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## Introduction

The purpose of this work is to show how the recent work of Waldspurger ([W-1] and [W-2]) can be fit into a general theory of using the oscillator representation to (1) construct certain L-functions of automorphic representations associated to classical groups and (2) give exact formulae for the special values of these L-functions at integer and half integer points. We are presenting a point of view that is the outgrowth of our results in [R-2] and [R-3]. We note that this work was initiated in an attempt to relate the results of [R-3] to Waldspurger's results in [W-2]. We describe the general idea.

Indeed we start with a dual reductive pair, $(G,G')$ and the associated oscillator representation of $G \times G'$ on a Schwartz space $S$. With a cusp form $f$ on $G'(\mathbb{A})$ and a general $\theta_\phi$-kernel ($\phi$ a Schwartz function), we can construct the lifting $\langle\theta_\phi(x,y)|f(y)\rangle_{G'} = \ell_{\phi,f}(x)$ which becomes an automorphic form on $G(\mathbb{A})$. The general problem studied by many authors is to characterize the space $\{\ell_{\phi,f} | \phi \in S, f \in \Pi\}$ ($S$, a Schwartz space, $\Pi$ an irreducible cuspidal representation of $G'(\mathbb{A})$). For instance we want (i) to show this space (if nonzero) is irreducible, (ii) to characterize the cuspidal properties of this space, and (iii) to determine when the space is nonzero and which specific automorphic representation of $G(\mathbb{A})$ we have. These questions are answered in specific cases in [W-1] and [Ps] for the dual pairs $(\overline{Sp}_1, O(2,1))$ and $(\overline{Sp}_1, O(3,2))$. We note from the results of [R-2] that it is easy to use the local Howe duality conjecture and multiplicity one for local oscillator representations to deduce (i) above. Moreover, question (ii) can be answered to some extent also from the results of [R-2]. Question (iii) is more difficult and has so far not yielded to a general solution. What is striking about the results in [W-1] is that one requires both global and local data to get a nonvanishing condition for the lift. In particular in [W-1] one can associate a certain L-function to an automorphic representation of $\overline{Sp}_1(\mathbb{A})$ given by the Shimura lift $S^*(\Pi)$ defined in [G-Ps-1]. Then the global data for the lifting of $\Pi$ (defined above) to be nonvanishing is that $L(S^*(\Pi), \frac{1}{2}) \neq 0$ ; the local data is that the local components of $\Pi_v$ have a prescribed Whittaker model for all primes $v$. It is remarkable that these two pieces of data are precisely the ones which fit into our general picture.

Our program in answering (iii) above is to give an effective way to compute the Petersson inner product

(1) $$\langle \ell_{\phi,f} | \ell_{\phi,f} \rangle_{G(\mathbb{A})} .$$

Indeed in [R-3] we determined (in special cases) a formula for this inner product in terms of the special values of a certain L-function associated to the automorphic representation defined by $\pi$. What is directly evident from this formula is the two pieces of data discussed above. Namely we have (i) certain information about the special value of a global L-function, i.e., the global data and (ii) at a finite number of places the occurrence of a local component $\pi_v$ in a local oscillator representation, i.e., the existence of certain Whittaker models for $\pi_v$ locally.

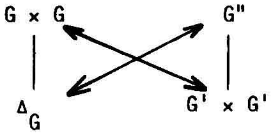
Our purpose here is to describe how it is possible to generalize the results of [R-3] to include part of the work of [W-1] and [W-2] (see [K-Z] for interpretation of results in [W-2]) and in the process develop the theory discussed in the first paragraph.

Indeed when we unwind the integration in $\langle \ell_{\phi,f} | \ell_{\phi,f} \rangle$ we get a formula of the form

(2) $$\langle f \otimes \bar{f} | \int \theta_{\phi}(x,z)\theta_{\phi}(x,z')dx \rangle_{(G' \times G')(\mathbb{A})} .$$

The main problem here is to interpret the second term above. First we take the tensor product $(S \otimes S) = \tilde{S}$ and consider the associated oscillator representation of $(G \times G, G' \times G')$. However it is possible to apply Kudla's idea of a see-saw dual pair [Ku-2]; that is, there exists another dual pair $(\Delta_G, G'')$ so that $\Delta_G = \{(g,g) | g \in G\}$ and so that $(\Delta_G, G'')$ acts on $\tilde{S}$. Moreover we have the fundamental relationship

(3)



where | denotes inclusion. Thus the second term in (2) above is essentially the lift of the identity representation of $\Delta_G$ to the group G" where we use the $\theta$-kernel function $\phi_{\phi \otimes \phi}(x,(z,z'))$ on $\Delta_G \times G''$ (restricted to $\Delta_G \times (G' \times G')$).

