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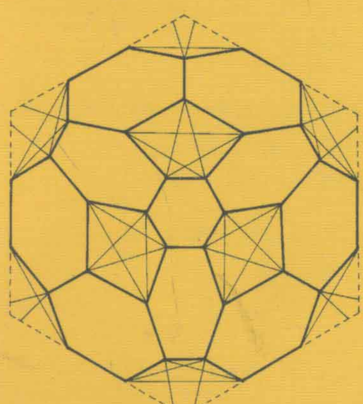
Wolfgang Kühnel

Tight Polyhedral Submanifolds and Tight Triangulations



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Preface

A first version of this monograph was written several years ago, while the author was a guest of the I.H.E.S. at Bures-sur-Yvette. He gratefully acknowledges numerous discussions about tight submanifolds with N.H. Kuiper, T.F. Banchoff and others. Later this first draft was distributed as preprint Nr. 108 of the Math. Dept., University of Duisburg. The author also acknowledges the warm hospitality of the Landau-Center at the Hebrew University of Jerusalem in 1990. At that time Gil Kalai introduced the author to some of the mysteries of the Upper and Lower Bound Conjecture for polytopes, and certain parts of Chapter 4 were being developed. Fortunately or unfortunately, there are still a number of conjectures left open.

Tightness is a concept from differential geometry that has many connections to other branches of mathematics. This monograph is a presentation of a part of mathematics sitting between various special disciplines such as differential geometry, topology, theory of convex polytopes, combinatorics. The main intention is to stimulate further fruitful interaction in this direction. The treatment of the 2-dimensional case in the – essentially self-contained – Chapter 2 is an example of an interplay between the theory of convex polytopes, graph theory, and elementary polyhedral topology.

Finally, the author should like to thank S. Lukas for typing the first preprint version, B. Dunkel for typing the main part of the present version in LaTeX (including numerous complicated formulas), Ch. Habbe for valuable support in drawing the figures, and D. Cervone and M. van Gemmeren for careful proofreading and support in handling the LaTeX system.

Only a few days before the manuscript was finished, we received the sad news that Professor Nicolaas Hendrik Kuiper passed away at the age of 74. He was not only a great mathematician and one of the pioneers in this area of mathematics, but also a longstanding friend. This volume is dedicated to his memory.

January 1995

W.K.

Contents

1. Introduction and basic notions

Tightness is a generalization of the notion of convexity that applies to objects other than topological balls and their boundary surfaces. In some sense it means that a submanifold or subset of E^d is embedded 'as convexly as possible' with respect to its topological properties.

The subject of tight embeddings began in 1938 with a paper of A.D. Alexandrov [A] on 'T-surfaces', studying the total absolute curvature for analytic torus-shaped surfaces in three-space. A T-surface in Alexandrov's sense is a torus with minimal total absolute curvature 8π. In 1950, J. Milnor [Mir] proved an inequality between the total absolute curvature of a hypersurface in Euclidean space and its total Betti number. This paper was not published until recently but it anticipated several developments of the subsequent years. Not much later, in 1957, S.S. Chern and R.K. Lashof [CL] initiated the study of total absolute curvature for smooth submanifolds of higher dimension and codimension in Euclidean space, and they proved the inequality between the total absolute curvature and the total Betti number in general. This was done by relating this question to numbers of critical points of height functions restricted to the submanifold. This aspect of tightness was further developed by N.H. Kuiper in a series of papers beginning in 1958 [Kui1].

N.H. Kuiper was the first one to interpret minimal total absolute curvature for non-smooth submanifolds, by extending the critical point theory in an appropriate way. In 1965, T.F. Banchoff [Ba1] introduced tightness and the Two-piece-property for polyhedral surfaces in higher dimensional space. This has been extended by the author to a variety of other contexts, including higher dimensional polyhedra and the relationship of tightness with the classical theory of convex polytopes.

After the early Lecture Notes by D. Ferus [Fe], the monograph by T. Cecil and P. Ryan [CR] gives the background of the theory of tightness for smooth submanifolds including related subjects such as tautness and isoparametric hypersurfaces. In this monograph, we will present the theory of tightness for polyhedral manifolds and their embeddings into Euclidean space, and the related subject of tight triangulations. These are triangulations such that any simplexwise linear mapping into any Euclidean space is tight.

