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George Isac

Complementarity Problems

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INTRODUCTION

In 1984 we were invited, by the Department of Mathematics of University of Limoges, to give several lectures on a subject considered interesting in Nonlinear Analysis and Optimization.

So, we decided to present the subject "Complementarity Problems (In Infinite Dimensional Spaces)".

After this course, we became quickly conscious that a volume on all mathematical aspects of these nice problems is necessary.

The literature on this subject is already impressive and the task to write this volume was not easy.

The Complementarity Problem is considered by many mathematicians, as a large independent division of Mathematical Programming Theory, but our opinion is quite different.

The Complementarity Problem represents a very deep, very interesting and very difficult mathematical problem. This problem is a very nice research domain because it has many interesting applications and deep connections with important chapters of the Nonlinear Analysis.

Our principal aim is to present the all principal mathematical aspects about the Complementarity Problems.

To be agree with this aim we consider generally, Nonlinear Complementarity Problems in infinite dimensional spaces. But, the finite dimensional case is not neglected and several important results about the linear or the Nonlinear Complementarity Problems specific for this case are also presented.

Several problems arising in various fields (for example: Economics, Game Theory, Mathematical Programming, Mechanics, Elasticity Theory, Engineering, and, generaly, several "Equilibrium Problems") can be stated in the following unified form:

given $f:R^n_+ \to R^n$ a mapping,

$$(1): \quad \left\| \begin{array}{l} \text{find } x_o \in R^n_+ \text{ such that,} \\ f(x_o) \in R^n_+ \text{ and } \langle x_o, f(x_o)\rangle = 0, \end{array} \right.$$

(where $< , >$ is the inner product, $\langle x,y \rangle = \sum_{i=1}^{n} x_i y_i$).

Problem (1) is called the <u>Complementarity Problem</u> and the origin of this problem is perhaps in the <u>Kuhn-Tucker Theorem</u> for nonlinear programming (which gives the necessary conditions of optimality when certain conditions of differentiability are met), or perhaps in the old and neglected Du Val's paper. [P. Du Val, The Unloading Problem for Plane Curves, Amer. J. Math., 62 (1940), 307-311.]

Certainly, one thing is clear: in 1961 Dorn showed that, if A is a positive-definite (but not necessarily symmetric) matrix, then the minimum value of the quadratic programming problem,

$$(2):\quad \left\| \begin{array}{l} \min \langle x,\ Ax + q \rangle \\ x \in \mathcal{D} \\ \quad \mathcal{D} = \{x \in R^n :\ 0 \le x\ ,\ 0 \le Ax + q\} \\ q \in R^n \end{array} \right.$$

is zero. [W.S. Dorn, Self-dual Quadratic Programs, SIAM J. Appl. Math., Vol.9, Nr.1, (1961), 51-54.]

Dorn's paper was the first step in treating the Complementarity Problem as an independent problem.

In 1963 Dantzig and Cottle generalized Dorn's result to the case when all the principal minors of the matrix A are positive. [G.B. Dantzig and R.W. Cottle, Positive semi-definite Programming, (Nonlinear Programming. A course. J. Abadie (ed.). North-Holland, Amsterdam, (1967), 55-73)].

The result annouced in 1963 by Dantzig and Cottle was generalized in 1964 and 1966 by Cottle to a certain class of nonlinear functions. [R.W. Cottle, a) Notes on a Fundamental Theorem in Quadratic Programming, SIAM J. of Appl. Math., Vol. 12, (1964), 663-665; b) Nonlinear Programs with Positively Bounded Jacobians, SIAM J. of Appl. Math. Vol. 14, (1966), 147-158].

Also, in 1965 Lemke proposed the Complementarity Problem as a method for solving matrix games. [C.E. Lemke, Bimatrix Equilibrium Points and Mathematical Programming, Manag. Sci., Vol. 11, Nr 7, (1965), 681-689].