This set up leads immediately to the Siegel-Weil formula given in [We-2]. That is, the second term in (2) becomes a particular Eisenstein series $(\phi(z,z'))$ on G" (restricted to $G' \times G'$) . Thus (2) equals

(4) $$\langle f \otimes \bar{f} | \mathbb{E}(\phi,( \; , \; ))\rangle_{G' \times G'(\mathbb{A})} .$$

However, recalling the method of formation of Eisenstein series it is possible to find an analytic family of Eisenstein series $\mathbb{E}_s(\phi, \;)$ (formed relative to a certain maximal parabolic subgroup of G") so that (4) is just the value at $s = 0$ of the meromorphic function

(5) $$s \rightsquigarrow \langle f \otimes \bar{f} | \mathbb{E}_s(\phi,( \; , \; ))\rangle .$$

It is with this family of meromorphic functions that we construct certain L-functions associated to the automorphic representation $\pi$ containing f . In particular in this paper we consider the dual pair $(Sp_1, O(Q))$ and show (Theorem 6.1) that (5) can be written as a product of two terms. The first term is the product of a certain Abelian zeta function times the "restricted" L-function of either (i) (if Q is even dimensional) the "symmetric square" of a certain $GL_2(\mathbb{A})$ representation associated to $\pi$ or (ii) (if Q is odd dimensional) the Shimura lift of $\pi$ in the sense of [G-Ps-1]. The second term is a finite Euler product in which each local factor is a rational function such that the denominator divides a power of the local factors associated to the symmetric square or Shimura lift mentioned above. More generally (i.e. for many dual pairs) the author in joint work with Piatetski-Shapiro can show that (5) splits into a product of 2 terms similar to the terms in the case discussed above. The global term will, in general, be an Abelian zeta function times the restricted L-function of the automorphic representation $\pi$ (in the sense of Langlands in [B]).

In any case, it is the first term in the formula for (5) that gives the "global" contribution to (2) (when we evaluate (5) at $s = 0$). That is, we get the <u>special value</u> of the <u>associated</u> L-<u>function</u>! On the other hand, when we evaluate the associated local factors (in the second term mentioned above) we get a term which defines a $G'_v \times G'_v$ invariant bilinear form on the space

$$A(S_v) \otimes A(\Pi_v)$$

($A(S_v)$ and $A(\Pi_v)$, the space of smooth matrix coefficients of the $G'_v$ representation on $S_v$ and $\Pi_v$ respectively). Then we know that (in the special case $(Sp_1, O(Q))$) the nonvanishing of this form is equivalent to the occurrence of $\Pi_v$ in $S_v$ (<u>Proposition</u> 5.1, <u>Proposition</u> 6.1 and <u>Corollary</u> 1 to <u>Proposition</u> 6.1). This is the local data referred to above (indeed the equivalence of this fact to the occurrence of certain Whittaker models of $\Pi_v$ is discussed in <u>Corollary</u> 1 to <u>Proposition</u> 6.1).

Thus we see that the nonvanishing of (2) is equivalent to (i) the nonvanishing of the associated L-function at a special value ("global data") and (ii) the occurrence of $\Pi_v$ in $S_v$ ("local data") for finitely many $v$ (<u>Theorem</u> 6.3).

We emphasize here that we have demonstrated the above results for the dual pair $(Sp_1, O(Q))$ where $O(Q)$ is a rank one group. The obstacles to extending to more general cases appear to be not totally of a technical nature. Indeed we emphasize here that we require the <u>Siegel-Weil formula</u> in a range of cases hitherto not covered by the classical results (see [We-2]). Indeed the problem is to give meaning to the integral

$$(6) \qquad \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \theta_\phi(x,y)dy$$

even though in general it may not be convergent. Indeed we note first that in the case when $G'(\mathbb{Q})\backslash G'(\mathbb{A})$ is compact ($G'$ is "anisotropic") then the above integral is convergent. In fact we show (for the case $G' = O(Q)$ and $G = Sp_n$ with $\dim Q \geq n$) that we can construct an analytic family of Eisenstein series $\{\mathbb{E}^*(\phi,\mu)\}$ similar to the family defining (5) so that the integral in (6) equals $\mathbb{E}^*(\phi,0)$ (here we mean the continuation of the series defining $\mathbb{E}^*(\phi,\mu)$) (<u>Theorem</u> 4.1).

