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# Equivariant Surgery Theories and Their Periodicity Properties

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# PREFACE

This book began as a pair of papers covering the authors' work on periodicity in equivariant surgery from late 1982 to mid 1984. Since our results apply to many different versions of equivariant surgery theory, it seemed desirable to provide a unified approach to these results based upon formal properties that hold in all the standard theories. Although workers in the area have been aware of these common properties for some time, relatively little has been written on the subject. Furthermore, it is not always obvious how the settings for different approaches to equivariant surgery are related to each other, and this can make it difficult to extract the basic properties and interrelationships of equivariant surgery theories from the literature. For these and other reasons we eventually decided to include a survey of equivariant surgery theories that would present their basic formal properties. Shortly after we began revising our papers to include such a survey, Wolfgang Lück and Ib Madsen began work on a more abstract—and in many respects more general—approach to equivariant surgery. It became increasingly clear that we should include their methods and results in our book, not only for the sake of completeness but also because their work leads to improvements in the exposition, simplifications of some proofs, and significant extensions of our main results and applications. The inclusion of Lück and Madsen's work and its effects on our own work are two excuses for the four year delay in revising our original two papers.

Our initial work on equivariant periodicity appears in Chapters III through V. The first two chapters summarize the main versions of equivariant surgery and the basic formal properties of these theories. A reader who is already familiar with equivariant surgery should be able to start with Chapter III and use the first two Chapters as reference material. On the other hand, a reader who wants to know what equivariant surgery is about should be able to use the first two chapters as an introduction to the subject. We shall assume the reader is somewhat familiar with the main results of nonsimply connected surgery and the basic concepts of transformation groups. Some standard references for these topics are discussed in the second paragraph of the introductory chapter entitled *Summary: Background Material and Main Results*.

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Questions about periodicity in equivariant surgery first arose in the nineteen seventies. In particular, Bill Browder presented some crucial observations in lectures at the 1976 A.M.S. Summer Symposium on Algebraic and Geometric Topology at Stanford University but never put them into writing. We appreciate for the openness and good will with which he has discussed this unpublished work on several separate occasions. We are also grateful to Mel Rothenberg and Shmuel Weinberger for describing some of their unpublished work in enough detail so that we could outline their methods and conclusions in Section I.6 and for their generous attitude towards our inclusion of this material. Similarly, we wish to thank Min Yan for discussions regarding his thesis currently in preparation and the relation of his work to ours.

Andrew Ranicki has been very helpful in several connections, commenting on an old draft of Chapter IV, making suggestions on numerous points related to his research, and furnishing copies of some recent papers. We appreciate his repeated willingness to provide assistance. Preprints of work by other topologists have led to many enhancements and even some major improvements in this book; in particular we would like to mention Frank Connolly, Wolfgang Lück, Ib Madsen, Jan-Alve Svensson, Shmuel Weinberger and Bruce Williams and express our thanks to them.

Several typists from the Purdue University Mathematics Department deserve credit and thanks for converting the original hanndwritten text into typewritten drafts and preliminary TEX files at various times over the past six years.

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Finally, we are grateful to Albrecht Dold and the editorial personnel at Springer for their patience with the four year delay between their initial decision to publish this book in July of 1985 and our submission of a revised manuscript in December of 1989.

# Table of Contents

# SUMMARY: BACKGROUND MATERIAL AND BASIC RESULTS

This book is about some surgery-theoretic methods in the theory of transformation groups. The main goals are to provide a unified description of equivariant surgery theories, to present some equivariant analogs of the fourfold periodicity theorems in ordinary surgery, and to study some implications of these periodicity theorems.

Since equivariant surgery theory involves both transformation groups and the theory of surgery on manifolds, we need to assume that the reader has some some familiarity with both subjects. Most of the necessary background on transformation groups appears in Chapters I–II and Sections VI.1–2 of Bredon's book [**Bre**] or in Chapter I and Sections II.1–2 of tom Dieck's book [**tD**]. The standard reference for surgery theory is Wall's book [**Wa**], particularly Chapters 1–6 and 9; most of the necessary background can also be found in survey articles by W. Browder [**Bro**], F. Latour [**Lat**], and J. A. Lees [**Lees**]. Additional background references for [**Wa**] are listed in a supplementary bibliography at the end of this Summary. Most of the algebraic topology that we use can be found in the books by Spanier [**Sp**] and Milnor and Stasheff [**MS**].

