Basic Concepts of Elementary Mathematics

SECOND EDITION

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To Peter and Bob

Preface

In the brief interval of years since the first appearance of this book, gratifying strides have been made in mathematical education, for pupils at all levels as well as for their teachers. It is now apparent that both teachers in service and teachers in training quickly embraced the "new" program with commendable professional spirit and even with enthusiasm. Hence any "modern" presentation of the basic concepts of mathematics for teachers of elementary arithmetic need no longer "tread softly," let alone apologize for a reasonable degree of rigor and abstractness in its treatment of the subject.

Thus in the present edition I have eliminated a few of the "practical applications" of mathematics and moderately increased the accent on theory and structure. Specifically, the discussion of sets (Chapter 1) now includes a discussion of operations and mappings; the chapter on logic now includes a discussion of truth tables; the chapter on Geometry has been reorganized and expanded; the chapter on the integers now includes some material on the theory of numbers and modular arithmetic; new material on Equations and Inequalities has been added; and the treatment of "permutations, combinations, and probability" has been moderately extended.

It is hoped that these minor changes will enhance the usefulness of the book, and that the slight deviation from the original sequence of chapters will not discourage students from carrying on.

W. L. SCHAAF

Boca Raton, Florida
January, 1965

Preface to the First Edition

The role of mathematics in contemporary society is unique. Our society is not only complex, it is apparently changing at an accelerated pace. One cannot therefore predict with assurance either the mathematical needs of today's learner, or the mathematical ideas required by tomorrow's society. Hence the most that one can reasonably expect education to contribute in this connection is an optimum concern with fundamental mathematical ideas and methods of mathematical thinking, together with a modicum of attention to mathematical information and specific mathematical skills and techniques. One of the chief tools for obtaining knowledge and arriving at conclusions is the deductive method, and, although much of mathematics is discovered or invented inductively, mathematics is still, *par excellence,* the science of deductive reasoning. It is clear, therefore, that we need to understand mathematical methods and the language of mathematics in order to apply them to the physical and social sciences.

Since instruction in secondary school mathematics rests squarely upon the foundations laid in elementary school arithmetic, it is obvious that the elementary school teacher should have an adequate understanding of elementary mathematics, including arithmetic, algebra, geometry, and related fields. It is an acknowledged truism that one cannot teach a subject effectively unless his knowledge and understanding go well beyond the scope of that which he is expected to teach. It is my purpose in this book, therefore, to supply some of the appropriate mathematical backgrounds so desperately needed by elementary school teachers of arithmetic. These basic backgrounds include, among other things, the nature of number and of systems of enumeration, the logical structure of arithmetic, the number system of arithmetic and algebra, informal and formal geometry, computation, measurement, trigonometry, functional relations, and certain concepts of statistics and probability. That this need is very real has been pointed out repeatedly. The effective teacher of any discipline should live intimately with that discipline. He should himself have far greater insight than he strives for in his students. Suppose one were to ask an elementary school teacher, "*Why* does $2 + 3 = 3 + 2$?" Is this equation a definition? Is it an undeniable fact? An arbitrary assumption? A remarkable coincidence? A fortunate accident? An eternal verity? How many readers, at this point, can answer correctly?

In short, teachers of arithmetic and junior high school mathematics require more than a conventional course in methods of teaching arithmetic. They need a *content* course in mathematics. Such a course should not be a simple review or refresher course in seventh and eighth year arithmetic, or a traditional course in algebra, geometry, trigonometry, and analytics. Nor should it be an experience designed to achieve desirable computational proficiency. On the contrary, this course should strive to give some insight into the nature and structure of mathematics, including not only arithmetic, but algebra and geometry as well.

As I write this preface, imminent changes in mathematics curricula, from the college on down, suggest that sooner or later even the elementary school will feel the impact of the "new look" in mathematics. Although the extent of these changes on the secondary level is not yet altogether clear, the newer curricula will surely differ in approach and in emphasis. Somewhat greater stress will be laid upon the abstract point of view, with explicit attention to axiomatics and mathematical systems. Despite the criticisms and dire predictions in some quarters that this change cannot be made, it begins to appear that it is feasible, and is in fact *being* done. One can scarcely question the thesis that the teacher, at all events, should be reasonably familiar with the nature of modern mathematics and its implications for education. To be sure, the term "modern mathematics" will mean different things to different people. As the term is used here it refers essentially to mathematical ideas which were unknown (or not widely accepted) as recently as one hundred years ago. Notable among such ideas are the logical foundations of mathematics, abstract algebra, symbolic logic, and the contemporary theory of probability and statistical inference. More specifically, this new approach will deal with the historical development of systems of numeration, the evolution of the number concept, the role of postulates and definitions in mathematics, generalization, abstraction, and formalism, the nature of mathematical proof, intuitive set theory, symbols, relations, and operations, the logical basis of the number system, measurement, approximations, variables and functions, statistical concepts.

It is my express purpose, in the following pages, to capture the spirit of contemporary mathematics and to integrate it with those aspects of "classical" mathematics which are pertinent to the elementary school. I sincerely hope that the day is not too far distant when the essence of mathematical thinking, the nature of mathematical relations, the significance of mathematical systems and models, and the relation to the rational numbers to the natural numbers, for example, will be as familiar to future elementary teachers as "concrete numbers," "borrowing in subtraction," and the "partition idea of division" have been to an earlier generation of teachers. In the interests of a more enlightened populace living in an increasingly complex culture predicated upon faith in science and technology, may that day soon arrive.

W. L. Schaaf

Flushing, New York
April, 1959

"Mathematics is the science that uses easy words for hard ideas."

—Edward Kasner

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Modern Mathematics

The originality of mathematics consists in the fact that in mathematical science connections between things are exhibited which, apart from the agency of human reason, are extremely unobvious.

—Alfred North Whitehead

THE AXIOMATIC POINT OF VIEW

Historical Retrospect

Popular opinion has it that mathematics is a precise science, that its truths are absolute, and that the facts of mathematics are final and unequivocal. How often do we hear the phrase "mathematically certain," or the remark, "just as surely as two and two are four!" Indeed, until comparatively recent times, this general attitude was prevalent among mathematicians themselves. Yet nothing could be more misleading concerning the nature of mathematical knowledge. It should be said at once that we are speaking here about "pure" mathematics rather than "applied" mathematics, although the distinction between the two is not always easily made.

Consider for a moment the older notions of the nature of mathematical knowledge. The word "mathematics" is derived from an ancient Greek word, *manthanein,* which meant "to learn." How did the mathematicians of an earlier day come by their knowledge? One of the oldest viewpoints was that of the classic Greek mathematicians and philosophers, who