We next examine the case when $G'$ is not anisotropic. We plan to relate this to a "smaller" anisotropic case. Indeed it is first a matter of appropriately defining (6). For this we use a trick similar to the Maas-Selberg inner product formulae. Namely we first consider an analytic family of Eisenstein series $R_s$ on $G'$ so that $\underset{s=*}{\text{residue}}\, R_s = 1$ (the constant functions on $G'(\mathbb{Q})\backslash G'(\mathbb{A})$). Then we consider the truncation operator $\Lambda^T$ applied to $R_s$. In particular this gives a convergent integral

$$(7) \qquad \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \theta_\phi(x,y)\Lambda^T R_s(y)dy \ .$$

We compute (7) as a function of $T$ and $s$ (again in special cases discussed in <u>Theorem</u> 2.1 and <u>Corollary</u> 1 to <u>Theorem</u> 2.1). The upshot of the calculation of (7) is that we get "asymptotically" in $T$ a sum of 2 terms; the first term does not involve $T$ and represents an Eisenstein series on $G(\mathbb{A})$ (which is the "lift" of $R_s$). The second term is a finite sum with simple exponentials (involving $T$ and $s$) times a fixed Eisenstein series on $G(\mathbb{A})$ (coming from the anisotropic <u>Siegel-Weil</u> formula case). In any event it is this formula that allows us to compute (2) when we cannot use the classical Siegel-Weil formula (<u>Proposition</u> 3.1).

At this point, we return to the importance of computing (1) in terms of the global and local data as mentioned above. Indeed if $G'$ is "large" compared to $G$, the global data is always nonvanishing (that is, the restricted $L$-function is nonvanishing at a large special value). Thus all we need for the nonvanishing of (1) is the local data. On the other hand, if the ranks of $G$ and $G'$ are close, then the "Waldspurger" phenomenon takes place. Namely the nonvanishing of the $L$-function at a point in the critical strip becomes an additional criterion for the nonvanishing of the lift of $\pi$ (defined by the oscillator representation). We contrast this point to lifting from $G'$ to $G$ given by Langlands functoriality. In such an instance we would always get a possible automorphic representation $\pi'$ of $G$. However the arithmetic nature of the lifting defined by the $\theta$-kernel method says simply that we cannot expect to always realize the Langlands lifting by the $\theta$-lifting method. Moreover we note here that it is reasonable to expect that the $\theta$-lifting actually satisfies Langlands functoriality when $G$ and $G'$ are comparable (see [R-1] for this).

We now describe the organization of this paper.

In §0 we give the notation and preliminaries for this paper.

In §1 we consider a special family $R_s$ of Eisenstein series on $O(Q)$ (when $Q$ is rank one) which has the property that $\operatorname{res}_{s=m/2-1} R_s = 1$. Indeed to study the meromorphic continuation of $R_s$ we note that it suffices to compute the constant term of $R_s$. For this, using the theory of dual pairs, we show the constant term of $R_s$ is equal to the constant term on a family $R_s^*$ of Eisenstein series on $\overline{Sp}_1$ (Theorem 1.1). Thus we reduce the calculation of the constant term of $R_s$ to that of $R_s^*$, which is relatively straightforward. This appears to be another instance where the correspondence given by the oscillator representation reduces a problem of calculation on a big group to a simple calculation on a smaller group!

In §2 we consider the Siegel-Weil formula discussed above. Namely we compute (7) (Theorem 2.1 and Corollary 1 to Theorem 2.1). We compute an asymptotic formula for (7). The main technical point is given in Lemma 2.1 - 2.3. We show that the appropriate remainder term in (7) goes to zero as $T \to \infty$ ($T$, the truncation variable defined above). We emphasize here that this formula essentially says that the lift of $R_s$ is another family $R_s^{**}$ of Eisenstein series on $Sp_n$ (Theorem 2.2).