Certainly, one of the first important papers on the Complementarity Problem is the Ingleton's paper, [A.W. Ingleton, A Problem in Linear Inequalities, Proc. London Math. Soc., Vol. 16, (1966), 519-536], which showed the importance of the Complementarity Problem in engineering applications.

It seems that the term "Complementarity" was proposed by Cottle, Habetler and Lemke, and the reason is the following observation.

A solution x^o of problem (1) is said to be nondegenerate if at most n components of 2n-components vector $(x^o, f(x^o))$ equal zero.

Otherwise, it is a degenerate solution.

We denote, $N_n = \{1, 2, \dots, n\}$. If $x^o = (x_i^o)_{i=1, 2, \dots, n}$ is a nondegenerate solution of problem (1) and $y^o = f(x^o)$, then the sets,

$A = \{i \mid x_i^o > 0\}$; $B = \{i \mid y_i^o > 0\}$, where $y^o = (y_i^o)_{i=1, 2, \dots, n}$)

are complementary subsets of N_n (that is, $A = C_{N_n} B$).

After 1970 the theory of the Complementarity Problem has known a strong and ascending development, based on several important results obtained by Cottle, Eaves, Karamardian, Mangasarian, Saigal, Gould, Garcia, Moré, Kojima, Megiddo, Kaneko and Kostreva, Pang, etc.

In Chapter 1 we present some preliminary definitions and the definitions of principal complementarity problems.

In Chapter 2 we give examples of practical problems which have as mathematical model a specific complementarity problem. Other models, especially in infinite dimensional spaces are presented in other chapters.

The important mathematical problems equivalent to complementarity problems are studied in Chapter 3.

In Chapter 4 we present the principal existence theorems and we study some properties of solution set.

The order Complementarity Problem and the Implicit Complementarity Problem are respective studied in Chapter 5 and 6.

In Chapter 7 we introduce the notion of isotone projection cone and we use this notion to study the Complementarity Problem.

The last chapter is devoted to the sutdy of several problems, about the Complementarity Problem not considered in other chapters and which are opened to new researchers.

The last time, two books on the Complementarity Problem were published: 1°) K.G. Murty: Linear complementarity, linear and nonlinear programming. Heldermann Verlag, Berlin (1988). 2°) R.W. Cottle, J.S. Pang and R.E. Stone: The linear complementarity problem. Academic Press (1992), but our volume is completely different and essentially it is a complementary book.

We hope that our notes form a satisfactory introduction to the study of the Complementarity Problem, which is a fascinating problem by its simplicity and profoundness, and by the fact that it is a cross-point of several chapters of fundamental and applied mathematics.

The part on numerical methods solving complementarity problems is not considered in this volume, since this part can be considered as a subject for another volume.

Many numerical methods for the Linear Complementarity Problem are studied in the cited books.

To select the subjects considered in this volume the author used the openness to new developments and his personal preferences, since the principal motor in mathematics is the pleasure to do mathematics.

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St-Jean, Québec
1992

CHAPTER 1

PRELIMINARIES AND DEFINTIONS OF PRINCIPAL COMPLEMENTARITY PROBLEMS

1.1 Notations, definitions and necessary results. In this section we specify some terms and notations used systematically in this paper.

First, we suppose known the definitions and the fundamental properties of Hilbert, Banach and locally convex spaces [B8], [B9], [B16].

We denote by $(H, \langle\, ,\, \rangle)$ a Hilbert space, by $(E, \|\ \|)$ a Banach space and by $E(\tau)$ a locally convex space. In this paper we consider only real vector spaces and it is clear that every Banach or Hilbert space is a locally convex space.

If $E(\tau)$ is a locally convex space, then E^* denotes the topological dual of E.

We say that $\langle E, F\rangle$ is a dual system if, E and F are vector spaces and $\langle\, ,\, \rangle$ is a bilinear functional on $E \times F$ such that,

1°) $\langle x, y\rangle = 0$, for each $x \in E \Longrightarrow y = 0$,

2°) $\langle x, y\rangle = 0$, for each $y \in F \Longrightarrow x = 0$.

