no difficulty, provided one skips Exercise 5 of §8-12, which involves the concept of normality.

Still another possible outline is the one suggested by the Committee on the Undergraduate Program in Mathematics (of the Mathematical Association of America) for a one-semester course in topology at the first-year graduate level. It would consist of Chapters 2, 3, and 4, followed by §5-1; §6-1, §6-3, §6-4; §7-1; §8-1 through §8-5, §8-8 through §8-11, and §8-14. This program assumes that the student has already had an introduction to set theory equivalent to our Chapter 1.

In a two-semester course, one can reasonably expect to cover the entire book.

Acknowledgements. Most of the topologists with whom I have studied, or whose books I have read, have contributed in one way or another to this book; I mention only Edwin Moise, Raymond Wilder, Gail Young, and Raoul Bott, but there are many others. For their helpful comments concerning this book, my thanks to Robert Mosher and John Hemperly, and to my colleagues George Whitehead and Kenneth Hoffman. My appreciation goes to Miss Viola Wiley, who deciphered my handwriting and converted it into neat copy, and to the employees of Bertrick Associate Artists, Inc., who drew the illustrations.

But most of all, to my students go my most heartfelt thanks. From them I learned at least as much as they did from me; without them this book would be very different.

J.R.M.

A Note to the Reader

Two matters require comment—the exercises and the examples.

Working problems is a crucial part of learning mathematics. No one can learn topology merely by poring over the definitions, theorems, and examples that are worked out for him in the text. He must work part of it out for himself. To provide that opportunity is the purpose of the exercises.

They vary in difficulty, with the easier ones usually given first. Some are routine verifications designed to test whether you have understood the definitions or examples of the preceding section. Others are less routine. You may, for instance, be asked to generalize a theorem of the text. While the result obtained may be interesting in its own right, the main purpose of such an exercise is to encourage you to work carefully through the proof in question, mastering its ideas thoroughly, more thoroughly (I hope!) than mere memorization would demand.

Some exercises are phrased in an "open-ended" fashion. Students often find this practice frustrating. When faced with an exercise which asks, "Is every regular Lindelöf space normal?" they respond in exasperation, "I don't know what I'm supposed to do! Am I supposed to prove it or find a counter-example or what?" But mathematics (outside textbooks) is usually like this. More often than not, all a mathematician has to work with is a conjecture or question, and he doesn't know what the correct answer is. You should have some experience with this situation.

A few exercises that are more difficult than the rest are marked with asterisks. But none are so difficult but that the best student in my class can usually solve them.

Another important part of mastering any mathematical subject is acquiring a repertoire of useful examples. One should of course come to know those major examples from whose study the theory itself derives, and to which the important applications are made. But he should also have a few counterexamples at hand with which to test plausible conjectures.

Now it is all too easy in studying topology to spend too much time dealing with "weird counterexamples." Constructing them requires ingenuity, and is often great fun. But they are not really what topology is about. Fortunately, one does not need too many such counterexamples for a first course; there is a fairly short list which will suffice for most purposes. Let me give it here:

- R^J, the product of the real line with itself, in the product, uniform, and box topologies.
- R_l, the real line in the topology having the intervals $[a, b)$ as a basis.
- S_Ω, the minimal uncountable well-ordered set.
- $I \times I$, the closed unit square, in the dictionary order topology.

These are the examples you should master and remember; they will be exploited again and again.

Contents

TOPOLOGY
A First Course

JAMES R. MUNKRES
Professor of Mathematics
Massachusetts Institute of Technology

PRENTICE-HALL, INC., Englewood Cliffs, New Jersey

Library of Congress Cataloging in Publication Data

MUNKRES, JAMES RAYMOND, (date)
Topology: a first course.

