INTRODUCTION TO ALGEBRAIC *K*-THEORY

BY

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PREFACE AND GUIDE TO THE LITERATURE

The name "algebraic K-theory" describes a branch of algebra which centers about two functors K_0 and K_1, which assign to each associative ring Λ an abelian group $K_0\Lambda$ or $K_1\Lambda$ respectively. The theory has been developed by many authors, but the work of Hyman Bass has been particularly noteworthy, and Bass's book *Algebraic K-theory* (Benjamin, 1968), is the most important source of information. Here is a selected list of further references:

D. S. Rim, *Modules over finite groups*, Annals of Math. 69 (1959), 700-712.

R. Swan, *Projective modules over finite groups*, Bull. Amer. Math. Soc. 65 (1959), 365-367.

H. Bass, K-theory and stable algebra, Publ. Math. I.H.E.S. 22 (1964), 5-60.

H. Bass, A. Heller, and R. Swan, *The Whitehead group of a polynomial extension*, Publ. Math. I.H.E.S. 22 (1964), 61-79.

H. Bass, *The Dirichlet unit theorem, induced characters, and Whitehead groups of finite groups*, Topology 4 (1966), 391-410.

H. Bass (with A. Roy), *Lectures on topics in algebraic K-theory*, Tata Institute, Bombay 1967.

H. Bass and M. P. Murthy, *Grothendieck groups and Picard groups of abelian group rings*, Annals of Math. 86 (1967), 16-73

R. Swan, *Algebraic K-theory*, Lecture Notes in Math. 76, Springer 1968.

R. Swan (with E. G. Evans), *K-theory of finite groups and orders*, Lecture Notes in Math. 149, Springer 1970.

L. N. Vaseršteĭn, *On the stabilization of the general linear group over a ring,* Mat. Sbornik 79 (121), 405-424 (1969). (Translation v. 8, 383-400 (A.M.S.).)

The main purpose of the present notes is to define and study an analogous functor K_2, also from associative rings to abelian groups. The definition is suggested by work of Robert Steinberg. This functor K_2 is related to K_1 and K_0 for example by means of an exact sequence

$$K_2\mathfrak{a} \to K_2\Lambda \to K_2(\Lambda/\mathfrak{a})$$
$$\to K_1\mathfrak{a} \to K_1\Lambda \to K_1(\Lambda/\mathfrak{a})$$
$$\to K_0\mathfrak{a} \to K_0\Lambda \to K_0(\Lambda/\mathfrak{a}),$$

associated with any two-sided ideal $\mathfrak{a}$ in the ring Λ; where $K_2\mathfrak{a}$, $K_1\mathfrak{a}$ and $K_0\mathfrak{a}$ are suitably defined relative groups. Here is a list of references for K_2:

R. Steinberg, *Générateurs, relations et revêtements de groupes algebriques,* Colloq. Théorie des groupes algebriques, Bruxelles 1962, 113-127.

R. Steinberg (with J. Faulkner and R. Wilson), *Lectures on Chevalley groups* (mimeographed), Yale 1967.

C. Moore, *Group extensions of p-adic and adelic linear groups,* Publ. Math. I.H.E.S. 35 (1969), 5-74.

H. Matsumoto, *Sur les sous-groupes arithmétiques des groupes semi-simples déployés,* Ann. Sci. Ec. Norm. Sup. 4e serie, 2 (1969), 1-62.

H. Bass, *K_2 and symbols,* pp. 1-11 of Lecture Notes in Math. 108, Springer 1969.

M. Kervaire, *Multiplicateurs de Schur et K-théorie,* pp. 212-225 of *Essays on Topology and Related Topics,* dedicated to G. de Rham (ed. A. Haefliger and R. Narasimhan), Springer 1970.