If $E(\tau)$ is a locally convex space, we denote by $\langle E, E^*\rangle$ the dual system defined by the bilinear functional, $\langle x, u\rangle = u(x)$; for every $x \in E$ and every $u \in E^*$.

Let E be a real vector space. A subset $K \subset E$ is said to be a convex cone if the following conditions are satisfied:

$c_1$1) $K + K \subset K$

$c_2$2) $(\forall\, \lambda \in R_+)(\lambda K \subset K)$.

If $K \subset E$ is a convex cone, then we can define a preorder on E by:

$$"x \leq y" \iff y - x \in K.$$

Always the preorder defined on E by K, will be denoted by "$\leq$ ".

We can prove that, if $K \subset E$ is a pointed convex cone, that is, K is a convex cone and satisfies in addition,

c_3) $K \cap (-K) = \{0\}$,

then the preorder "$\leq$ " is an order, that is, it is a reflexive, transitive and antisymmetric relation.

Also, it is important to remark that, if $K \subset E$ is a pointed convex cone, then the order "$\leq$" is compatible with the linear structure, that is, the following two conditions are satisfied:

0_1) $x \leq y \Longrightarrow x + z \leq y + z$; $\forall\, x, y, z \in E$,

0_2) $x \leq y \Longrightarrow ax \leq ay$; $\forall\, a \in R_+$, $\forall\, x, y \in E$.

If for the vector space E is defined a pointed convex cone $K \subset E$, then we say that (E, K) is an ordered vector space.

Conversely, if on the vector space E is defined an order "$\leq$" satisfying 0_1) and 0_2) then the set, $K = \{x \in E | x \geq 0\}$ is a pointed convex cone.

An ordered vector space (E, K) is said to be a vector lattice if in addition, every non-empty finite subset of E has greatest lower bound.

Hence, if (E, K) is a vector lattice, then in particular, there exists sup $(x, y) \in E$ for every $x, y \in E$. In this case there exists also, inf $(x, y) \in E$ and we have, $\inf(x, y) = -\sup(-x, -y)$.

Obviously, if we consider the n-dimensional real vector space,

$R^n = \{x | x = (x_1, x_2, \ldots, x_n), x_i \in R; \forall i = 1, 2, \ldots, n\}$ then

$R^n_+ = \{(x_1, x_2, \ldots, x_n) | x_i \geq 0; \forall i = 1, 2, \ldots, n\}$ is a pointed convex cone and "$\leq$" is exactly the usual order considered on R^n and (R^n, R^n_+) is a vector lattice.

We observe that, R^n is a Hilbert space with respect to the inner product, $\langle x,y \rangle = \sum_{i=1}^{n} x_i y_i$, where, $x = (x_i)_{i=1,2,\ldots,n}$ and $y = (y_i)_{i=1,2,\ldots,n}$

Several interesting examples and properties of ordered vector spaces we find in [B12], [B16].

If $\langle E,F \rangle$ is a dual system and $K \subset E$ is a convex cone, then we denote,

$K^* = \{u \in F | \langle x,u \rangle \geq 0 \ ; \ \forall x \in K\}$,

$K^\circ = \{u \in F | \langle x,u \rangle \leq 0 \ ; \ \forall x \in K\}$

and we observe that K^* (resp. K°) is a convex cone.

The cone K^* (resp. K°) is called the dual (resp. the polar) cone of K. If K is closed then $K = K^{**} = (K^*)^*$.

Consider a dual system $\langle E,E^* \rangle$, where $E(\tau)$ is a locally convex space and $f:E \to R$ a functional.

The functional f is said to have a Gâteaux derivative at $x_o \in E$ if there exists $u(x_o) \in E^*$ such that,

(G) : $\lim_{\lambda \to 0} [f(x_o + \lambda x) - f(x_o)]/\lambda = \langle x, u(x_o) \rangle ; \ \forall x \in E.$

We denote, $\nabla f(x_o) = u(x_o)$ and $\nabla f(x_o)$ is called the Gâteaux derivative, or the gradient of f at x_o.

