ELEMENTS OF PLANE TRIGONOMETRY

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ELEMENTS OF
PLANE TRIGONOMETRY

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PREFACE

The purpose of this textbook is to present trigonometry in the language and spirit of modern mathematics. The vocabulary of elementary mathematical analysis is used exclusively throughout.

The over-all plan of the book is simple. The first eight chapters emphasize *the theory of the trigonometric functions;* the final three chapters emphasize *applications of trigonometry*. Logarithms and related topics are discussed in appendices, which may be tapped or not depending upon individual requirements. Coordinate methods are used wherever possible, particularly in formal proofs. Nearly all conventional topics are discussed, but the arrangement and approach are intended to focus attention on *periodicity* and *graphs* rather than on the traditional *right triangle*.

There are many problems which stress the importance of trigonometry in various fields of science and engineering. The final section in each of Chapters 1, 4, 7, and 10 is devoted to a *Special Topic,* which is related to the material in the chapter but is not essential to the continuity of presentation. These topics have been chosen to demonstrate applications of trigonometry to physical problems both ancient and modern.

The author gratefully acknowledges his indebtedness to teachers, colleagues, and friends who have influenced the preparation of this textbook. In addition, a special word of thanks is due both Dr. A. A.

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H. S., Jr.

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1

INTRODUCTION

The word "trigonometry" is a combination of two Greek words that, taken together, mean "triangle measurement." In this sense, the study of trigonometry has an unbroken history reaching from its source in ancient Greece to the present day. But the name of our subject, although historically appropriate, is deceptive, for triangle measurement is not the only important application of trigonometry in the modern world. It is our purpose in this introduction to sketch the origin of trigonometry and to indicate its uses, both ancient and modern.

1. EARLY HISTORY

In its earliest stages trigonometry was closely related to geometry. In fact, it seems to have originated more than 2,000 years ago in Egypt and Greece with the application of geometric principles to problems arising in land surveys and astronomy. The individuals associated with its foundation as a systematic study are Hipparchus (Greek, second century B.C.) and Ptolemy (Greek, residing in Alexandria, *c.* A.D. 150). Isolated instances of the use of trigonometric ideas appear much earlier. The Egyptians, for example, had made use of certain trigonometric rules in re-establishing along the Nile River the boundary markers and lines usually destroyed by annual floods.

Hipparchus is generally credited with developing this and many other examples into a coherent theory by which more difficult problems could be solved. Most persons unacquainted with classical Greek history are amazed to learn how far advanced scientific knowledge was in that age. To illustrate, by the time of Hipparchus the Greeks had already discovered that the earth is spherical, and by geometry and trigonometry they had estimated its diameter and that of the moon with surprising accuracy. These discoveries were afterward forgotten or overlooked for more than a thousand years, and they were not popularly revived until the time of Columbus.

Ptolemy, who is famous primarily as an astronomer, refined the theory inherited from Hipparchus. After Ptolemy there were few important additions to trigonometry until about the seventeenth century. Since that time, new mathematical ideas have exerted an entirely different and nongeometrical influence on the subject.

2. PERIODIC EVENTS

The physical world is dominated by periodic events: the alternations of day and night, the phases of the moon, the appearance of certain comets, the flow and ebb of the tides, and, on a smaller scale, the swinging of a pendulum, the operation of the pistons in an internal combustion engine, the rotation of the wheels in a watch movement — these are all examples of events that occur periodically (or very nearly so). One of the basic purposes of mathematics is to furnish a symbolic language through which events in the physical world may be concisely and elegantly described. It is natural, then, to look for a mathematical scheme by which periodic motions can be represented. The algebraic ideas with which we may already be familiar are not well suited to the description of periodic motions. On the other hand, it will become apparent as the subject is developed that trigonometric expressions are particularly well adapted to this purpose.

3. GENERAL COMMENTS

Trigonometry encompasses in a single theory two widely different kinds of application. The methods of trigonometry can be used on the one hand to study the numerical relationships between the sides and angles of triangles, and on the other hand to analyze problems relating to periodic events. Questions of the former type arise, for example, in surveying, astronomy, navigation, and mechanics; questions of the latter type occur in the study of electrical phenomena, the theory of vibrations, and in many other branches of modern science and engineering.