J. Wagoner, *On K_2 of the Laurent polynomial ring,* to appear.

B. J. Birch, *K_2 of global fields,* Proc. Symp. Pure Math. 20, Amer. Math. Soc. 1970.

J. Tate, *Symbols in arithmetic*, Proc. Int. Congr. Math. Nice, to appear.

M. Stein, *Chevalley groups over commutative rings*. Bull. Amer. Math. Soc. 77 (1971), 247-252.

It should be pointed out that definitions of K_n for all integers $n \geq 0$ have been proposed by several authors. See the following:

A. Nóbile and O. Villamayor, *Sur la* K-*théorie algébrique*, Ann. Sci. Éc. Norm. Sup. 4e série 1(1968), 581-616.

R. Swan, *Non-abelian homological algebra and* K-*theory*, Proc. Symp. in Pure Math. 17, 88-123, A.M.S. 1970.

M. Karoubi and O. Villamayor, *Foncteurs* K^n *en algèbre et en topologie*, C. R. Acad. Sc. Paris 269 (1969), 416-419.

S. Gersten, *Stable* K-*theory of discrete rings:* I and II, to appear.

J. Milnor, *Algebraic* K-*theory and quadratic forms*, Inventiones math. 9 (1970), 318-344.

D. Quillen, *The* K-*theory associated to a finite field:* I (mimeographed), 1970.

R. Swan, *Some relations between higher* K-*functors*, to appear.

These definitions are not mutually compatible, in general. Much work remains to be done in clarifying the relationships between various definitions. Note also that functors K_n for $n < 0$ have been defined by Bass (*Algebraic* K-*theory*, pp. 657-677).

The functors K_0 and K_1 are both important to geometric topologists. In the topological applications the ring Λ is always an integral group ring $Z\Pi$, where Π is the fundamental group of the object being studied. This theory had its beginnings in J.H.C. Whitehead's definition of the *torsion* associated with a homotopy equivalence between finite complexes. The Whitehead torsion lies in a certain factor group of $K_1 Z\Pi$. An important further step was taken by C. T. C. Wall. Consider a topological space A which is dominated by a finite complex. Then one can define a generalized "euler characteristic" $\chi(A)$, belonging to $K_0 Z\Pi$. Wall showed that A has the homotopy type of a finite complex if and only if $\chi(A)$ is an integer. Siebenmann and Golo have shown that similar obstructions exist to the problem of fitting a boundary onto an open manifold.

Recent work by J. Wagoner and A. Hatcher indicates that the functor K_2 has similar topological applications. If one is given a "pseudo-isotopy" of a closed manifold, then an obstruction to deforming it into an isotopy lies in an appropriate factor group of $K_2 Z\Pi$. Here is a list of references:

C. T. C. Wall, *Finiteness conditions for CW-complexes* I, Annals of Math. 81 (1965), 56-59; and II, Proc. Royal Soc. A 295 (1966), 129-139.

L. Siebenmann, *The structure of tame ends,* Notices Amer. Math. Soc. 13 (1966), 862.

J. Milnor, *Whitehead torsion,* Bull. Amer. Math. Soc. 72 (1966), 358-426.

G. de Rham, S. Maumary, and M. Kervaire, *Torsion et type simple d'homotopie,* Lecture Notes in Math. 48, Springer 1967.

V. L. Golo, *An invariant of open manifolds,* Izv. Akad. Nauk SSSR Ser. Mat. 31 (1967), 1091-1104. (Translation v. 1, 1041-1054 (A.M.S.).)

L. Siebenmann, *Torsion invariants for pseudo-isotopies on closed manifolds,* Notices Amer. Math. Soc. 14 (1967), 942.

R. M. F. Moss and C. B. Thomas (editors), *Algebraic K-theory and its geometric applications,* Lecture Notes in Math. 108, Springer 1969.

J. Wagoner, *Algebraic invariants for pseudo-isotopies, Proceed- of Liverpool Singularities Symposium II*, Lecture Notes in Math., Springer, to appear.

A strong impetus to the development of algebraic K-theory has been provided by work on the congruence subgroup problem, that is the problem of deciding whether every subgroup of finite index in an arithmetic group (such as SL(n, Λ) where Λ is the ring of integers in a number field) contains a congruence subgroup. This is closely related to the problem of computing $K_1\mathfrak{a}$ for an arbitrarily small ideal $\mathfrak{a} \subset \Lambda$. See the following, as well as the papers of Moore and Matsumoto mentioned earlier:

J. Mennicke, *Finite factor groups of the unimodular group,* Annals of Math. 81 (1965), 31-37.

J.-P. Serre, *Groupes de congruence,* Séminaire Bourbaki, 19e année (1966-67), n° 330.