If for every $x_o \in E$ holds (G), the functional f is said to be differentiable in the Gâteaux sens in E and the operator $\partial_G : E \to E^*$, which with every x_o associates $\partial_G(x_o) = \nabla f(x_o)$ is said to be the Gâteaux differential of f in E.

Let $f:E \to R \cup \{+\infty\}$ be a convex functional and $x_o \in E$.

The functional f is said to be subdifferentiable at x_0 if the set,

$$\partial f(x_0) = \{u \in E^* \mid f(x)-f(x_0) \geq \langle x-x_0, u\rangle;\ \forall x \in E\}$$

is non-empty.

The set $\partial f(x_0)$ is said to be the subgradient of f at x_0.

If for every $x_0 \in E$, $\partial f(x_0) \neq \phi$, we say that f is subdifferentiable in E and the application, $\partial f: E \to 2^{E^*}$, which with every $x_0 \in E$ associates $\partial f(x_0)$ 2^{E^*} is called the subdifferential of f.

We remark that, if f is Gâteaux differentiable at x_0 then $\partial f(x_0) = \{\nabla f(x_0)\}$.

Let $K \subset E$ be a convex set. The indicator of K is the function defined by:

$$\Psi_K(x) = \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{if } x \notin K \end{cases}$$

If $K \subset E$ is a convex cone, then we have,

$$\partial \Psi_K(x_0) = \{x^* \in E^* \mid x^* \in K^\circ \text{ and } \langle x_0, x^*\rangle = 0\}$$

Consider again the dual system $\langle E, E^*\rangle$ and $f: E \to 2^{E^*}$ a mapping. The effective domain of f is, $D(f) = \{x \in E \mid f(x) \neq 0\}$ and its graph is, $Gr(f) = \{(x, f(x)) \mid x \in D(f)\}$.

The mapping f is said to be monotone if,

$$(\forall x,y\ D(f))(\forall x^* \in f(x))(\forall y^* \in f(y))(\langle x-y, x^*-y^*\rangle \geq 0).$$

and f is said to be maximal monotone if it is monotone and there does not exist $f: E \to 2^{E^*}$ such that f is monotone and $Gr(f) \underset{\neq}{\subset} Gr(f)$.

The mapping f is said to be α-monotone if there exists a strictly increasing function $\alpha: [0, +\infty[\longrightarrow [0, +\infty[$ such that,

α_1) $\alpha(0) = 0$,

α_2) $\lim_{t \to +\infty} \alpha(t) = +\infty$,

α_3) $(\forall x, y \in D(f))(\forall x^* \in f(x))(\forall y^* \in f(y))(\langle x-y, x^*-y^*\rangle \geq \|x-y\| \alpha(\|x-y\|))$.

Also, the mapping $f: E \to 2^{E^*}$ is said to be strictly monotone if:

m_1) f is monotone,

m_2) $\langle x - y, x^* - y^*\rangle > 0$, if $x \neq y$, $x^* \in f(x)$ and $y^* \in f(y)$.

If f is α-monotone, where $\alpha(t) = \rho t^2$, $\rho \in R_+ \setminus \{0\}$ then we say that f is strongly monotone.

A mapping $f: C \to E^*$, where C is a convex subset of E is said to be hemicontinuous if it is continuous from the line segment of C to the weak topology of E^*.

We recall that an operator, $f: E \to F$, where $(E, \|\ \|)$, $(F, \|\ \|)$ are two Banach spaces, is k-Lipschitz if there exists a constant $k > 0$ such that,

$\|f(x) - f(y)\| \leq k\|x - y\|$; $\forall$ $x, y \in E$.

If $0 < k < 1$ then f is said to be a contraction.

Finally, we denote by M_{nxn} (R) the space of n x n real matrices. If for every $n \in N$ we denote, $P_n = \{1, 2, \ldots, n\}$ and $P = \{P_n\}_{n \in N}$ then a matrix $A \in M_{nxn}$ (R) is a function $A: P_n \times P_n \to R$.