In §3 we compute the inner product $\langle \ell_{\phi,f} | \ell_{\phi,f} \Lambda^T(R_s) \rangle$ using the data from §2 (Lemma 3.1). The asymptotic formula in Theorem 2.1 shows that this inner product equals a sum of two terms:

$$(8) \qquad \langle f \otimes \bar{f} | \mathbb{E}_s(\phi^*, ( \ , \ )) \rangle_{Sp_n \times Sp_n} + \{\text{a linear combination of simple exponentials (in } T \text{ and } s) \text{ times } \langle \ell_{\tilde{\phi},f} | \ell_{\tilde{\phi},f} \rangle\}$$

where $\phi^*$, a Schwartz function related to $\phi$ and $\langle \ell_{\tilde{\phi},f} | \ell_{\tilde{\phi},f} \rangle$ is an inner product associated to a lifting of $f$ to a smaller dimensional $O(Q')$. Thus assuming knowledge of the lower rank cases it suffices to examine the first term of the sum above. The analytic family $\mathbb{E}_s$ of Eisenstein series is constructed from data relative to a maximal parabolic subgroup of $Sp_n$; this parabolic subgroup has Levi factor of the form $Sp_{n-i} \times G\ell_i$, and $\mathbb{E}_s$ is formed from the tensor product of a Siegel-Weil integral (relative to a dual pair of form $O(Q') \times Sp_{n-i}$) and a one dimensional character on $G\ell_i$. It is precisely at this point that we apply our version of the Siegel-Weil formula in the anisotropic case (§4). This means that we look at a new family $\mathbb{E}^*_{s,\mu}$ of Eisenstein series on $Sp_n$ (formed relative to a codimension 2 parabolic subgroup of $Sp_n$) and examine the integral

$$(9) \qquad \langle f \otimes \bar{f} | \mathbb{E}^*_{s,\mu}((\ ,\ ))\rangle .$$

Then in Lemma 3.2 we show that $\mathbb{E}^*_{s,0} = \mathbb{E}_s$. The problem is thus to evaluate (9). It is at this juncture that we restrict to the $n = 1$ case. Indeed we use the orbit decomposition of the affine symmetric triple $(B_2, Sp_2, Sp_1 \times Sp_1)$ to compute (9). This idea comes from the methods employed in [R-3]. Then we compute (9) in Theorem 3.3 to get a "mixed" integral over $Sp_1(\mathbb{A})$ of the product of a matrix coefficient $(\langle \pi_Q(G_1)\beta | \beta\rangle)$ of a lower dimensional oscillator representation times a Rankin integral involving $(f * G_1) \otimes \bar{f} \otimes \{\mathbb{F}(\ ,s,\mu) * G_1\}$ where $\mathbb{F}$ is an Eisenstein series on $Sp_1(\mathbb{A})$ with $\{\mathbb{F}(\ ,s,\mu) * G_1\}$, the translation by $G_1$ in $Sp_1(A)$ of this series in the bigger group $Sp_2$ (i.e. $Sp_1 \to Sp_2$ via $x \to (x,1)$). Indeed we note that if $G_1 = e$, then we get the classical Rankin integral of $f \otimes \bar{f} \otimes \mathbb{F}(\ ,s,\mu)$. In any case the problem of computing (9) has been reduced to dealing with a classical Eisenstein series on $Sp_1(\mathbb{A})$. Now the fundamental trick in evaluating (1) is to realize that

$$\operatorname*{res}_{s=\frac{m}{2}-1} \ \operatorname*{val}_{\mu=0} (9) = \operatorname*{val}_{\mu=0} \ \operatorname*{res}_{s-\mu=\frac{m}{2}-1} (9) \text{ and that}$$

$$\operatorname*{res}_{s-\mu=m/2-1} \{\mathbb{F}(\ ,s,\mu) * G_1\} = \theta_\mu(G_1),$$ a certain exponential function on $Sp_1(\mathbb{A})$ (which is independent of the automorphic form variable of $\mathbb{F}(\ ,s,\mu)$). Thus we

deduce in Theorem 3.5 that (1) equals the convolution of the matrix coefficient $\langle f * G_1 | f \rangle_\Pi$ with the function $\theta_\mu(G_1)\langle \pi_Q(G_1)\beta|\beta\rangle$. The technical aspects in the proof of Theorem 3.5 require much care. We need certain analytic continuations and subsequent delicate bounds (in order to compute the integrals in Theorem 3.3). These steps are given in Lemma 3.4 and the Lemma in the Appendix. Then finally in Corollary 1 to Theorem 3.5 we reduce the calculation of (9) to the computation of an Euler product given by the right hand side of (3-39).