H. Bass, *The congruence subgroup problem,* pp. 16-22 of *Local fields,* edited by T. A. Springer, Springer 1967.

H. Bass, J. Milnor, and J.-P. Serre, *Solution of the congruence subgroup problem for* SL_n ($n \geq 3$) *and* Sp_{2n} ($n \geq 2$), Publ. Math. I.H.E.S. 33 (1967).

L. N. Vaserštein, K_1*-theory and the congruence subgroup problem,* Mat. Zametki 5 (1969), 233-244 (Russian).

J.-P. Serre, Le problème des groupes de congruence pour SL_2, Annals of Math. 92 (1970), 487-527.

I want to thank Hyman Bass, Robert Steinberg, and John Tate for many valuable conversations, and particularly for access to their unpublished work. Also I want to thank Jeffrey Joel for a number of suggestions, and for his lecture notes (based on lectures at Princeton University in 1967), which provided the starting point for this manuscript. Finally I want to thank Princeton University, U.C.L.A., M.I.T., and the Institute for Advanced Study, as well as the National Science Foundation (grants G.P. -7917, -13630, and -23305) for their support during the preparation of this manuscript.

CONTENTS

§1. Projective Modules and $K_0\Lambda$

The word *ring* will always mean associative ring with an identity element 1.

Consider left modules over a ring Λ. Recall that a module M is *free* if there exists a basis $\{m_\alpha\}$ so that each module element can be expressed uniquely as a finite sum $\Sigma\lambda_\alpha m_\alpha$, and *projective* if there exists a module N so that the direct sum $M \oplus N$ is free. This is equivalent to the requirement that every short exact sequence $0 \to X \to Y \to M \to 0$ must be split exact, so that $Y \cong X \oplus M$.

The *projective module group* $K_0\Lambda$ is an additive group defined by generators and relations as follows. There is to be one generator [P] for each isomorphism class of finitely generated projective modules P over Λ, and one relation

$$[P] + [Q] = [P \oplus Q]$$

for each pair of finitely generated projectives. (Compare the proof of 1.1 below.)

Clearly every element of $K_0\Lambda$ can be expressed as the difference $[P_1] - [P_2]$ of two generators. (In fact, adding the same projective module to P_1 and P_2 if necessary, we may even assume that P_2 is free.) We will give a criterion for the equality of two such differences.

First another definition. Let Λ^r denote the free module consisting of all r-tuples of elements of Λ. Two modules M and N are called *stably isomorphic* if there exists an integer r so that

$$M \oplus \Lambda^r \cong N \oplus \Lambda^r.$$

LEMMA 1.1. *The generator* $[P]$ *of* $K_0\Lambda$ *is equal to the generator* $[Q]$ *if and only if* P *is stably isomorphic to* Q. *Hence the difference* $[P_1] - [P_2]$ *is equal to* $[Q_1] - [Q_2]$ *if and only if* $P_1 \oplus Q_2$ *is stably isomorphic to* $P_2 \oplus Q_1$.

Proof. The group $K_0\Lambda$ can be defined more formally as a quotient group F/R, where F is free abelian with one generator $\langle P\rangle$ for each isomorphism class of finitely generated projectives P, and where R is the subgroup spanned by all $\langle P\rangle + \langle Q\rangle - \langle P \oplus Q\rangle$. (Thus we are reserving the symbol $[P]$ for the residue class of $\langle P\rangle$ modulo R.) Note that a sum $\langle P_1\rangle + \ldots + \langle P_k\rangle$ in F is equal to $\langle Q_1\rangle + \ldots + \langle Q_k\rangle$ if and only if

$$P_1 \cong Q_{\pi(1)}, \ldots, P_k \cong Q_{\pi(k)}$$

for some permutation π of $\{1,\ldots,k\}$. If this is the case, note the resulting isomorphism

$$P_1 \oplus \ldots \oplus P_k \cong Q_1 \oplus \ldots \oplus Q_k.$$

Now suppose that $\langle M\rangle \equiv \langle N\rangle \bmod R$. This means that

$$\langle M\rangle - \langle N\rangle = \Sigma(\langle P_i\rangle + \langle Q_i\rangle - \langle P_i \oplus Q_i\rangle)$$
$$- \Sigma(\langle P_j'\rangle + \langle Q_j'\rangle - \langle P_j' \oplus Q_j'\rangle)$$

for appropriate modules P_i, Q_i, P_j', Q_j'.

