DEVELOPMENTS IN CIVIL ENGINEERING, 17

POST-BUCKLING OF ELASTIC STRUCTURES

EDITED BY

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PREFACE

This volume contains the lectures of the EUROMECH COLLOQUIUM 200 on Post-Buckling of Elastic Structures held at Mátrafüred, Hungary in the autumn of 1985. Some of the lectures have already been published elsewhere, so only their abstracts are given here.

Undoubtedly, these papers provide worthwhile contributions with possible applications in theoretical investigations and algorithms.

There were 33 lectures at the colloquium followed by lively discussions. The themes were divided into 3 groups:

I. General theory of buckling and post-buckling
II. Buckling and post-buckling of particular structures
III. Special topics

This enumeration, however, does not offer a true picture of the main topics of the colloquium, which is outlined below.

Professor Koiter could not take part in the colloquium personally, but of course, almost all the authors referred to him and mentioned his famous Ph.D. thesis, "On the Stability of Elastic Equilibrium" published 40 years ago. This thesis can still be considered a pioneering work in the exact analysis of the non-linear phenomenon of the post-buckling behaviour of structures.

Change of post-critical state and its stability can be generally investigated by numerical methods. At the same time, the classification of critical points is gaining ground, while new and old models with improved numerical methods are becoming.

important, aided by the increasingly popular catastrophe theory. Accordingly, a change in quality is taking place in the description of equilibrium paths; discussion of critical regions is replacing that of critical points.

These tendencies could be observed at the colloquium and they are reflected in this book. Progress has been made in the direction of theoretical investigations justified by the results of special, proved, numerical procedures and suitable for wider generalization.

As a last point, any data on experiments performed in the laboratory or on full-size structures, justifying theoretical or numerical investigations are most welcome. Data contradicting theoretical and numerical investigation, and in this way challenging further progress, are also welcome.

We feel this volume will be an inspiration for further advances in this branch of science.

Professor J. Szabó

Member of the Hungarian Academy
of Sciences

Chairman of the Colloquium

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ABSTRACTS

EUROMECH 200, Mátrafüred, Hungary, 1985
J. Szabó (ed.)

DYNAMIC STABILITY OF ORTHOTROPIC PLATES RESTING ON PASTERNAK TYPE FOUNDATIONS

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ABSTRACT

In this paper, the dynamic stability of orthotropic rectangular plates resting on Pasternak type foundation is investigated. Linear plate theory is used to study the vibration of such a plate with and without in-plane forces. Dynamic stability of the plate under harmonic in-plane forces is treated and the regions of dynamic stability is discussed.

I. INTRODUCTION

During the last twenty years, advanced composites have become established as high performance structural materials and their use is increasing rapidly. Fiber-reinforced laminated composite plates are receiving greater attention in a variety of engineering structures. Many applications involve thin plates and shells under loadings causing failure by buckling. Static stability of orthotropic plates have been the aubject of numerous investigations such as (Chamis, 1969; Chao etal 1975; Harris 1975; Harris 1976; Dickinson, 1978; and Brunelle 1983). On the dynamic stability of plates the work by (Ekstrom, 1973) and (Boduroğlu and Uzman, 1980) can be considered. For the vibration of such plates (Bradford and Dong 1975), can be given as a reference.

In this paper, the effect of Pasternak type elastic foundation (Pasternak, 1954) on the dynamic stability of an orthotropic plate is investigated. This type of foundation yields the load-displacement relation

$$p = Kw - G_o \nabla^2 w \tag{1}$$

where ∇^2 is the Laplace operator in x and y ; K is the elastic spring constant and G_o is a constant showing the effect of the shear interactions of the vertical elements.

2. GOVERNING EQUATIONS

The small deflection theory of plates is utilized subjected to harmonic in-plane forces N_x, N_y and N_{xy} given in the appendix. The resulting differential equation of an orthotropic rectangular plate is obtanied as

$$L_w + m \frac{\partial^2 w}{\partial t^2} + K_w - G_o \nabla^2 w = N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial x^2} \qquad (2)$$

where L is the operator given by,

$$L = D_x \frac{\partial^4}{\partial x^4} + 2(D_x \nu_{yx} + 2D_{66}) \frac{\partial^4}{\partial x^2 \partial y^2} + D_y \frac{\partial^4}{\partial y^4} \qquad (3)$$

w(x,y,t) is the out-of-plane displacement, m is the unit mass of the plate, D_x, D_y, D_{66} are the plate rigidities and ν_{yx} is Poisson's ratio.