We have attempted to explain the specialized notation and terminology that we use; a subject index and an index for symbols can be found at the end of the book. For the sake of completeness we shall mention some standard conventions that are generally not stated explicitly. The unit interval $\{t \in \mathbb{R} | 0 \leq t \leq 1\}$ will be denoted by $I$ or $[0,1]$, and the order of a finite group $G$ will be denoted by $|G|$. Theorem X.88.99 will denote Theorem 99 in Chapter X, Section 88, and similary for Propositions, Corollaries, or Lemmas. References to the first and second author correspond to alphabetical order (so the first author is Dovermann and the second is Schultz).

### *Guide to the contents of this book*

The first two chapters are a survey of equivariant surgery theory, and the last three chapters contain the periodicity theorems and some applications to analyzing the role of a basic technical condition in equivariant surgery called the Gap Hypothesis (this is defined in Section I.5). However, most of Chapter IV deals exclusively with A. Ranicki's theory of algebraic surgery (see [**Ra1**] and [**Ra2**]), and this portion of the book is independent of the remaining chapters.

As indicated by its title, Chapter I summarizes the main ideas in equivariant surgery theory. We do not assume any previous familiarity with equivariant surgery. In Section 1 we attempt to motivate the passage from ordinary to equivariant surgery by summarizing some applications of ordinary surgery to free actions of compact Lie groups; most of this work dates back to the nineteen sixties and seventies. Sections 2 through 5 are basically a survey of an equivariant surgery theory developed by the first author and M. Rothenberg [**DR**]. We look at this theory first because it illustrates all the main ideas but avoids various technical complications that appear in other equivariant surgery theories. In Section 5 we also include some extensions of [**DR**] that will be needed in subsequent chapters. Most of these results deal with relative or *adjusted* versions of the equivariant surgery obstruction groups in [**DR**]; a similar construction appears in work of W. Lück and I. Madsen [[**LM**, Part II, Section 1] where such groups are called *restricted*. Sections 5 and 7 also discuss some exceptional phenomena in limiting cases of the Gap Hypothesis that were discovered by M. Morimoto [**Mor**]. Finally, in Section 6 we describe some unpublished results of M. Rothenberg and S. Weinberger showing that a major result in Section 5—the equivariant $\pi - \pi$ theorem—fails to hold if one does not assume the Gap Hypothesis.

In Chapter II we summarize the various approaches to equivariant surgery in the literature and describe their relationships with each other. Section 1 introduces the *transverse-linear-isovariant* surgery theory constructed by W. Browder and F. Quinn in the early nineteen seventies [**BQ**]. This theory deals exclusively with equivariant maps satisfying very strong restrictions, but it does not require the Gap Hypothesis. The setting of [**BQ**] relies on the existence of a smooth stratification (in the sense of Thom and Mather) for the orbit space of a smooth manifold. We shall indicate how the necessary properties can be extracted from work of M. Davis [**Dav**], and in Section 4 we shall fill a gap in the literature by outlining a direct proof of the stratification theorem along the lines of W. Lellmann's 1975 *Diplomarbeit* [**Ll**]. In Section 2 we shall discuss several variants of the theory considered in Chapter I. The objective of the Chapter I theory is to convert an equivariant map of manifolds into an equivariant homotopy equivalence, where the manifolds in question have smooth group actions, the Gap Hypothesis holds, and certain simple connectivity conditions also hold. For certain problems in transformation groups it is preferable to have theories in which the objective or the conditions on the manifolds are modified. In particular, the objective might be to obtain an equivariant homotopy equivalence whose generalized equivariant Whitehead torsion (in the sense of [**DR2**] or [**I**]) is trivial, or the objective might be to obtain a map that is an equivariant homotopy equivalence after a suitable equivariant localization. Similarly, one might want to weaken the condition on the group action to piecewise linear or topological local linearity or to remove the simple connectivity assumption. However, in all these cases it is necessary to retain the Gap Hypothesis. Our discussion includes the relatively recent approach to equivariant surgery developed by Lück and Madsen [**LM**].