We denote, $a_{ik} = A(i, k)$.

If $P \subset P_n$ and $A \in M_{nxm}$ (R), we denote by $A(P) = A_{|PxP}$ (the restriction of A to PxP).

The real number, det A(P) is called the principal minor of order P of A.

The following fundamental results are necessary in the development of this paper.

Let $\langle E, E^* \rangle$ be a dual system where $E(\tau)$ is a locally convex space.

We say that, an element $x^* \in E^*$ is normal to a convex set $K \subset E$ at a point x if:

n_1) $x \in K$,

n_2) $(\forall\ y \in K)(\langle y - x, x^* \rangle \leq 0)$

For each $x \in E$, the set of all x^* normal to K at x is called the normal cone to K at x.

The normal cone to K at x is weak* closed convex cone in E^*, it is empty when $x \notin K$ and it contains at least the zero element of E^* when $x \in K$.

The multivalued mapping from E to E^* which assigns to each $x \in E$ the normal cone to K at x is called the normality operator for K.

The normality operator for K is actually the subdifferential of the indicatrix Ψ_K of K, so that it is a maximal monotone operator with effective domain K(if K is non-empty).

The following result was proved by Rockafellar and it is a theorem of Browder-Stampacchia type.

Theorem [Browder-Stampacchia-Rockafellar]. [B.14]

Let $(E, \|\ \|)$ be a Banach space and let $K \subset E$ be a non-empty closed convex subset. Let $f_1: E \to E^*$ be the normality operator for K and let $f_2 2: E \to E^*$ be a monotone operator (not multivalued and not necessarily maximal) such that, $D(f_2) \supset K$.

If f_2 is hemicontinuous then $f_1 + f_2$ is a maximal monotone operator.

A mapping $f: E \to E^*$ is said to be bounded, if for every bounded set $B \subset E$, f(B) is bounded.

Theorem [MOSCO] [B10]

Let $(E, \| \|)$ be a reflexive Banach space and let $K \subset E$ be a closed convex cone.

Suppose that $f:K \to E^*$ is a bounded, hemicontinuous strictly monotone operator and consider $\{K_r\}$ a family of non-empty closed convex subsets of K.

Then, for every r there exists a unique element $x_r \in K_r$ such that, $\langle z-x_r, f(x_r)\rangle \geq$ ≥ 0; $\forall z \in K_r$.

Theorem [Rockafellar] [B14]

Let $(E, \| \|)$ be a reflexive Banach space and let $f:E \to E^*$ be a maximal monotone operator.

If there exists a real number $\beta>0$ such that, $(\forall x \in D(f))(\|x\|>\beta)(\forall x^* \in f(x))(\langle x,x^*\rangle>0)$ then there exists an element $x_o \in E$ such that, $0 \in f(x_o)$.

1.2 Complementarity problems. (Definitions and problems). Dorn [A75] considered in 1961 the following optimization problem,

(1.2.1):
$$\left| \begin{array}{l} \min\limits_{x \in \mathcal{F}} f(x) \\ \text{where: } \mathcal{F} = \{x \in R^n \mid x \geq 0 \ , \ Ax + b \geq 0\} \\ \qquad A \in \mathcal{M}_{nxn}(R), b \in R^n \text{ and } f(x) = \langle x, Ax + b \rangle \end{array} \right.$$

and he showed that, when A is a positive definite (thought not necessarily symmetric) matrix, then the quadratic program (1.2.1) must have an optimal solution and $\min\limits_{x \in \mathcal{F}} f(x) = 0$.

Dorn's paper was the first step in treating the complementarity problems.

In 1964 Cottle [A42] studied problem (1.2.1) under the assumption that A is a positive semidefinite matrix and he remarked that, in this case it is not true that (1.2.1) must possess an optimal solution.