Bibliography: p. 399
1. Topology. I. Title.
QA611.M82 514 74-3465
ISBN 0-13-925495-1

PRENTICE-HALL INTERNATIONAL, INC., *London*
PRENTICE-HALL OF AUSTRALIA, PTY. LTD., *Sydney*
PRENTICE-HALL OF CANADA, LTD., *Toronto*
PRENTICE-HALL OF INDIA PRIVATE LIMITED, *New Delhi*
PRENTICE-HALL OF JAPAN, INC., *Tokyo*

10 9 8 7 6 5

Printed in the United States of America

Part One

1. Set Theory and Logic

We adopt, as most mathematicians do, the naive point of view regarding set theory. We shall assume that what is meant by a *set* of objects is intuitively clear, and we shall proceed on that basis without analyzing the concept further. Such an analysis properly belongs to the foundations of mathematics and to mathematical logic, and it is not our purpose to initiate the study of those fields.

Logicians have analyzed set theory in great detail, and they have formulated axioms for the subject. Each of their axioms expresses a property of sets that mathematicians commonly accept, and collectively the axioms provide a foundation broad enough and strong enough that the rest of mathematics can be built on them.

It is unfortunately true that careless use of set theory, relying on intuition alone, can lead to contradictions. Indeed, one of the reasons for the axiomatization of set theory was to formulate rules for dealing with sets that would avoid these contradictions. Although we shall not deal with the axioms explicitly, the rules we follow in dealing with sets derive from them. In this book, you will learn how to deal with sets in an "apprentice" fashion, by observing how we handle them and by working with them yourself. At some point of your studies you may wish to study set theory more carefully and in greater detail; then a course in logic or foundations will be in order.

1-1 Fundamental Concepts

Here we introduce the ideas of set theory, and establish the basic terminology and notation. We also discuss some points of elementary logic that, in our experience, are apt to cause confusion.

Basic Notation

Commonly we shall use capital letters $A, B, \ldots$ to denote **sets**, and lowercase letters $a, b, \ldots$ to denote the **objects** or **elements** belonging to these sets. If an object a belongs to a set A, we express this fact by the notation

$$a \in A.$$

If a does not belong to A, we express this fact by writing

$$a \notin A.$$

The equality symbol $=$ is used throughout this book to mean *logical identity*. Thus when we write $a = b$, we mean that "a" and "b" are symbols for the same object. This is what one means in arithmetic, for example, when one writes $\frac{2}{4} = \frac{1}{2}$. Similarly, the equation $A = B$ states that "A" and "B" are symbols for the same set; that is, A and B consist of precisely the same objects.

If a and b are different objects, we write $a \neq b$; and if A and B are different sets, we write $A \neq B$. For example, if A is the set of all nonnegative real numbers, and B is the set of all positive real numbers, then $A \neq B$, because the number 0 belongs to A and not to B.

We say that A is a **subset** of B if every element of A is also an element of B; and we express this fact by writing

$$A \subset B.$$

Nothing in this definition requires A to be different from B; in fact, if $A = B$, it is true that both $A \subset B$ and $B \subset A$. If $A \subset B$ and A is different from B, we say that A is a **proper subset** of B and we write

$$A \subsetneq B.$$

How does one go about specifying a set? If the set has only a few elements, one can simply list the objects in the set, writing "A is the set consisting of the elements a, b, and c." In symbols, this statement becomes

$$A = \{a, b, c\},$$

where braces are used to enclose the list of elements.

The usual way to specify a set, however, is to take some set A of objects and some *property* that elements of A may or may not possess, and to form the set consisting of all elements of A having that property. For instance, one might take the set of real numbers and form the subset B consisting of all even integers. In symbols, this statement becomes

$$B = \{x \mid x \text{ is an even integer}\}.$$

Here the braces stand for the words "the set of," and the vertical bar stands for the words "such that." The equation is read, "B is the set of all x such that x is an even integer."

The Union of Sets and The Meaning of "or"

Given two sets A and B, one can form a set from them that consists of all the elements of A together with all the elements of B. This set is called the **union** of A and B and is denoted by $A \cup B$. Formally, we define

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

But we must pause at this point and make sure exactly what we mean by the statement "$x \in A$ *or* $x \in B$."