Transposing all negative terms to the opposite side of the equation and then applying the remark above, we get

$$M \oplus (\Sigma (P_i \oplus Q_i) \oplus \Sigma P_j' \oplus \Sigma Q_j') \cong N \oplus (\Sigma P_i \oplus \Sigma Q_i \oplus \Sigma (P_j' \oplus Q_j')),$$

or briefly $M \oplus X \cong N \oplus X$, since the expressions inside the long parentheses are clearly isomorphic. Now choose Y so that $X \oplus Y$ is free, say $X \oplus Y \cong \Lambda^r$. Then adding Y to both sides we obtain $M \oplus \Lambda^r \cong N \oplus \Lambda^r$. Thus M is stably isomorphic to N.

The rest of the proof of 1.1 is straightforward. ■

If the ring Λ is commutative, note that the tensor product over Λ of (finitely generated projective) left Λ-modules is again a (finitely generated projective) left Λ module. Defining

$$[P]\cdot[Q] = [P \otimes Q]$$

we make the additive group $K_0\Lambda$ into a commutative ring. The identity element of this ring is the class $[\Lambda^1]$ of the free module on one generator.

In order to compute the group $K_0\Lambda$ it is necessary to ask two questions.

Question 1. Is every finitely generated projective over Λ actually free (or at least stably free)?

Question 2. Is the number of elements in a basis for a free module actually an invariant of the module? In other words if $\Lambda^r \cong \Lambda^s$ does it follow that $r = s$?

If both questions have an affirmative answer then clearly $K_0\Lambda$ *is the free abelian group generated by* $[\Lambda^1]$. *This will be true, for example, if* Λ *is a field, or a skew field, or a principal ideal domain.*

Of course Questions 1 and 2 may have negative answers. For example if Λ is the ring of endomorphisms of a finite dimensional vector space of dimension greater than 1, then Question 1 has a negative answer; and if Λ is the ring of endomorphisms of an infinite dimensional vector space then Question 2 has a negative answer. (The group $K_0\Lambda$ is infinite cyclic but not generated by $[\Lambda^1]$ in the first case, and is zero in the second.)

Here is an important example in which $K_0\Lambda$ is free cyclic.

LEMMA 1.2. *If* Λ *is a local ring, then every finitely generated* projective is free, and* $K_0\Lambda$ *is the free cyclic group generated by* $[\Lambda^1]$.

First recall the relevant definitions. A ring element u is called a *unit* if there exists a ring element v with $uv = vu = 1$. The set $\Lambda^\bullet$ consisting of all units in Λ evidently forms a multiplicative group.

Λ is called a *local ring* if the set $\mathfrak{m} = \Lambda - \Lambda^\bullet$ consisting of all non-units is a left ideal. It follows that $\mathfrak{m}$ is a right ideal also. For if some

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* Compare Kaplansky, *Projective modules*, Annals of Mathematics 68 (1958), 372-377.

product $m\lambda$ with $m \in \mathfrak{m}$ and $\lambda \in \Lambda$ were a unit, then clearly m would have a right inverse, say $mv = 1$. This element v certainly cannot belong to the left ideal $\mathfrak{m}$. But v cannot be a unit either. For if v were a unit, then the computation

$$m = m(vv^{-1}) = (mv)v^{-1} = v^{-1}$$

would show that m must be a unit.

This contradiction shows that $\mathfrak{m}$ is indeed a two-sided ideal. The quotient ring $\Lambda/\mathfrak{m}$ is evidently a field or skew-field.

Note that a square matrix with entries in Λ is non-singular if and only if the corresponding matrix with entries in the quotient $\Lambda/\mathfrak{m}$ is non-singular. To prove this fact, multiply the given matrix on the left by a matrix which represents an inverse modulo $\mathfrak{m}$, and then apply elementary row operations to diagonalize. This shows that the matrix has a left inverse, and a similar argument constructs a right inverse.