However, if A is positive semidefinite and $\mathcal{F} \neq \emptyset$ then an optimal solution for (1.2.1) exists and again, $\min\limits_{x \in \mathcal{F}} f(x) = 0$.

After three years, Dantzig and Cottle [A71] constructively showed that, if A is a square (not necessarily symmetric) matrix, all of whose principal minors are positive, then problem (1.2.1) has an optimal solution x_* satisfying the following equation,

(1.2.2): $\langle x_*, Ax_* + b \rangle = 0$.

This result was later generalized by Cottle [A43].

More precisely, Cottle considered the following nonlinear program associated to a continuously differentiable mapping $h:R^n \to R^n$,

(1.2.3):
$$\begin{cases} \min f(x) \\ x \in F \\ \text{where: } f(x) = \langle x, h(x) \rangle \text{ and} \\ F = \{x \in R^n \mid x \geq 0,\ h(x) \geq 0\} \end{cases}$$

and he showed that, if x_o is an optimal solution of program (1.2.3) and the Jacobian matrix $J_h(x_o)$ has positive principal minors, then x_o satisfies the following conditions:

(1.2.4):
$$\begin{cases} x_o \geq 0;\ h(x_o) \geq 0 \quad \text{and} \\ \langle x_o, h(x_o) \rangle = 0 \end{cases}$$

Thus, in relation with program (1.2.3) Cottle obtained in 1966 the first nice result on the Nonlinear Complementarity Problem. This result is the following.

Consider a differentiable mapping $h:R^n \to R^n$.

We say that h has a positively bounded Jacobian matrix $J_h(x)$, if there exists a real number $0 < \delta < 1$ such that for every $x \in R^n$, each principal minor of $J_h(x)$ is an element of the interval $[\delta, \delta^{-1}]$.

A solution (y, x) of the equation $y - h(x) = 0$ is nondegenerate if at most n of its 2n components are zero.

Theorem 1.2.1 [Cottle] [A43]

If $h:R^n \to R^n$ is a continuous differentiable mapping such that the solutions of $y - h(x) = 0$ are non degenerate and if h has a positively bounded Jacobian matrix $J_h(x)$, then there exists an element $x_o \in R^n_+$ such that, $h(x_o) \geq 0$ and $\langle x_o, h(x_o) \rangle = 0$.

We note that another important result which contributed to the development of the Complementarity Theory is the Lemke's paper [A 173]. In this paper Lemke proposed the Complementarity Theory as method for solving matrix games.

Certainly, the development of the Complementarity Theory was imposed by a large variety of applications in fields as: Optimization, Economics, Games Theory, Mechanics, Variational Calculus, Stochastic Optimal Control Theory etc.

The Complementarity Theory is closely linked with two other problems, the solution of variational inequalities and the determination of the fixed points for a given mapping.

Thus, the existence theorems and the methods used in the study of the last two problems are widely used in the Complementarity Theory and conversely, the ideas and methods developed specially for complementarity problems are used to solve variational inequalities or to solve fixed point problems.

In this section we present the principal complementarity problems studied till now.

Some of these problems has been much studied till now, but other ones are very little known.

Concerning the Complementarity Problem we distinguish two entirely distinct class of problems: the Topological Complementarity Problem (T.C.P.) and the Order Complementarity Problem (O.C.P.).

A. Topological Complementarity Problems.

In this class we have the following problems.

A.1-The generalized complementarity problem.

Let $<E, F>$ be a dual system of locally convex spaces.

For a given closed convex cone $K \subset E$ and a mapping $f:K \to F$, the Generalized Complementarity Problem (associated to K and f) is,

$$(G.C.P.): \left\| \begin{array}{l} \text{find } x_o \in K \text{ such that,} \\ \\ f(x_o) \in K^* \text{ and } <x_o, f(x_o)> = 0. \end{array} \right.$$

Remarks

(1.2.1) If $f(x) = L(x) + b$, where $L:E \to F$ is a linear mapping and b an element of F then we have the Linear Complementarity Problem (L.C.P.).