In Chapter 2 we start with the following quite elementary problem: Is it possible to embed compact and connected surfaces (without boundary) as polyhedra into Euclidean space such that any hyperplane separates the surface into at most two pieces? This is called the two-piece-property; in this special case it is equivalent to tightness. We have to distinguish between the case of small codimension 1,2,3 (Section 2B) and higher codimension (Section 2C); in fact, the methods are quite different. It was the idea of T.F. Banchoff to construct tight polyhedral surfaces as subcomplexes of higher dimensional cubes or simplices. In this case the crucial condition is that the surface contains the complete 1-dimensional skeleton of the cube or simplex (or any polytope in general). We shall improve Banchoff's results by applying several tools from the theory of convex polytopes. With a few exceptions, it is always possible to find a tight polyhedral surface $M \to E^d$ for arbitrary given M and d subject to just one essential condition: the Heawood inequality between d and $\chi(M)$. The origin of this inequality is the Map Color Problem for surfaces of genus greater than zero, studied by Heawood at the end of the 19^{th} century. Tight triangulations of surfaces are characterized by the case of equality in the Heawood inequality (Section 2D). Finally, the cases of surfaces with boundary and with singularities are discussed in Section 2E and Section 2F.

Chapter 3 gives the general definition and basic facts about tightness of higher dimensional polyhedra. Here it is quite natural to take advantage of a certain critical point theory which generalizes the classical Morse theory and which is adapted to the polyhedral case. With respect to this critical point theory, we follow the ideas and results of M. Morse and N.H. Kuiper. Section 3C presents a general method how to construct tight subcomplexes of higher dimensional cubes. This is an application of L. Danzer's construction of a 'power-complex' 2^K for a given simplicial complex K. It turns out that 2^K is always tight in the ambient Euclidean space, independently of K. This leads to unexpected types of examples.

Chapter 4 is the central chapter of this monograph. It deals with tightly embedded $(k-1)$-connected $2k$-manifolds, as a natural generalization of the results of Chapter 2 to the case of arbitrary k. Several versions of generalized Heawood inequalities are given. In particular, as in the case of surfaces, the case of equality is exactly the case of tight triangulations: subcomplexes of the simplex containing its complete k-dimensional skeleton. Special examples are the unique 9-vertex triangulation of CP^2 and three different 15-vertex triangulations of an 8-manifold. Various conjectures are made, and various open problems are mentioned.

Chapter 5 deals with the odd-dimensional case. Here it seems that there is no easy way to transform the tightness into a purely combinatorial condition. Therefore the results are much weaker than the ones in Chapter 4. However, there are nontrivial examples. In particular there is a 9-vertex triangulation of a nonorientable handle found by D. Walkup and independently by A. Altshuler and

L. Steinberg. This leads to a tight embedding of the 3-dimensional Klein bottle into E^8. Moreover, for any dimension d there is a tight polyhedral embedding of the sphere product $S^1 \times S^{d-1}$ or the corresponding twisted product into $(2d+2)$-space by a tight triangulation.

Chapter 6 describes a method how to embed connected sums tightly into Euclidean space if the summands admit appropriate tight embeddings. This is a generalization of Banchoff's construction of a tight Klein bottle from two copies of a tight Möbius band. In particular it leads to tight embeddings of connected sums of $\mathbf{C}P^2$ into E^8. This problem is closely related with the discussion of tight embeddings of manifolds with boundary.

Chapter 7 gives some generalizations to the case of odd-dimensional manifolds and pseudomanifolds with isolated singularities. We obtain an inequality of the Heawood type which in a certain sense is an odd-dimensional analogue of the one in Chapter 4. This inequality deals with $(k-1)$-connected $(2k-1)$-pseudomanifolds with isolated singularities, and once again the case of equality coincides with the case of tight triangulations. Slices by hyperplanes are tight polyhedral manifolds if the hyperplane does not meet any of the vertices.

Throughout this volume (except for Chapter 2 which is essentially selfcontained), the following basic notions will be used:

1.1 Definition (Polytopes): A convex d-polytope is the convex hull of finitely many points in E^d not lying in a common hyperplane. This has the structure of a complex built up by the partially ordered set of faces of dimension $0, 1, \ldots, d-1$. The faces are defined as the intersections of the convex set with supporting hyperplanes. The $(d-1)$-dimensional faces are often called facets, the 1-dimensional faces are the edges, the 0-dimensional faces are the vertices. The d-simplex Δ^d is the convex hull of $d+1$ points in general position in E^d. A d-polytope is called simplicial if for any k each of its k-dimensional faces is a k-simplex; it is called simple if its dual is simplicial, or equivalently, if at each vertex there meet exactly d edges, the minimum number. Each face of a simple polytope is also simple. For each vertex of a convex d-polytope, the vertex figure is defined as the $d-1$-polytope which occurs as a slice by a hyperplane through the d-polytope, separating this vertex from all the other vertices.
The k-dimensional skeleton (or k-skeleton) of a polytope P, denoted by $Sk_k(P)$, is the union of all k-dimensional faces of P. In particular $Sk_1(\Delta^{n-1})$ is combinatorially equivalent to the complete graph with n vertices, denoted by K_n. $Sk_k(\Delta^{n-1})$ is also called the complete k-complex with n vertices, see [Du].