Most of the preceding material is expository. The new results of this book begin in Section II.3, where we study the relations between various theories and show that certain pairs of theories yield the same equivariant surgery obstruction groups. Much of this has been known to workers in the area for some time, but little has been written down. The central idea of equivariant surgery theory is to view an equivariant surgery problem as a sequence of ordinary surgery problems. As in the obstruction theory for extending continuous functions, one assumes that all problems up to a certain point can be solved and considers the next problem in the sequence. This problem determines a *stepwise surgery obstruction* that usually takes values in an ordinary surgery obstruction group; if this obstruction vanishes, one can solve the given problem in the sequence and proceed to the next one. The value groups for stepwise obstructions can also be viewed as adjusted obstruction groups for the equivariant surgery theory in question, and this suggests that a relation between equivariant surgery theories should involve a family of homomorphisms from the adjusted obstruction groups of one theory to the adjusted obstruction groups of another. Further analysis suggests that these homomorphisms should have compatibility properties like those of a natural transformation between homology or cohomology theories (see II.2.0 and II.3.1.A-B). In this setting it will follow that

> a transformation from one equivariant surgery theory to another is an isomorphism (modulo problems with borderline cases) if and only if the transformation induces isomorphisms on stepwise obstruction groups.

A precise statement of this principle appears in Theorem II.3.4. We apply this result to compare Browder-Quinn theories, the theory of Chapter I and some of its variants, and the Lück-Madsen theories. Specifically, we construct transformations between these theories and conclude that analogous theories of the different types determine the same groups (see Subsections II.3.A, E, and F). We shall also construct transformations between the different versions of Browder-Quinn and Lück-Madsen theories as well as the theory of Chapter I and its variants, and in Subsections II.B and C we shall use Theorem II.3.4 to obtain some qualitative information about the kernels and cokernels of those mappings.

In Chapter III we come to the main results of this book. Browder and Quinn showed in [**BQ**] that their transverse-linear-isovariant surgery theories have fourfold periodicity properties like those of ordinary surgery; in particular, the periodicity isomorphism is given geometrically by crossing with $\mathbf{CP}^2$. The link between algebraic and geometric periodicity arises from an integer valued invariant of an oriented manifold called the signature; for the complex projective plane this invariant equals $+1$. Similar results hold to a limited extent for Lück-Madsen theories and variants of the theories in Chapter I, and these periodicity isomorphisms are compatible with the transformations defined in Section II.3. However, there is one difficulty; namely, if one is not working with Browder-Quinn groups it is necessary to use $G$-manifolds $X$ such that both $X$ and $X \times \mathbf{CP}^2$ satisfy the Gap Hypothesis. Since it is always possible to find some $k(X) > 0$ such that $X \times (\mathbf{CP}^2)^k$ does not satisfy the Gap Hypothesis if $k \geq k(X)$, the $\mathbf{CP}^2$-periodicity for Lück-Madsen and Chapter I type theories is finite; in contrast, the $\mathbf{CP}^2$-periodicity properties of ordinary surgery obstruction groups and Browder-Quinn groups are infinite. On the other hand, Browder suggested an alternative in lectures at the 1976 A.M.S. Summer Symposium on Algebraic and Geometric Topology at Stanford. Specifically, if we let $\mathbf{CP}^2 {\uparrow} G$ denote a product of $|G|$ copies of $\mathbf{CP}^2$ and let $G$ act on $\mathbf{CP}^2$ by permuting the coordinates, then $\mathbf{CP}^2 {\uparrow} G$ is a smooth $G$-manifold and for all $k > 0$ the product $X \times (\mathbf{CP}^2 {\uparrow} G)^k$ will satisfy the Gap Hypothesis if $X$ does. Furthermore, if $G$ has odd order then $\mathbb{CP}^2\uparrow G$ satisfies an analog of the signature result for $\mathbb{CP}^2$; specifically, an equivariant refinement of the ordinary signature known as the *G-signature* of $\mathbb{CP}^2\uparrow G$ is the unit element of the real representation ring $RO(G)$. Thus if $G$ has odd order it is reasonable to ask whether crossing with $\mathbb{CP}^2\uparrow G$ induces isomorphisms of equivariant surgery obstruction groups. The central results of Chapter III show this is true for many equivariant surgery theories. Statements of the periodicity theorems appear in Section III.2; the most important special cases are Theorems III.2.7–9. To prove such results we shall