In §5 we start the calculation of (3-39). We begin with an arbitrary local factor of (3-39) given by (5-3). In Proposition 5.1 we determine the analyticity properties of (5-3) and show that we can take $\operatorname{val}_{\mu=0} (\operatorname{val}_{s-\mu=\frac{m}{2}-1} (5\text{-}3))$. We show for finite primes that (5-3) is a rational function in the variables $q^{-s}$ and $q^{-\mu}$ (Corollary 1 to Proposition 5.1) admitting a prescribed common denominator (independent of the input data defining (5-3)) which depends on the local component $\Pi_v$ in $\Pi$. On the other hand we show in Corollary 2 to Proposition 5.1 that $\operatorname{val}_{\mu=0} (\operatorname{val}_{s-\mu=\frac{m}{2}-1} (5\text{-}3))$ defines a $Sp_1 \times Sp_1$ invariant bilinear form on $A(\Pi_v) \otimes A(S)$. The bilinear form is the local integral over $Sp_1(Q_v)$ of the product of a matrix coefficient of $\Pi_v$ times a matrix coefficient of the local oscillator representation. This integral appears already in [R-3]. Next we consider the calculation of (5-3) in the spherical case (i.e. $\Pi_v$, a class one representation of $Sp_1(Q_v)$). We compute this local factor in Proposition 5.2. We interpret this local factor in terms of the L-functions defined by Langlands for spherical representations (see (5-29), (5-30) and (5-31)). In Remarks 5.3 and 5.4 we consider the restricted Euler product of the spherical factors and interpret the corresponding global L-function as either (i) a symmetric square of a representation of $GL_2$ (associated to $\Pi$) if dim Q is even, or (ii) a Shimura lift of $\Pi$ if dim Q is odd. We also include a discussion of the general holomorphicity properties of such L-functions. Finally in Proposition 5.3 we summarize the calculation of (1) in terms of the product of the special value of the L-functions given above times the local factors (given in Corollary 2 to Proposition 5.1).

In §6 we collect together the results of §3 to §5. Indeed in Theorem 6.1 we compute $\operatorname{res}_{s-\mu=m/2-1}$ (9) (as a function of $\mu$) in terms of the $L$-functions and local factors given in §5. Then in Theorem 6.2 we deduce the identity between the inner product in (1) and the product of the special value of the restricted $L$-function (discussed above) times a finite Euler product of local terms defining the bilinear form on $A(\pi_v) \otimes A(S_v)$ given above. It is at this point that we show in Proposition 6.1 that the nondegeneracy of such a local form is equivalent to the occurrence of $\pi_v$ in $S_v$ (as an $Sp_1$ module). Moreover we show in Corollary 1 to Proposition 6.1 which possible noncuspidal representations occur in $S_v$; for supercuspidal and discrete series representations we give a criterion for the occurrence of such representations in $S_v$ in terms of the existence of certain Whittaker models of these representations. Finally in Theorem 6.3 we give the criterion for the nonvanishing of (1) in terms of the global and local data mentioned earlier!

In §4 we discuss the Siegel-Weil formula in the anisotropic case. We consider first the question of what extent the valuation of an Eisenstein series $\mathbb{E}(\ ,s)$ on $Sp_n(\mathbb{A})$ at a given value $s = s_0$ is a Hecke intertwining map from the induced representation $\operatorname{Ind}_{P_n(\mathbb{A})}^{Sp_n(\mathbb{A})}(|\det|^s\omega(\det))$ to the space $\mathcal{A}$ of slowly measuring autormorphic forms on $Sp_n(\mathbb{A})$. In particular this leads to a certain filtration on a subspace $\Pi_*$ of $\mathcal{A}$ determined by taking the various terms in a Taylor expansion of $\mathbb{E}(\ ,s)$ at $s = s_0$. Then assuming that the $K_v$ types admit multiplicity one properties in a local induced representation $X_{s_0} = \operatorname{ind}_{P_n(Q_v)}^{Sp_n(Q_v)}(|\ |^{s_0}\omega(\ ))$ it is possible to define a nonzero Hecke intertwining map from a Hecke stable and irreducible subspace $Z_{s_0} \subseteq X_{s_0}$ to a certain quotient $\mathcal{A}/\Pi_*$.