1.2 Definition (Polyhedra): A polyhedron is a finite union of convex polytopes of arbitrary dimensions (called faces) such that the intersection of any two of them is either empty or a common face of both of them. A special case is a simplicial complex which can always be regarded as a subcomplex of the boundary complex of a simplex of sufficiently high dimension. Sometimes one wants to allow also non-convex faces. By subdivision one can always reduce this case to the case of convex faces. However, note that we have to distinguish between polyhedra and polytopes even in the convex case: the ordinary cube consisting of six squares is a 3-polytope (= the convex hull of its eight vertices). A subdivided cube consisting of twelve triangles, each square subdivided by a diagonal, is a convex polyhedron but not a polytope in this sense. More precisely, the diagonals are considered as edges of the polyhedron but not as edges of the polytope. In the literature such distinction is also made between so-called proper and improper edges or vertices. Every triangulated d-sphere with at most $d+4$ vertices is the boundary complex of a certain $(d+1)$-polytope (see [M]), but there are triangulated 3-spheres with 8 vertices which are non-polytopal (see [Bar2], [Grü-S]).

1.3 Definition (Tightness): A compact and connected subset $M \subset E^d$ is called tight with respect to a field F if for every open or closed half space $h \subset E^d$ the induced homomorphism

$$H_*(h \cap M; F) \to H_*(M; F)$$

is injective where H_* denotes an appropriate homology theory. For polyhedra M it is convenient to use singular homology or simplicial homology because $h \cap M$ has always the homotopy type of a finite polyhedron. M is called tight if it is tight with respect to at least one field, where the standard case in the literature is the field $\mathbf{Z}_2$ with two elements. M is called substantial in E^d if it is not contained in any hyperplane.

1.4 Definition (top-sets, Tight triangulations): Let M be a polyhedron in E^d and $\mathcal{H}$ its convex hull, regarded also as a convex polytope. For each k-dimensional face A_k of $\mathcal{H}$ we call $M \cap A_k$ a k-top-set of M. A top-set is a k-top-set for some k. Moreover, if M is tight then every top-set of M is also tight because for any half space h the intersection $h \cap M \cap A_k$ is a deformation retract of $h' \cap M$ for some nearby half space h'. A top-set is called essential if it is not a convex set.

If in addition M is a subcomplex of $\mathcal{H}$ then we call it a tight complex. This carries a structure analogous to that of the d-polytope $\mathcal{H}$ if we replace the k-faces of $\mathcal{H}$ by the tight k-top-sets of M: it is just the complex built up by the tight top-sets. A subcomplex M of the boundary complex of a d-dimensional simplex is called a tight simplicial complex if M is tight regarded as a polyhedron in the ambient Euclidean space. In this case every top-set is again a tight simplicial complex.
If it is in addition a triangulation of a manifold with or without boundary, then we call this simplicial complex a tight triangulation of the manifold.

1.5 Definition (combinatorial manifolds): If the underlying set of a simplicial complex is a topological manifold then it is usually called a triangulated manifold. However, in general these two structures (topological and combinatorial) need not be compatible. There is the strange example of the so-called Edwards sphere [Ed] which is a double suspension of an homology 3-sphere and which is not PL. A PL structure on a manifold is an atlas of charts which are compatible with each other by piecewise linear coordinate transformations. Therefore, a slightly stronger notion is introduced as follows: A simplicial complex is called a combinatorial manifold of dimension d if the link (or vertex figure) of any k-dimensional simplex is a triangulated $(d-k-1)$-dimensional sphere. This condition implies that this manifold carries a PL structure and that each vertex star is a neighborhood of the corresponding vertex and that this vertex star is PL homeomorphic to a d-ball. Conversely, every PL manifold admits a triangulation which is a combinatorial manifold in this sense. For details see [Hud], [Kui7], [Scht], [RS]. As usual, $\chi(M)$ denotes the Euler characteristic of M.