In ordinary everyday English, the word "or" is ambiguous. Sometimes the statement "P or Q" means "P or Q, or both" and sometimes it means "P or Q, but not both." Usually one decides from the context which meaning is intended. For example, suppose I spoke to two students as follows:

> "Miss Smith, every student registered for this course has taken either a course in linear algebra or a course in analysis."
> "Mr. Jones, either you get a grade of at least 70 on the final exam or you will flunk this course."

In the context, Miss Smith knows perfectly well that I mean "everyone has had linear algebra or analysis, or both," and Mr. Jones knows I mean "either he gets at least 70 or he flunks, but not both." Indeed, Mr. Jones would be exceedingly unhappy if both statements turned out to be true!

In mathematics, one cannot tolerate such ambiguity. One has to pick just one meaning and stick with it, or confusion will reign. Accordingly, mathematicians have agreed that they will use the word "or" in the first sense, so that the statement "P or Q" always means "P or Q, or both." If one means "P or Q, but not both," then one has to include the phrase "but not both" explicitly.

With this understanding, the equation defining $A \cup B$ is unambiguous; it states that $A \cup B$ is the set consisting of all elements x that belong to A or to B or to both.

The Intersection of Sets, The Empty Set, and The Meaning of "If . . . Then"

Given sets A and B, another way one can form a set is to take the common part of A and B. This set is called the **intersection** of A and B and is denoted by $A \cap B$. Formally, we define

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

But just as with the definition of $A \cup B$, there is a difficulty. The difficulty is not in the meaning of the word "and"; it is of a different sort. It arises when the sets A and B happen to have no elements in common. What meaning does the symbol $A \cap B$ have in such a case?

To take care of this eventuality, we make a special convention. We introduce a special set which we call the **empty set,** denoted by $\varnothing$, which we think of as "the set having no elements."

Using this convention, we express the statement that A and B have no elements in common by the equation

$$A \cap B = \varnothing.$$

We also express this fact by saying that A and B are **disjoint.**

Now some students are bothered by the notion of an "empty set." "How," they say, "can you have a set with nothing in it?" The problem is similar to that which arose many years ago when the number 0 was first introduced.

The empty set is only a convention, and mathematics could very well get along without it. But it is a very convenient convention, for it saves us a good deal of awkwardness in stating theorems and in proving them. Without this convention, for instance, one would have to prove that the two sets A and B do have elements in common before one could use the notation $A \cap B$. Similarly, the notation

$$C = \{x \mid x \in A \text{ and } x \text{ has a certain property}\}$$

could not be used if it happened that no element x of A had the given property. It is much more convenient to agree that $A \cap B$ and C equal the empty set in such cases.

Since the empty set $\varnothing$ is merely a convention, we must make conventions relating it to the concepts already introduced. Because $\varnothing$ is thought of as "the set with no elements," it is clear we should make the convention that for each object x, the relation $x \in \varnothing$ does not hold. Similarly, the definitions of union and intersection show that for every set A we should have the equations

$$A \cup \varnothing = A \quad \text{and} \quad A \cap \varnothing = \varnothing.$$

The inclusion relation is a bit more tricky. Given a set A, should we agree that $\varnothing \subset A$? Once more we must be careful about the way mathematicians use the English language. The expression $\varnothing \subset A$ is a shorthand way of writing the sentence, "Every element that belongs to the empty set also belongs to the set A." Or to put it more formally, "For every object x, if x belongs to the empty set, then x also belongs to the set A."

Is this statement true or not? Some might say "yes" and others say "no." You will never settle the question by argument, only by agreement. This is a statement of the form "If P, then Q," and in everyday English the meaning of the "if . . . then" construction is ambiguous. It always means that if P is true, then Q is true also. Sometimes that is *all* it means; other times it means something more: that if P is false, Q must be false. Usually one decides from the context which interpretation is correct.