(1) show that crossing with certain smooth $G$-manifolds can be viewed algebraically as a transformation of equivariant surgery theories,
(2) use the results of Chapter II to reduce the proofs to questions about the effects of products on stepwise obstructions,
(3) interpret the equivariant products as the twisted products in ordinary surgery that were studied by T. Yoshida [**Yo**],
(4) use the results of [**Yo**] and a few elementary computations to show that products induce isomorphisms of stepwise obstruction groups.

In his Stanford lectures Browder noticed one further property of $\mathbb{CP}^2\uparrow G$ when $G$ has odd order. Namely, for each smooth $G$-manifold $X$ one can find some positive integer $n(X)$ such that $X \times (\mathbb{CP}^2\uparrow G)^n$ satisfies the Gap Hypothesis if $n \geq n(X)$. Browder also noted that this yields equivariant surgery invariants for $G$-surgery problems outside the Gap Hypothesis range; it suffices to look at the product of such a $G$-surgery problem with $(\mathbb{CP}^2\uparrow G)^n$ for suitable values of $n$. Our periodicity theorems imply that these obstructions are essentially the same. In Section 5 we shall develop this systematically and obtain a few general results. For example, if $G$ has odd order we show that the Browder-Quinn groups are direct summands of the Lück-Madsen groups. We shall view the process of crossing with $(\mathbb{CP}^2\uparrow G)^n$ for sufficiently large $n$ as a *periodic stabilization* of an equivariant surgery problem. This appears to be a useful first step in analyzing $G$-equivariant surgery without the Gap Hypothesis if $G$ has odd order; some further results in this direction are discussed in [**Sc**].

Recent work of M. Yan [**Ya**] yields periodicity theorems analogous to III.2.9 for the isovariant stratified surgery groups defined by S. Weinberger [**Wb**]. In fact, the results of [**Ya**] establish periodicity properties for the isovariant structure sequences of [**Wb**] that include the analog of III.2.9 and reduce to the usual fourfold periodicity of the topological surgery sequence from [**KiS**] (also see [**Ni**]) if $G$ is the trivial group.

In the final two chapters we formulate and prove periodicity theorems for other equivariant surgery theories. Chapter IV considers equivariant homology surgery with coefficients, both in the elementary sense of [**An**] and the more sophisticated sense of [**CS**]. The methods of Chapter III show that such periodicity theorems hold if there are analogs of Yoshida's work on twisted products for surgery with coefficients. Precise statements of the periodicity theorems for equivariant homology surgery appear in Section IV.6; the remaining sections of Chapter IV establish the necessary generalizations of Yoshida's results. Since these extensions of [**Yo**] are potentially of interest in other contexts, these sections are written so that they can be read independently of the rest of the book.

Finally, in Chapter V we consider the theories of **[DP1]** and **[DP2]** for equivariant surgery up to pseudoequivalences (*i.e.*, a homotopy equivalence that is $G$-equivariant but not necessarily a $G$-homotopy equivalence). These theories are significantly more difficult to handle than the others, and special considerations are needed at many steps. For example, more complicated notions of reference data are required, and extra restrictions must be placed on the surgery problems to be considered; these include the Euler characteristic and connectivity hypotheses of **[DP1]** and **[DP2]**. Section 1 summarizes these points. In order to extend the periodicity results and their proofs it is necessary to examine some new algebraic invariants that do not arise in the other theories. These invariants take values in subquotients of the projective class group $\widetilde{K}_0(\mathbf{Z}[G])$, and the behavior of such invariants with respect to products must be analyzed. Our results on this question appear in Sections 4 and 5, the conclusions are weaker than we would like, but they suffice to prove the periodicity theorems in some important cases (for example, if $G$ is abelian).