Galerkin's method can be used to solve the above differential equation. Then the out-of-plane displacement can be considered as

$$w(x,y,t) = \sum_{i=1}^{m} \sum_{j=1}^{m} f_{ij}(t)\, \phi_{ij}(x,y) \qquad (4)$$

where $\phi_{ij}(x,y)$ are the coordinate functions satisfying the plate boundary conditions in the form of

$$\phi_{ij}(x,y) = \phi_i^x(x)\, \phi_j^y(y) \qquad (5)$$

Here ϕ_i^x and ϕ_i^y are the eigenfunctions of a vibrating beam satisfying the boundary conditions in x and y respectively and $f_{ij}(t)$ are the coefficients as a function of time t. Application of Galerkin's method yields the following set of simultaneous differential equations in vector form

$$m\, \underline{I}\, \ddot{\underset{\sim}{f}} + \left[\frac{\sqrt{D_x D_y}}{a^2 b^2} - \underline{R} + K\underline{I} - \frac{G_o}{ab} \underline{S} - \frac{\alpha}{ab} \underline{P} - \frac{\beta \cos\theta t}{ab} \underline{Q} \right] \underset{\sim}{f} = \underset{\sim}{0} \qquad (6)$$

in which I in the unit matrix and the matrices R, K, S, P, Q are given in the appendix and f and $\ddot{f}$ are the vectors of the coefficients f_{ij} and their second time derivatives respectively.

3. SPECIAL CASES

Eq. (6) can be studied for the following special cases :

Case-1 Vibration of the plate under static in-plane forces when β is zero. In this case, the frequencies Ω_α of the vibration of the plate are obtained from the roots of the determinant,

$$\left[\frac{\sqrt{D_x D_y}}{a^2 b^2} \underline{R} + K\underline{I} - \frac{G_o}{ab} \underline{S} - \frac{\alpha}{ab} \underline{P} - (\Omega_\alpha) m\, \underline{I} \right] = 0 \qquad (7)$$

Case-2 Vibration of the plate under static in-plane forces when α and θ are zero. The frequencies Ω_β are obtained from the iegenvalues of the determinant

$$\left[\frac{\sqrt{D_x D_y}}{a^2 b^2} \underset{\sim}{R} + K\underset{\sim}{I} - \frac{G_o}{ab} \underset{\sim}{S} - (\Omega_\beta)^2 m \underset{\sim}{I} \right] = 0. \tag{8}$$

Case-3 Vibration of the plate under static in-plane forces when θ is zero. The corresponding frequencies $\Omega_{\alpha\beta}$ are obtanied from the eigenvalues of the determinant

$$\left[\frac{\sqrt{D_x D_y}}{a^2 b^2} \underset{\sim}{R} + K\underset{\sim}{I} - \frac{G_o}{ab} \underset{\sim}{S} - \frac{\alpha}{ab} \underset{\sim}{P} - \frac{\beta}{ab} Q - (\Omega_{\alpha\beta})^2 m \underset{\sim}{I} \right] = 0. \tag{9}$$

Case-4 Boundaries of the dynamic stability of the plate when θ is non zero. For certain values of θ , the stability of the plate is lost. The boundaries of θ responsible for the instability can be determined. It is known that the solution of the differential equation (2) turns ont to be periodic when the frequency θ of the forcing function is chosen to be at these boundaries. The period is given by $T = 4\pi/\theta$. Values of θ outside of these boundaries will not cause instabilities of the related solutions.

For periodic solutions of the Eq.(2), f(t) can be expressed as

$$\underset{\sim}{f}(t) = \frac{1}{2} \underset{\sim}{b}_o + \sum_{n=1}^{\infty} (\underset{\sim}{a}_n \sin \frac{n\theta t}{2} + \underset{\sim}{b}_n \cos \frac{n\theta t}{2}) . \tag{10}$$

We will determine the values of θ giving such solutions. Substituting Eq.(10) into Eq.(6) and equating the coefficients of the harmonics, a lineer set of homogeneous equations in a_n and b_n is obtained. Thus the determinant of the the coefficient matrix will give us the values of θ for periodic solutions. These values of θ are the boundaries for the dynamic stability problem.