4. SPECIAL TOPIC: THE SIZE OF THE EARTH

For almost a thousand years after the time of Alexander the Great, the city of Alexandria, founded by him near one mouth of the Nile River, was unparalleled as a center of learning. About 300 B.C. a university was established there and a great library built, into which poured many of the writings and observations of the ancient world. During their lifetimes, Archimedes, Euclid, Ptolemy, and probably Hipparchus were associated with the university, along with many other lesser known scholars. One of these was Eratosthenes (Greek, 275–194 B.C.), who for a long period was university librarian at Alexandria.

Long before the time of Eratosthenes, convincing arguments had been given that the earth is spherical. Two of these arguments were: (1) the shadow of the earth cast on the moon during an eclipse always appears circular, and (2) a relatively small change in position north or south on the earth's surface produces an appreciable change in the height of certain stars above the horizon. Belief that the earth is a sphere led Eratosthenes to search for a method of finding its size.

Eratosthenes discovered in his library records that there was a most unusual phenomenon associated with a deep well near Syene, an Egyptian city that he believed to be about 500 miles due south of Alexandria. (He used, of course, a different measure of distance.) At noon on only one day of the year, the sun could be observed to shine straight down the well, producing a reflection on the water. He reasoned that when this occurred the sun, the well, and the earth's center lay on the same straight line. He was able to determine that, at the same time as the sun reflected in the well, a vertical column in Alexandria cast a shadow indicating that the sun was 7°12′ south of zenith. By assuming that Alexandria and Syene lie along the same meridian and that the sun's rays are parallel, he could then determine the earth's circumference (Fig. 1.1).

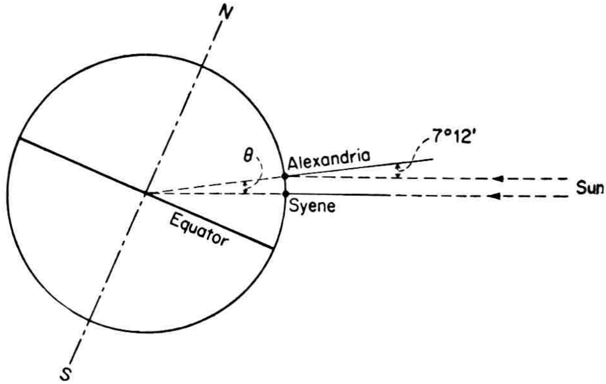

Fig. 1.1

Knowing that the angle at the earth's center is 7°12′ and that this angle subtends an arc of length 500 miles, he reasoned that the entire circumference C must be given by the equation

$$\frac{C}{500} = \frac{360^\circ}{7.2^\circ} \qquad \text{or} \qquad C = (500)(50) = 25{,}000 \text{ miles.}$$

Suggested Exercises

1. The city of Rhodes is about 400 miles directly north of Alexandria. A bright star, *Canopus*, is barely visible above the horizon at Rhodes, whereas at Alexandria its highest point is about 5°30′ above the horizon. Draw a sketch of this situation and estimate the circumference of the earth. (This method was used over a century later by Posidonius, another Alexandrian scholar.)

2. Hipparchus calculated that the distance from the surface of the earth to the moon is about $33\frac{1}{2}$ earth diameters. Using this information, with Eratosthenes' value for the size of the earth, estimate the diameter of the moon if it is observed near zenith to intercept an arc of $\frac{1}{2}^\circ$.

3. How do modern values compare with those found here for earth diameter, moon diameter, and distance from earth to moon?

4. Assuming the earth to be 93,000,000 miles from the sun, what is approximately the diameter of the sun if it is observed to subtend an angle of $\frac{1}{2}^\circ$?

2

FUNDAMENTAL CONCEPTS

In this chapter we discuss several concepts essential to an understanding of modern mathematics. The vocabulary introduced in this chapter will be used extensively in the pages ahead.

1. SETS

In the study of mathematics it is frequently convenient and indeed necessary to refer specifically to a whole collection of distinct objects. The objects in the collection are characterized by sharing in common a particular property that distinguishes each from objects not in the collection. *A collection that is to be considered as a whole is called a* ***set,*** *and the objects in the collection are called* ***elements*** *of the set. Each element is said to* ***belong to*** *the set, and the set is said to* ***contain*** *each element.*

We shall postulate the existence of one *set containing no element at all, which for this reason is called the* ***null*** *set or the* ***void*** *set.* In certain more advanced mathematical topics this set behaves very much like the zero in our ordinary system of numbers.