We are now ready to prove Lemma 1.2. If the module P is finitely generated and projective over Λ then we can choose Q so that $P \oplus Q \cong \Lambda^r$. Thinking of the quotients $P/\mathfrak{m}P$ and $Q/\mathfrak{m}Q$ as vector spaces over the skew-field $\Lambda/\mathfrak{m}$, we can choose bases. Choose a representative in P or in Q for each basis element. The above remark on matrices then implies that the elements so obtained constitute a basis for $P \oplus Q$. Clearly it follows that P and Q are free. Since the dimension of the vector space $P/\mathfrak{m}P$ is an invariant of P, this completes the proof. ■

Next consider a homomorphism

$$f : \Lambda \to \Lambda'$$

between two rings. (It is always assumed that $f(1) = 1$.) Then every module M over Λ gives rise to a module

$$f_{\#}M = \Lambda' \otimes_{\Lambda} M$$

over Λ'. Clearly if M is finitely generated, or free, or projective, or splits as a direct sum over Λ, then $f_{\#}M$ is finitely generated, or free, or projective, or splits as as a corresponding direct sum over Λ'. Hence the correspondence

$$[P] \mapsto [f_\# P]$$

gives rise to a homomorphism

$$f_* : K_0\Lambda \to K_0\Lambda'$$

of abelian groups. Note the functorial properties

$$(\text{identity})_* = \text{identity}, \ (f \circ g)_* = f_* \circ g_*.$$

Example 1. Let Z be the ring of integers. Then for any ring Λ there is a unique homomorphism

$$i : Z \to \Lambda.$$

The image

$$i_* K_0 Z \subset K_0\Lambda$$

is clearly the subgroup generated by the free module $[\Lambda^1]$. The co-kernel

$$K_0\Lambda/(\text{subgroup generated by } [\Lambda^1]) = K_0\Lambda/i_* K_0 Z$$

is called the *projective class group* of Λ.

Example 2. Suppose that Λ can be mapped homomorphically into a field or skew-field F. This is always possible, for example, if Λ is commutative. Then we obtain a homomorphism

$$j_* : K_0\Lambda \to K_0 F \cong Z.$$

In the commutative case, this homomorphism is clearly determined by the kernel of j, which is a prime ideal in Λ. Hence one can speak of the *rank* of a projective module at a prime ideal $\mathfrak{p}$. If $\mathfrak{p} \supset \mathfrak{p}'$, note that the rank at $\mathfrak{p}$ is equal to the rank at $\mathfrak{p}'$. For if we localize the integral domain $\Lambda/\mathfrak{p}'$ at the ideal corresponding to $\mathfrak{p}$ (that is adjoin the inverses of all elements not belonging to $\mathfrak{p}$) we obtain a local ring which embeds in the quotient field of $\Lambda/\mathfrak{p}'$ and maps homomorphically into the quotient field of $\Lambda/\mathfrak{p}$. Using Lemma 1.2, it follows that the ranks are equal. *In particular, if Λ is an integral domain, then the rank of a projective module is the same at all prime ideals.*

In any case, choosing some fixed homomorphism $j : \Lambda \to F$, since $j_* i_*$ is an isomorphism, we obtain a direct sum decomposition

$$K_0\Lambda = (\text{image } i_*) \oplus (\text{kernel } j_*).$$

The first summand is free cyclic, and the second maps bijectively to the projective class group of Λ.

In the commutative case, note that (kernel j_*) is an ideal in the ring $K_0\Lambda$. We will denote this ideal by $\tilde{K}_0\Lambda$, and write

$$K_0\Lambda \cong Z \oplus \tilde{K}_0\Lambda.$$

Example 3. Suppose that Λ splits as a cartesian product

$$\Lambda_1 \times \Lambda_2 \times \ldots \times \Lambda_k$$

of rings. Then the projection homomorphisms

$$K_0\Lambda \to K_0\Lambda_i$$

give rise to a corresponding cartesian product structure

$$K_0\Lambda \cong K_0\Lambda_1 \times K_0\Lambda_2 \times \ldots \times K_0\Lambda_k.$$

The proof is not difficult.