(1.2.2) If $f:K \to F$ is a nonlinear mapping, then we have the Nonlinear Complementarity Problem (N.C.P.).

(1.2.3) If $E = F = R^n$, $K = R^n_+$, $<x, y> = \sum_{i=1}^{n} x_i y_i$, where $x = (x_i)$, $y = (y_i) \in R^n$, $A \in M_{nxm}(R)$ and $b \in R^n$, then we obtain the classical Linear Complementarity Problem,

$$(L.C.P.): \left\| \begin{array}{l} \text{find } x_o \geq 0 \text{ such that} \\ \\ Ax_o + b \geq 0 \text{ and } <x_o, Ax_o + b> = 0 \end{array} \right.$$

(In this case, $K = K^*$).

(1.2.4) The following Special Linear Complementarity Problem is used in the study of some structural engineering problems [A185], [A152].

We consider, $E = F = R^n$, $K = R^n_+$, $<x, y> = \sum_{i=1}^{n} x_i y_i$, (where $x = (x_i)$, $y = (y_i)$), $M, N, P \in M_{nxm}(R)$, $q, r \in R^n$ and supposing that $f(x, v) = q + Mv + Nx$ and $y(v) = r - Pv$, we are interested to study the following complementarity problem:

$$(S.L.C.P.): \left\| \begin{array}{l} \text{find } x_o, v_o \in R^n_+ \text{ such that,} \\ f(x_o, v_o) \in R^n_+, \; y(v_o) \in R^n_+, \\ <v_o, f(x_o, v_o)> = 0 \text{ and } <x_o, y(v_o)> = 0 \end{array} \right.$$

We can show that problem (S.L.C.P.) is equivalent to an ordinary linear complementarity problem.

Indeed, if we set,

$$Z = \begin{bmatrix} v \\ x \end{bmatrix}; \; q_o = \begin{bmatrix} q \\ r \end{bmatrix}; \; L = \begin{bmatrix} M & N \\ -P & 0 \end{bmatrix}; \; F(Z) = LZ + q_o$$

and if we consider, $E = F = R^{2n}$, $K = R^{2n}_+$, $X, Y = \sum_{i=1}^{2n} X_i Y_i$; $X = (X_i)$; $Y = (Y_i)$, then problem (S.L.C.P.) is equivalent to the following linear complementarity problem:

$$(1.2.5): \left| \begin{array}{l} \text{find } Z_o \in R_+^{2n} \text{ such that,} \\ \\ \text{find } F(Z_o) \in R_+^{2n} \text{ and } < Z_o, F(Z_o) > = 0 \end{array} \right.$$

A.2-The Generalized Multivalued Complementarity Problem.

This problem is important in the study of some practical problems, as for example some problems defined in Economics.

If $<E, F>$ is a dual system of locally convex spaces, $K \subset E$ a closed convex cone and $f:K \to F$ a multivalued mapping (that is $f:K \to 2^F$), then the Generalized Multivalued Complementarity Problem, associated to f and K is:

$$(G.M.C.P.): \left| \begin{array}{l} \text{find } x_o \in K \text{ and } y_o \in F \text{ such that,} \\ y_o \in f(x_o) \cap K^* \text{ and } < x_o, y_o > = 0. \end{array} \right.$$

A.3-Parametric Complementarity Problems.

Supposing defined a dual system of locally convex spaces $<E, F>$, a closed convex cone $K \subset E$, a topological space T and a mapping $f:K \times T \to F$, the Generalized Parametric Complementarity Problem is:

$$(G.P.C.P.): \left| \begin{array}{l} \text{find the point-to-set mapping} \\ X_o:T \to E, \text{ such that, for every } X_o(t) \neq \phi, \\ \text{if } x_o(t) \in X_o(t) \text{ then, } x_o(t) \in K, \ f(x_o(t), t) \in K^* \\ \text{and } < x_o(t), f(x_o(t), t) > = 0 \end{array} \right.$$

Remarks

(1.2.6) If $E = F = R^n$, $T = R_+$, $< x, y > = \sum_{i=1}^{n} x_i y_i$; where $x = (x_i)$, $y = (y_i)$, $K = R_+^n$, $A \in \mathcal{M}_{nxm}(R)$, $q, p \in R^n$ and $f(x, t) = Ax + q + tp$, then we obtain the Parametric Complementarity Problem defined in the Elastoplastic Structures Theory [A185], [A186].