EXAMPLE 1. The positive integers less than 100 comprise a set. The elements of this set could be written out explicitly without too much labor, but mathematicians prefer to denote this set by the symbol $\{1, 2, \cdots, 99\}$. It is evident that this set contains exactly 99 elements.

EXAMPLE 2. We may also think of all positive integers as comprising a set. In this case, of course, it is impossible to list all the elements explicitly, and the usual symbol is $\{1, 2, \cdots\}$.

When a particular set is to be mentioned several times in the course of a discussion, it is desirable to assign some symbol as an abbreviation for the set. Capital letters are frequently used as such abbreviations. Let M and N be two sets. We say that *M and N are **equal*** ($M = N$) *if each element of M is also an element of N and, furthermore, if each element of N is also an element of M.* In other words, M and N are equal if they contain precisely the same elements.

There is an essential difference between the sets in Examples 1 and 2 that we may indicate in the following intuitive way. Suppose it were possible to line up, either physically or by imagination, all the elements of a given set. If we then start counting these elements, one of two possibilities may occur. A definite number is reached beyond which no more are needed because each element has already been counted; or no such number is reached because there are always elements of the set that have not been previously counted. In the former case the set is called **finite,** in the latter case the set is called **infinite.**

Problem Set 2.1

1. How else can you describe the set of all states larger than Texas?

2. How else can you describe the set of all planets closer to the sun than is the earth?

3. Is the set $\{1, 2, 3\}$ equal to the set $\{2, 1, 3\}$?

4. List all different three-digit numbers formed from the digits 1, 2, and 3,
(a) allowing repetitions (for example, 111 or 122),
(b) not allowing repetitions.

5. Specify the elements belonging to the set of all fractions a/b for which a can be any of the numbers 1, 3, or 5, while b can be either 2 or 4.

6. Give an intuitive argument for the fact that the set of fractions $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is infinite.

REMARK 1. Finite and infinite sets may be defined in the following way. *If the elements of a set A are related to the elements of a set B in such a way that to each element of A there corresponds one and only one element of B, and if each element in B is related to one and only one element of A, then A and B are said to be in a **one-to-one correspondence.***

Let A be a set. If C is a non-void *subset* of A (that is, each element of C is also an element of A), let B be the set consisting of those elements of A that are not in C. *If for some such C there exists a one-to-one correspondence between A and B, then A is called* ***infinite***; *otherwise A is called* ***finite.***

REMARK 2. Let A be the set of all positive integers, let C be the set of all odd positive integers, and let B be the set of all even positive integers. We establish a correspondence between A and B as follows: to each element of A corresponds its double. Thus 1 is related to 2, 2 to 4, 3 to 6, and so on. This correspondence is one-to-one, and C is non-void, hence the set of all positive integers is infinite.

7. Show by writing out all possible correspondences that the set $\{1, 2, 3\}$ is finite.

8. Prove that the set given in Problem 6 is infinite.

9. Show geometrically that the set of all points on a line is infinite.

2. REAL NUMBER SYSTEM

We shall designate the set consisting of zero and the positive integers as the set of ***natural numbers,*** *to be denoted by N. Thus* $N = \{0, 1, 2, 3, \cdots\}$. When the operations of subtraction and division are applied to the elements in N it appears immediately that this set must be extended in order to make these operations meaningful for all numbers. For example, $5 - 7$ is not a natural number, and $\frac{1}{2}$ is not a natural number. To overcome this difficulty, we extend the set N by defining first the set of whole numbers and next the set of rational numbers.

The set consisting of zero and the positive and negative integers is called the set of ***whole numbers,*** *to be denoted by W.*

Any number that can be represented in the form a/b, where a and b are elements of W and $b \neq 0$, is called a ***rational number.*** A rational number of the form $a/1$ may be considered identical to the whole number a. Thus, in a sense, the rational numbers include the whole numbers. With the rational numbers we have arrived at a set so large that the operations of addition, subtraction, multiplication, and division (with one exception) are all possible within the set. These four are called the **elementary rational operations** of arithmetic. It is easy to show that division by zero leads to contradiction, therefore we exclude $a/0$ as a number.