Then we review the theory of Tamagawa measures and the computation of the nondegenerate Fourier coefficient of a special Eisenstein constructed from Schwartz functions (see (4-1)). The method of proof of the Siegel-Weil formula in the anisotropic case is to compare Fourier coefficients of the Siegel-Weil integral with the special Eisenstein series. Using the classical calculations of Siegel we determine in Lemma 4.1 the local unramified factors of a nondegenerate Fourier coefficient of the Eisenstein series. In Proposition 4.1 we show that each nondegenerate Fourier coefficient of the above Eisenstein series is analytic at $\mu = 0$ (this specified value of $\mu$ is adapted to the Siegel-Weil formula). Moreover, we show that if $Y$ is not represented by the form $Q$ in question, then such type of Fourier coefficient vanishes at $\mu = 0$. In Lemma 4.2 we prove a uniqueness principle about distributions on $S(M_{mn}(\mathbb{A}))$ which are $O(Q)(\mathbb{A}) \times U_n(\mathbb{A})$ quasi invariant (explained in text). From this we deduce in Proposition 4.2 that the nondegenerate Fourier coefficients of the Siegel-Weil integral and the particular Eisenstein series (at $\mu = 0$) are proportional (independent also of the particular symmetric matrix $X$ determining the Fourier coefficient). Finally in Theorem 4.1 we prove the Siegel-Weil formula for the case $m \geq n$ ($m$ even or if $n = 1$ then all $m$) with $Q$ anisotropic. We assume in the proof that the Eisenstein series in question is analytic at $\mu = 0$. The remainder of this section is devoted to proving this point (Proposition 4.3). The key ideas in the proof of Theorem 4.1 and Proposition 4.3 are (i) the use of singular automorphic forms (a la Howe) and (ii) the construction of the Hecke intertwining maps discussed above. In the Appendix to §4 we prove the commutativity of a certain Hecke algebra; this proves that $I_{s_0}$ is a multiplicity free $K_v$ module. The Siegel-Weil formula is valid for Schwartz functions $\phi$ which are $K$-finite. We note that in the case $n = 1$, Theorem 4.1 is sharper (namely we show Siegel-Weil is true for all Schwartz functions $\phi$). This point is necessary for the applications in this paper!

At this point we would like to thank Professor Piatetski-Shapiro for several discussions that crystallized our point of view on L-functions in the preparation of this manuscript. Also we thank Professor Steve Kudla. He helped correct errors in Chapter 4 of an earlier version of this manuscript. In fact as an outgrowth of discussions with Professor Kudla, the Siegel-Weil formula has now been proved for the general anisotropic case (Kudla, S. and Rallis, S., "On the Weil-Siegel Formula, " preprint (1986)).

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# TABLE OF CONTENTS

# CHAPTER ZERO

## Notation and Preliminaries

(I)

Let $k$ be a local field of characteristic 0. We fix a nontrivial additive character $\tau$ on $k$. Let $< , >_k$ be the usual Hilbert symbol on $k$. Let $dx$ be a Haar measure on $k$ which is self dual relative to $\tau$. We let $| |_k$ be an absolute value of $k$.

If $k$ is a nonarchimedean field, we let $\mathcal{O}_k$ = ring of integers of $k$, $\pi_k$ = the maximal ideal in $\mathcal{O}_k$, and $q$ = the cardinality of $\mathcal{O}_k/\pi_k$.

(II)

Let $K$ be a number field (i.e. finite degree extension of $\mathbb{Q}$, the rational numbers). Let $\mathbb{A}_K$ be the corresponding adelic group. Then embed $K$ as a discrete subring in $\mathbb{A}_K$. Let $K_v$ be the completion of $K$ relative to a prime $v$ in $K$. Let $\tau$ be a nontrivial character on $\mathbb{A}_K$ which equals 1 on $K$; then there exist compatible characters $\tau_v$ on $K_v$ (for all primes $v$ in $K$) such that $\tau(X) = \prod_v \tau_v(X_v)$. Let $dX$ be the measure (Tamagawa measure) on $\mathbb{A}_K$ such that the group $\mathbb{A}_K/K$ is self dual relative to $\tau$ and $\mathbb{A}_K/K$ has mass 1. When the context is clear, we drop $K$ in $\mathbb{A}_K$ and just use $\mathbb{A}$ for $\mathbb{A}_K$.