(1.2.7) The Parametric Complementarity Problem has close relations with the Sensitivity Theory of nonlinear programming problems and in these cases is interesting to study the existence of continuously or differentiable selections of the solution mapping X_o.

In finite dimensional spaces we distinguish another interesting parametric complementarity problem. This problem was considered by Meister [A212].

Let $\mathcal{D} \subset R^n$ be a set of the form, $\mathcal{D} = R_+^k \times R_+^k \times Q$, where $\mathcal{D}$ takes the form $R_+^k \times R_+^k$ for $m = 2k$ and $\mathcal{D}$ is an arbitrary interval in R^m for $k = 0$.

We remark that $\mathcal{D}$ is supposed to be with non-empty interior.

Given a mapping $f: \mathcal{D} \to R^n$ and supposing, $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$, we consider the Special Parametric Complementarity Problem,

$$(\text{S.P.C.P.}): \left\| \begin{array}{l} \text{find } (x_o, y_o, z_o) \in \mathcal{D} \text{ such that,} \\ f(x_o, y_o, z_o) = 0 \text{ and } \langle x_o, y_o \rangle = 0 \end{array} \right.$$

Certainly, we can define this problem in infinite dimensional spaces but its study is more complicated.

A.4-Implicit Complementarity Problems.

The origin of the Implicit Complementarity Problems is the dynamic programming approach of stochastic impulse and of continuous optimal control.

It is not without interest to know that there exist deep and interesting relations between the Implicit Complementarity Problems and the Quasivariational Inequalities Theory [A18], [A19], [A20], [A21], [A22], [A37], [A222].

We consider a locally convex space $E(\tau)$, a closed convex cone $K \subset E$, an element $b \in E$ and two mappings $A, M: E \to E$.

Given a bilinear functional $\langle\, ,\, \rangle$ on $E \times E$, the Implicit Complementarity Problem is,

$$(\text{I.C.P.}): \left\| \begin{array}{l} \text{find } x_o \in E \text{ such that,} \\ M(x_o) - x_o \in K,\ b - A(x_o \in K \text{ and} \\ \langle A(x_o) - b,\ x_o - M(x_o \rangle = 0. \end{array} \right.$$

If $\langle E, F \rangle$ is a dual system of locally convex spaces and $K \subset E$ is a closed convex cone, then given $M: E \to E$ and $A: E \to E$ two mappings and $b \in F$ an arbitrary element, the Generalized Implicit Complementarity Problem is,

$$(\text{G.I.C.P.}): \left\| \begin{array}{l} \text{find } x_o \in E \text{ such that,} \\ M(x_o) - x_o \in K,\ b - A(x_o) \in K^* \\ \text{and } \langle A(x_o) - b,\ x_o - M(x_o) \rangle = 0 \end{array} \right.$$

More general, we can consider the following multivalued implicit complementarity problem.

Let $\langle E, F \rangle$ be a dual system of locally convex spaces and consider:

$M: E \to E$, a point-to-point mapping,

$f: E \to F$, a point-to-set mapping and

$L: E \to E$, a cone-valued mapping, that is for every $x \in E$, $L(x) \subset E$ is a closed convex cone.

The Multivalued Implicit Complementarity Problem is,

$$(\text{M.I.C.P.}): \left\| \begin{array}{l} \text{find } x_o \in M(x_o) + L(x_o) \text{ and } y \in F \\ \text{such that, } y \in f(x_o) \cap L(x_o)^* \text{ and} \\ \langle y,\ x_o - M(x_o) \rangle = 0. \end{array} \right.$$