Such a splitting of Λ occurs for example whenever Λ is commutative and artinian,* but is not local. For since Λ is commutative, the set of all nilpotent elements forms an ideal. If Λ is not local, there must exist an element λ which is neither a unit nor a nilpotent element. Since Λ is artinian, the sequence of principal ideals

$$(\lambda) \supset (\lambda^2) \supset (\lambda^3) \supset \ldots$$

must terminate, say $(\lambda^n) = (\lambda^{n+1}) = \ldots$ so that $\lambda^n = \rho\lambda^{2n}$ for some ρ. But this implies that the element $e = \rho\lambda^n$ is idempotent $(ee = e)$, and hence that Λ splits as a cartesian product

$$\Lambda \cong \Lambda/(e) \times \Lambda/(1-e).$$

This splitting is not trivial since the hypothesis that λ is neither a unit nor nilpotent implies that $e \neq 1, 0$. This procedure can be continued inductively until Λ has been expressed as a cartesian product of local rings. It then follows that

$$K_0\Lambda \cong Z \times Z \times \ldots \times Z.$$

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* A ring is *artinian* if every descending sequence of ideals must terminate.

Dedekind Domains

Important examples in which the ring $K_0\Lambda$ has a more interesting structure are provided by Dedekind domains. We will discuss these in some detail, starting for variety with a non-standard version of the definition.*

DEFINITION. A *Dedekind domain* is a commutative ring without zero divisors such that, for any pair of ideals $\mathfrak{a} \subset \mathfrak{b}$, there exists an ideal $\mathfrak{c}$ with $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$.

REMARK 1.3. Note that the ideal $\mathfrak{c}$ is uniquely determined, except in the trivial case $\mathfrak{a} = \mathfrak{b} = 0$. In fact if $\mathfrak{b}\,\mathfrak{c} = \mathfrak{b}\,\mathfrak{c}'$, then choosing some non-zero principal ideal $b_0\Lambda \subset \mathfrak{b}$ we can express $b_0\Lambda$ as a product $\mathfrak{x}\,\mathfrak{b}$ and conclude that $\mathfrak{x}\,\mathfrak{b}\mathfrak{c} = \mathfrak{x}\,\mathfrak{b}\,\mathfrak{c}'$, hence $b_0\mathfrak{c} = b_0\mathfrak{c}'$, from which the equality $\mathfrak{c} = \mathfrak{c}'$ follows.

DEFINITION. Two non-zero ideals $\mathfrak{a}$ and $\mathfrak{b}$ in the Dedekind domain Λ belong to the same *ideal class* if there exist non-zero ring elements x and y so that $x\mathfrak{a} = y\mathfrak{b}$.

Clearly the ideal classes of Λ form an abelian group under multiplication, with the class of principal ideals as identity element. We will use the notation $C(\Lambda)$ for the ideal class group of Λ, and the notation $\{\mathfrak{a}\} \in C(\Lambda)$ for the ideal class of $\mathfrak{a}$.

Note that $\{\mathfrak{a}\} = \{\mathfrak{b}\}$ if and only if $\mathfrak{a}$ is isomorphic, as Λ-module, to $\mathfrak{b}$. For if $\phi : \mathfrak{a} \to \mathfrak{b}$ is an isomorphism, then choosing $a_o \in \mathfrak{a}$, the computation $a_o\phi(a) = \phi(a_oa) = \phi(a_o)a$ shows that $a_o\mathfrak{b} = \phi(a_o)\mathfrak{a}$.

Important examples of Dedekind domains can be constructed as follows. Let F be a finite extension of the field Q of rational numbers. An element of F is called an *algebraic integer* if it is the root of a monic polynomial

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* The usual definition is of course equivalent to the one given here. For further information, see Zariski and Samuel, *Commutative Algebra* I, Van Nostrand 1958; or Lang, *Algebraic Number Theory*, Addison-Wesley 1970; as well as Cartan and Eilenberg, *Homological Algebra*, Princeton University Press 1956.