The situation is similar to the ambiguity in the use of the word "or." One can reformulate the examples involving Miss Smith and Mr. Jones to illustrate the ambiguity. Suppose I said the following:

> "Miss Smith, if any student registered for this course has not taken a course in linear algebra, then he has taken a course in analysis."
> "Mr. Jones, if you get a grade below 70 on the final, you are going to flunk this course."

In the context, Miss Smith understands that if a student in the course has not had linear algebra, then he has taken analysis, but if he has had linear algebra, he may or may not have taken analysis as well. And Mr. Jones knows that if he gets a grade below 70, he will flunk the course, but if he gets a grade of at least 70, he will pass.

Again, mathematics cannot tolerate ambiguity, so a choice of meanings must be made. Mathematicians have agreed always to use "if . . . then" in the first sense, so that a statement of the form "If P, then Q" means that if P is true, Q is true also, but if P is false, Q may be either true or false.

As an example, consider the following statement about real numbers:

$$\textit{If } x > 0, \textit{ then } x^3 \neq 0.$$

It is a statement of the form, "If P, then Q," where P is the phrase "$x > 0$" (called the **hypothesis** of the statement) and Q is the phrase "$x^3 \neq 0$" (called the **conclusion** of the statement). This is a true statement, for in every case for which the hypothesis $x > 0$ holds, the conclusion $x^3 \neq 0$ holds as well.

Another true statement about real numbers is the following:

$$\textit{If } x^2 < 0, \textit{ then } x = 23;$$

in every case for which the hypothesis holds, the conclusion holds as well. Of course, it happens in this example that there are *no* cases for which the hypothesis holds. A statement of this sort is sometimes said to be **vacuously true.**

To return now to the empty set and inclusion, we see that the inclusion $\varnothing \subset A$ does hold for every set A. Writing $\varnothing \subset A$ is the same as saying, "If $x \in \varnothing$, then $x \in A$," and this statement is vacuously true.

Contrapositive and Converse

Our discussion of the "if . . . then" construction leads us to consider another point of elementary logic that sometimes causes difficulty. It concerns the relation between a *statement*, its *contrapositive*, and its *converse.*

Given a statement of the form "If P, then Q," its **contrapositive** is defined to be the statement "If Q is not true, then P is not true." For example, the contrapositive of the statement

$$\textit{If } x > 0, \textit{ then } x^3 \neq 0,$$

is the statement

$$\textit{If } x^3 = 0, \textit{ then it is not true that } x > 0.$$

Note that both the statement and its contrapositive are true. Similarly, the statement

$$\textit{If } x^2 < 0, \textit{ then } x = 23,$$

has as its contrapositive the statement

$$\textit{If } x \neq 23, \textit{ then it is not true that } x^2 < 0.$$

Again, both are true statements about real numbers.

These examples may make you suspect that there is some relation between a statement and its contrapositive. And indeed there is; they are two ways of saying precisely the same thing. Each is true if and only if the other is true; they are *logically equivalent.*

This fact is not hard to demonstrate. Let us introduce some notation first. As a shorthand for the statement "If P, then Q," we write

$$P \Longrightarrow Q,$$

which is read "P implies Q." The contrapositive can then be expressed in the form

$$(\text{not } Q) \Longrightarrow (\text{not } P),$$

where "not Q" stands for the phrase "Q is not true."

Now the only way in which the statement "$P \Rightarrow Q$" can fail to be correct is if the hypothesis P is true and the conclusion Q is false. Otherwise it is correct. Similarly, the only way in which the statement (not Q) $\Rightarrow$ (not P) can fail to be correct is if the hypothesis "not Q" is true and the conclusion "not P" is false. This is the same as saying that Q is false and P is true. And this, in turn, is precisely the situation in which $P \Rightarrow Q$ fails to be correct. Thus we see that the two statements are either both correct or both incorrect; they are logically equivalent. Therefore, we shall accept a proof of the statement "not $Q \Rightarrow$ not P" as a proof of the statement "$P \Rightarrow Q$."

There is another statement that can be formed from the statement $P \Rightarrow Q$. It is the statement

$$Q \Longrightarrow P,$$