At various points throughout the book we mention questions that are related to the results of this book and seem to deserve further study. A few especially noteworthy examples appear in the next to last paragraph of the introduction to Chapter III and the final paragraphs of Sections III.4 and IV.4.

## References


[**An**] G. A. Anderson, "Surgery with Coefficients," Lecture Notes in Mathematics Vol. 591, Springer, Berlin-Heidelberg-New York, 1977.

[**Bre**] G. Bredon, "Introduction to Compact Transformation Groups," Pure and Applied Mathematics Vol. 46, Academic Press, New York, 1972.

[**Bro**] W. Browder, *Manifolds and homotopy theory*, in "Manifolds—Amsterdam 1970 (Proc. NUFFIC Summer School, Amsterdam, Neth., 1970)," Lecture Notes in Mathematics Vol. 197, Springer, Berlin-Heidelberg-New York, 1971, pp. 17–35.

[**BQ**] W. Browder and F. Quinn, *A surgery theory for G-manifolds and stratified sets*, in "Manifolds–Tokyo, 1973," (Conf. Proc. Univ. of Tokyo, 1973), University of Tokyo Press, Tokyo, 1975, pp. 27–36.

[**CS**] S. Cappell and J. Shaneson, *The codimension two placement problem and homology equivalent manifolds*, Ann. of Math. **99** (1974), 277–348.

[**tD**] T. tom Dieck, "Transformation Groups," de Gruyter Studies in Mathematics Vol. 8, W. de Gruyter, Berlin and New York, 1987.

[**DP1**] K. H. Dovermann and T. Petrie, *G-Surgery II*, Memoirs Amer. Math. Soc. **37** (1982), No. 260.

[**DP2** ] ________, *An induction theorem for equivariant surgery (G-Surgery III)*, Amer. J. Math. **105** (1983), 1369–1403.

[**DR**] K. H. Dovermann and M. Rothenberg, *Equivariant Surgery and Classification of Finite Group Actions on Manifolds*, Memoirs Amer. Math. Soc. **71** (1988), No. 379.

[**DR2**] ________, *An algebraic approach to the generalized Whitehead group*, in "Transformation Groups (Proceedings, Poznań, 1985)," Lecture Notes in Mathematics Vol. 1217, Springer, Berlin-Heidelberg-New York, 1986, pp. 92–114.

[**I**] S. Illman, *Equivariant Whitehead torsion and actions of compact Lie groups*, in "*Group Actions on Manifolds (Conference Proceedings, University of Colorado, 1983)*," Contemp. Math. Vol. 36, American Mathematical Society, 1985, pp. 91-106.

[**KiS**] R. C. Kirby and L. C. Siebenmann, "Foundational Essays on Topological Manifolds, Smoothings, and Triangulations," Annals of Mathematics Studies Vol. 88, Princeton University Press, Princeton, 1977.

[**Lat**] C. Latour, *Chirurgie non simplement connexe (d'après C. T. C. Wall)*, in "Séminaire Bourbaki Vol. 1970–1971," Lecture Notes in Mathematics Vol. 244, Exposé 397, Springer, Berlin-Heidelberg-New York, 1971, pp. 289–322.

[**Lees**] J. A. Lees, *The surgery obstruction groups of C. T. C. Wall*, Advances in Math. **11** (1973), 113–156.

[**Ll**] W. Lellmann, *Orbiträume von G-Mannigfaltigkeiten und stratifizierte Mengen*, Diplomarbeit, Universität Bonn, 1975.

**[LM]** W. Lück and I. Madsen, *Equivariant L-theory I*, Aarhus Univ. Preprint Series (1987/1988), No. 8; [*same title*] *II*, Aarhus Univ. Preprint Series (1987/1988), No. 16.

**[MS]** J. Milnor and J. Stasheff, "Characteristic Classes," Annals of Mathematics Studies Vol. 76, Princeton University Press, Princeton, 1974.

**[Ni]** A. Nicas, *Induction theorems for groups of homotopy manifold structures*, Memoirs Amer. Math. Soc. **39** (1982). No. 267.