Considering one term in the series, i,e, $n = 1$, this determinant becomes

$$\begin{vmatrix} \left(\frac{\sqrt{D_x D_y}}{a^2 b^2} \underset{\sim}{R} + K\underset{\sim}{I} - \frac{G_o}{ab} \underset{\sim}{S} - \frac{\alpha}{ab} \underset{\sim}{P} + \frac{1}{2} \beta \underset{\sim}{Q} + \frac{1}{4} \theta^2 M\underset{\sim}{I} \right) & 0 \\ 0 & \left(\frac{\sqrt{D_x D_y}}{a^2 b^2} \underset{\sim}{R} + K\underset{\sim}{I} - \frac{G_o}{ab} \underset{\sim}{S} - \frac{\alpha}{ab} \underset{\sim}{P} - \frac{1}{2} \beta \underset{\sim}{Q} - \frac{1}{4} \theta^2 M\underset{\sim}{I} \right) \end{vmatrix} = 0 \tag{11}$$

The values of θ making the first term of the determinant equal to zero give the lower boundaries and while the last term of the determinant yields the upper boundaries. For each vibration mode one lower and one upper boundary value exist. When n is 1, for each vibration mode first resonance region is determined. Comparison of the Eqs. (9) and (11) yields a relationship between θ's and Ω's as

$$\frac{\theta^2}{4} = \Omega^2(\alpha, -\frac{\beta}{2}) \qquad \text{for lower boundaries}$$

$$\frac{\theta^2}{4} = \Omega^2(\alpha, \frac{\beta}{2}) \qquad \text{for upper boundaries} \tag{12}$$

4. EXAMPLE

A plate simply supported along four edges is taken as an example. Making use of the following coordinate transformation similar to the one suggested by (Krenk, 1979),

$$\bar{x} = x \sqrt[8]{\frac{D_y}{D_x}} \qquad \text{and} \qquad \bar{y} = y \sqrt[8]{\frac{D_x}{D_y}} \tag{13}$$

Eq.(6) can be written in a simpler form. In this case, the out-of-plane displacement can be considered as

$$w(x,y,t) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}(t) \frac{2}{\sqrt{\bar{a}\,\bar{b}}} \sin\frac{i\pi x}{\bar{a}} \sin\frac{j\pi y}{\bar{b}} \tag{14}$$

which satisfies the geometric and as well as the dynamic boundary conditions along the edges of the plate. Substituting Eqs.(13) and (14) into Eq.(6) and applying the Galerkin's method together with the orthogonality conditions of the harmonic functions,

$$m\ddot{f}_{ij} + D\left[\left(\frac{i\pi}{a}\right)^4 + 2D^*\left(\frac{ij\pi^2}{a\,b}\right)^2 + \left(\frac{j\pi}{b}\right)^4 + \left(K + (\bar{N}_x - G_o)\left(\frac{i\pi}{a}\right)^2 + (\bar{N}_y - G_o)\left(\frac{j\pi}{b}\right)^2\right]\right) f_{ij} = 0 \tag{15}$$

is obtained. Please see the appendix for the transformed quantities. As examples the following cases are considered.

Case I : N_x and N_y are taken to be constant. In this case the solution of Eq.(15) is

$$f_{ij} = A\cos\Omega_{ij}t + B\sin\Omega_{ij}t \tag{16}$$

The natural frequency Ω_{ij} corresponding to the mode ij when the in-plane forces are peresent is

$$\Omega_{ij} = \sqrt{\frac{D}{m}}\,\frac{ij\pi^2}{\bar{a}\,\bar{b}}\sqrt{\frac{1}{r_{ij}^2} + 2D^{*} + r_{ij}^2 + \left[\left(\frac{\bar{a}\,\bar{b}}{ij\pi^2}\right)^2 K + \frac{\bar{a}\,\bar{b}}{ij\pi^2}(\bar{N}_x - G_o)\frac{1}{r_{ij}} + (N_y - G_o)\frac{\bar{a}\,\bar{b}}{ij\pi^2}\, r_{ij}\right]\frac{1}{D}} \tag{17}$$

where $r_{ij} = \dfrac{aj}{bi}$,

Case II : N_x and N_y are taken to be harmonic. In this case, Eq.(15) becomes

$$m\,\ddot{f}_{ij} + D\left[\left(\frac{i\pi}{\bar{a}}\right)^4 + 2D^{*}\left(\frac{ij\pi^2}{\bar{a}\,\bar{b}}\right)^2 + \left(\frac{j\pi}{\bar{b}}\right)^4 + \{K + (\alpha X_o - G_\sigma)\left(\frac{i\pi}{\bar{a}}\right)^2 + (\alpha Y_o - G_\sigma)\left(\frac{j\pi}{\bar{b}}\right)^2\right]$$

$$+\left[\beta X_t - G_\sigma)\left(\frac{i\pi}{\bar{a}}\right)^2 + (\beta Y_t - G_o)\left(\frac{j\pi}{\bar{b}}\right)^2\right]\cos\theta t\}\, f_{ij} = 0 \tag{18}$$