(III)

Let $Q$ be a nondegenerate quadratic form on $K^m$. Let $Q_v$ be the corresponding local versions on $K_v^m$. If $Q_v$ is a totally split form which is the direct sum of $r$ hyperbolic planes, then we let $Q_v = H_r$.
Let $O(Q)$ be the orthogonal group of $Q$. Then we can form the corresponding adelic group $O(Q)(\mathbb{A})$ and the corresponding local orthogonal groups $O(Q_v)$ of $Q_v$ at $K_v$. Let $O(Q)(K)$ = the $K$ rational points in $O(Q)$ and embed $O(Q)(K)$ into $O(Q)(\mathbb{A})$ in the standard way. Choose a Tamagawa measure on the quotient of $O(Q)(K)\backslash O(Q)(\mathbb{A})$ as given in [Ar].

Similarly let $A$ be a nondegenerate alternating form on $K^{2n}$. Let $Sp_n$ be the corresponding symplectic group and $Sp_n(\mathbb{A})$, $Sp_n(K_v)$ the associated adelic and local objects. Let $Sp_n(K)$ = the $K$ rational points in $Sp_n$ and embed $Sp_n(K)$ into $Sp_n(\mathbb{A})$ again in the standard fashion and choose a Tamagawa measure on the quotient $Sp_n(K)\backslash Sp_n(\mathbb{A})$ as given in [Ar].

Moreover we let $\tilde{Sp}_n(\mathbb{A})$ = the two fold cover of $Sp_n(\mathbb{A})$ ; we know that $Sp_n(K)$ embeds into $\tilde{Sp}_n(\mathbb{A})$ and that we can choose an $\tilde{Sp}_n(\mathbb{A})$ invariant measure on the quotient $Sp_n(K)\backslash \tilde{Sp}_n(\mathbb{A})$ .

Where it is clear from the context we shall omit the ~ over $Sp_n$ (in referring to the two fold cover).

(IV)

We consider the category of smooth representations for the local and global objects in question. That is, $\pi_v$ is smooth locally for $G_v$ (a local group) if (1) at the Archimedean primes, $\pi_v$ is a differentiable module for $G_v$, i.e., $(\pi_v)_\infty = C^\infty$ vectors in $\pi_v = \pi_v$ and (2) at the finite non-Archimedean primes, $\pi_v$ is a smooth module for $G_v$ in the sense of [Cs]. Then we consider also the category of admissible modules as given in [Cs].

In the non-Archimedean case, we use the notion of Jacquet functor given in [Cs]. That is, if $N_v \subset G_v$ is any closed subgroup, and if $\pi_v$ is any smooth $G_v$ module, we have a functor from $\pi_v$ to $(\pi_v)_{N_v} = \pi_v/\pi_v(N_v)$, where $\pi_v(N_v)$ = {all linear combinations of the form $x - \pi_v(n)x$ as $x$ varies in $\pi_v$ and $n$ varies in $N_v$}. Moreover we consider the category of admissible $G_{\mathbb{A}}$ modules as given in [B]. In this context, we note the well known relation in [B] between unitary irreducible modules of $G_{\mathbb{A}}$ and admissible irreducible modules of $G_{\mathbb{A}}$. We also use the notion of automorphic irreducible representations of $G_{\mathbb{A}}$ as given in [B].

(V)

If $G_{\mathbb{A}}$ is a global group, then we denote the space of cusp forms on $G_K\backslash G_{\mathbb{A}}$ by $L^2_{cusp}(G_{\mathbb{A}})$. We note (by our convention) that if $G_K\backslash G_{\mathbb{A}}$ is compact, then $L^2_{cusp}(G_{\mathbb{A}})$ = {all functions $f \perp$ constants}. We know that $L^2_{cusp}(G_{\mathbb{A}})$ is discretely decomposable as a $G_{\mathbb{A}}$ module, and each unitary irreducible representation occurring in $L^2_{cusp}(G_{\mathbb{A}})$ has a finite multiplicity. We denote