**[Ra1]** A. A. Ranicki, *The algebraic theory of surgery I: Foundations*, Proc. London Math. Soc. **3:40** (1980), 87–192.

**[Ra2]** ———, *The algebraic theory of surgery II: Applications to topology*, Proc. London Math. Soc. **3:40** (1980), 193–283.

**[Sc]** R. Schultz, *An infinite exact sequence in equivariant surgery*, Mathematisches Forschungsinstitut Oberwolfach Tagungsbericht 14/1985 (Surgery and $L$-theory), 4–5.

**[Sp]** E. H. Spanier, "Algebraic Topology," McGraw-Hill, New York, 1967.

**[Wa]** C.T.C. Wall, "Surgery on Compact Manifolds," London Math. Soc. Monographs Vol. 1, Academic Press, London and New York, 1970.

**[Ya]** M. Yan, *Periodicity in equivariant surgery and applications*, Ph. D. Thesis, University of Chicago, in preparation.

**[Yo]** T. Yoshida, *Surgery obstructions of twisted products*, J. Math. Okayama Univ. **24** (1982), 73–97.

*Addendum: Additional references related to* [**Wa**]

Wall's book [**Wa**] assumes the reader is familiar with a considerable amount of material from differential and PL topology. The following references cover most of the material upon which [**Wa**] is based and might provide some helpful background information.

**[BJ]** Th. Bröcker and K. Jänich, "Introduction to Differential Topology," (Transl. by C. B. and M. J. Thomas), Cambridge University Press, Cambridge, U. K., and New York, 1982.

**[Bro2]** W. Browder, "Surgery on Simply Connected Manifolds," Ergeb. der Math. (2) 65, Springer, New York, 1972.

**[Coh]** M. Cohen, "A Course in Simple Homotopy Theory," Graduate Texts in Mathematics Vol. 10, Springer, Berlin-Heidelberg-New York, 1973.

[**HP**] A. Haefliger and V. Poenaru, *La classification des immersions combinatoires*, I. H. E. S. Publ. Math. **23** (1964), 75–91.

[**Hi1**] M. W. Hirsch, *Immersions of differentiable manifolds*, Trans. Amer. Math. Soc. **93** (1959), 242–276.

[**Hi2**] ————, "Differential Topology," Graduate Texts in Mathematics Vol. 33, Springer, Berlin-Heidelberg-New York, 1976.

[**Hud**] J. F. P. Hudson, "Piecewise Linear Topology," W. A. Benjamin, New York, 1969.

[**Krv**] M. Kervaire, *Le théorème de Barden-Mazur-Stallings*, Comment. Math. Helv. **40** (1965), 31–42.

[**Miln**] J. Milnor, "Lectures on the $h$-cobordism Theorem," Princeton Mathematical Notes No. 1, Princeton University Press, Princeton, N. J., 1965.

[**Mun**] J. R. Munkres, "Elementary Differential Topology (Revised Edition)," Annals of Mathematics Studies Vol. 54, Princeton University Press, Princeton, N. J., 1966.

[**Ph**] A. V. Phillips, *Submersions of open manifolds*, Topology **6** (1967), 171–206.

[**RS**] C. P. Rourke and B. J. Sanderson, "Introduction to Piecewise Linear Topology," Ergebnisse der Math. Bd. 69, Springer, Berlin-Heidelberg-New York, 1972.

# CHAPTER I

# INTRODUCTION TO EQUIVARIANT SURGERY

One of the main themes in topology is the development of techniques for passing back and forth between algebraic and geometric information. In particular, the phrase "surgery theory" generally refers to a collection of techniques for studying the structure of manifolds by means of homotopy theory and algebraic structures involving quadratic forms. For the most part, these techniques were developed during the past thirty years. At a very early point, topologists realized that surgery theory could be applied effectively to study a fairly wide range of problems involving group actions on manifolds; many ways of doing this have been developed. Several approaches can be described as *equivariant surgery theories* that are formally parallel to ordinary surgery theory as in Wall's book [**Wl**]. The underlying idea is simple: In ordinary surgery theory one considers maps between manifolds, and in equivariant surgery one attempts to consider equivariant maps between manifolds with group actions by similar methods. Some aspects of ordinary surgery theory extend easily to the category of manifolds with group actions. On the other hand, new types of difficulties appear when surgery theory is extended to manifolds with group actions, and effective means for dealing with such problems presently exist only if the manifolds or mappings satisfy some additional hypotheses. There are several possible choices of conditions that are useful in different contexts, and each choice has an associated version of equivariant surgery theory.