Eq.(18) can be rearranged to give the Mathieu equation and the boundaries of stability of the Mathieu function is easy to obtain (Bolotin,1964).

5. CONCLUSIONS

From the examples given above one can see that the elastic foundation constant K increases the frequencies while the constant G_o decreases them in a qualitative manner. The quantitative manner can be obtained by studying the numerical examples. The boundaries of the dynamic stability can be determined using the generalized Jacobi algorithm for the eigenvalues of the determinants mentioned above. This problem is still being investigated.

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Appendix

Parameter appearing in Eqs.(2) and (3) are given below.

$$D_x = \frac{E_x h^3}{12(1-\nu_{xy}\nu_{yx})} \qquad D_y = \frac{E_y h^3}{12(1-\nu_{xy}\nu_{yx})} \qquad D_{66} = \frac{G_{xy} h^3}{12}$$

in which E_x , E_y are the modulus of elasticity in x and y directions respectively and G_{xy} is the shear modulus h is the plate thickness; ν_{xy} and ν_{yx} are Poisson's ratios.

$$N_x(x,y,t) = -(\alpha X_o + \beta Y_t \cos\theta t)$$
$$N_y(x,y,t) = -(\alpha Y_o + \beta Y_t \cos\theta t)$$
$$N_{xy}(x,y,t) = -(\alpha Z_o + \beta Z_t \cos\theta t)$$

in which X_o, X_t, Y_o, Y_t, Z_o and Z_t are constants; α and β are load parameters.

The elements of the matrices K, R, S, P and Q are

$$K_{ij} = \int_o^a \int_o^b \phi_{ij}\, \phi_{rs}\, dxdy$$

$$R_{ij} = \int_o^a \int_o^b \left[\sqrt{\frac{D_x}{D_y}}\, \frac{\partial^4 \phi_{ij}}{\partial x^4}\, \phi_{rs} + 2\, \frac{(D_x \nu_{yx} - 2D_{66})}{\sqrt{D_x D_y}}\, \frac{\partial^4 \phi_{ij}}{\partial x^2 \partial y^2}\, \phi_{rs} + \sqrt{\frac{D_y}{D_x}}\, \frac{\phi_{ij}}{\partial y^4}\, \phi_{rs} \right] dxdy$$

$$S_{ij} = \int_o^a \int_o^b \left[\frac{\partial^2 \phi_{ij}}{\partial x^2}\, \phi_{rs} + \frac{\partial^2 \phi_{ij}}{\partial y^2}\, \phi_{rs} \right] dxdy$$

$$R_{ij} \quad \int_o^a \int_o^b \left[X_o\, \frac{\partial^2 \phi_{ij}}{\partial x^2}\, \phi_{rs} + 2Z_o\, \frac{\partial^2 \phi_{ij}}{\partial x \partial y}\, \phi_{rs} + Y_c\, \frac{\partial^2 \phi_{ij}}{\partial y^2}\, \phi_{rs} \right] dxdy$$

$$Q_{ij} = \int_o^a \int_o^b \left[X_t\, \frac{\partial^2 \phi_{ij}}{\partial x^2}\, \phi_{rs} + 2Z_t\, \frac{\partial^2 \phi_{ij}}{\partial x \partial y}\, \phi_{rs} + Y_t\, \frac{\partial^2 \phi_{ij}}{\partial y}\, \phi_{rs} \right] dxdy$$

Transformed quantities appearing in Eq.(15) is given below.

$$D = \sqrt{D_x D_y}$$

$$D^{*} = (D_x \nu_{yx} + 2D_{66})/D$$

$$\bar{N}_x = N_x \sqrt[4]{\frac{D_y}{D_x}} \quad , \quad \bar{N}_y = N_y \sqrt[4]{\frac{D_x}{D_y}}$$