Although there are often major differences between the various approaches to equivariant surgery, such theories all satisfy some basic formal properties. In this chapter we shall describe a version of equivariant surgery that has relatively few technical complications but still illustrates the formal properties that such theories have in common. The specific theory to be considered is called $I^{ht,DIFF}$ in [**DR**], where it is studied at length; the same theory also appears in several other references including [**PR**]. A survey of other approaches to equivariant surgery appears in Chapter II of this book.

### *Applications of equivariant surgery*

One of the principal reasons for introducing equivariant surgery theories is their usefulness in studying some basic problems in transformation groups. Unfortunately, an account of the applications would require a considerable amount of extra mathematical material that is not closely related to the main topics of this book, and consequently we have not attempted to treat this important aspect of equivariant surgery theory. Many of the various types of applications are presented in the books by M. Davis [**Dav2**] and T. Petrie and J. Randall [**PR**], a survey article by T. Petrie and the authors [**DPS**], and articles by the second author [**Sc3**], V. Vijums [**Vj**], the first author and L. Washington

[**DW**], and M. Rothenberg and G. Triantafillou [**RT**]. Of course, there are also other applications of surgery to transformation groups beyond the sorts described in these articles; the latter were chosen as a representative selection with references to other work in the area.

## 1. Ordinary surgery and free actions

Equivariant surgery theories evolved from earlier applications of ordinary surgery theory to transformation groups (see [**Brdr**], [**RS**]). In fact, some of these conclusions are essentially special cases of the equivariant surgery theories that were developed subsequently. In this section we shall describe some results of this type in order to provide motivation and background information.

An action of a group $G$ on a space $X$ is said to be *free* if for each $x \in X$ the isotropy subgroup $G_x$ is trivial. If $X$ is a reasonable space and $G$ is a compact Lie group that acts freely and continuously on $X$, then the orbit space projection $X \to X/G$ is a principal $G$-bundle projection (compare [**Bre2**]). Furthermore, if $X$ is a smooth manifold and $G$ acts smoothly and freely on $X$, then $X/G$ has a canonical smooth structure such that the orbit space projection is a smooth principal $G$–bundle (see [**GL**]).

These considerations "reduce" the study of free $G$–actions on spaces to the study of their orbit spaces and (the homotopy classes of) their classifying maps into the universal base space $BG$. The following special case of this reduction principle reflects the relevance of surgery theory to questions involving smooth $G$–manifolds:

PROPOSITION 1.1. *Let $M^n$ be a closed smooth manifold, and let $G \to E \to X$ be a principal $G$–bundle over a finite complex $X$ such that $E$ is homotopy equivalent to $M$ (perhaps $E = M$). Then passage to the orbit space defines a $1-1$ correspondence*

$$\left[\begin{array}{c}\textit{free smooth}\\ G-\textit{manifolds}\\ \textit{equivariantly}\\ \textit{homotopy equivalent}\\ \textit{to } E\end{array}\right] \cong \left[\begin{array}{c}\textit{smooth}\\ \textit{manifolds}\\ \textit{homotopy}\\ \textit{equivalent to}\\ X = E/G\end{array}\right]$$

■

COMPLEMENT 1.2. *A similar result holds in the topological category if we restrict attention to free actions that are locally linear (i.e., the orbit space is a topological manifold).*

REMARK: If $G$ is finite, then every free action of a finite group on a manifold is locally linear. On the other hand, if $G$ is a positive–dimensional compact Lie group, then the results of [**Lgr**] yield large, systematic families of free $G$–manifolds that are not locally linear (compare [**KS**]).

Proposition 1.1 and Complement 1.2 provide the basis for applying surgery theory to questions about manifolds with free actions of compact Lie groups. The special cases