2. Tight polyhedral surfaces

This chapter is essentially self-contained. It deals with the 2-dimensional case where tightness is a quite elementary property. A surface is a compact and connected 2-dimensional manifold without boundary, unless stated otherwise. At the end of this chapter we consider surfaces with boundary also. A surface M is embedded into E^d if it is homeomorphic to a subset of E^d. Similarly, M is immersed into E^d by a continuous mapping $f\colon M \to E^d$ if f is locally injective. Roughly speaking, 'immersion' means 'local embedding'. A polyhedral surface $M \subset E^d$ is a surface embedded in E^d such that M is a finite union of planar polygons (or faces) where any two of them have no interior parts in common. By subdivision one can always assume that the polygons are convex. Then the intersection of two such convex polygons is either empty or one point (a vertex) or a line segment (an edge), and that the intersection of two edges is either empty or one vertex. Similarly, one can talk about polyhedral immersions of surfaces. Note that in this case the self-intersections of faces are not automatically considered as edges or vertices.

A polyhedral surface can be regarded as the partially ordered set of its vertices, edges and convex faces. We call it triangulated if all faces are triangles. An abstract triangulation of an abstract surface is a decomposition into triangles such that any two of them meet along a vertex or an edge, or not at all. Such an abstract triangulation can always be regarded as a polyhedron in a Euclidean space of higher dimension. $\chi(M)$ denotes the Euler characteristic of M.

2A. 0-tightness

2.1 Definition: A surface $M \subset E^d$ (or an arbitrary connected polyhedron in E^d) is called 0-tight if for any open or closed half space $h \subset E^d$ the intersection $M \cap h$ is connected (Banchoff's Two-piece-property, TPP). For an immersion $f\colon M \to E^d$ the 0-tightness means that for any h the preimage $f^{-1}(M \cap h)$ is connected. A subset (or an immersion) is called substantial if it (or its image, respectively) is not contained in any hyperplane. For a compact polyhedron M in E^d let $\mathcal{H} = \mathcal{H}(M)$ denote the convex hull of M, the smallest convex set which contains M. $\mathcal{H}(M)$ is a convex d-polytope if M is substantial in E^d, compare 1.1.

The following necessary condition for the 0-tightness of a polyhedron is originally due to T. Banchoff. In his terminology the 1-skeleton of the convex hull is called the set of extreme vertices and extreme edges.

2.2 Lemma (T. Banchoff [Ba1;Lemma 3.1]): *Let $M \subset E^d$ be a 0-tight and connected polyhedron. Then M contains the 1-skeleton of its convex hull:*

$$Sk_1(\mathcal{H}) \subset M.$$

PROOF: Let e be an edge of $\mathcal{H}$ with endpoints v, w. By construction M contains v and w. There is a half space h of E^d such that $h \cap \mathcal{H} = e$. Consequently we have $\{v, w\} \subset h \cap M \subset h \cap \mathcal{H} = e$. By the 0-tightness $h \cap M$ must be connected. It follows that $h \cap M = e$.

2.3 Lemma: *A polyhedral surface M with convex faces is 0-tight if and only if its 1-skeleton is 0-tight. The same holds for any connected polyhedron M in the sense of Definition 1.2.*

PROOF: If the 1-skeleton is 0-tight then M is 0-tight because adding higher dimensional faces preserves the connectedness of $M \cap h$. Vice versa, if $M \cap h$ is connected then $Sk_1(M) \cap h$ must be connected because the faces are convex. Note that this is not true if there are non-convex faces.

By 2.3 all the information about 0-tightness depends only on the 1-dimensional skeleton of the polyhedron. This 1-skeleton is nothing but a graph whose edges are straight line segments of Euclidean space, a so-called polyhedral graph. Therefore it is useful to characterize the 0-tightness for such graphs as follows:

2.4 Lemma: *An embedded and connected polyhedral graph $G \subset E^d$ is 0-tight if and only if the following conditions are satisfied:*

(i) G contains the 1-skeleton of its convex hull $\mathcal{H}G$,

(ii) every vertex of G which is not a vertex of $\mathcal{H}G$ lies in the relative interior of some of its neighbors.

PROOF: Let G be 0-tight. Then (i) follows from 2.2. Now let v be a vertex of G which is not a vertex of $\mathcal{H}G$. By the two-piece-property it is impossible to separate this vertex from its neighbors by a closed half space. This implies that the neighbors of v can never lie in an open half space, and (ii) follows.
To see the converse direction, let h be a closed half space such that $h \cap G$ is disconnected. One of these components certainly contains vertices of $\mathcal{H}$. If

there are several of those components then (i) is violated. Otherwise there is an 'interior component' of $h \cap G$. Move h orthogonal to its boundary until finally this interior component becomes as small as possible but still nonempty. It follows that this component contains a vertex which contradicts (ii).

EXAMPLES: Figure 1 gives three embeddings of the same graph G. In the second one G coincides with the 1-skeleton of its convex hull, the third one is not 0-tight according to 2.4.

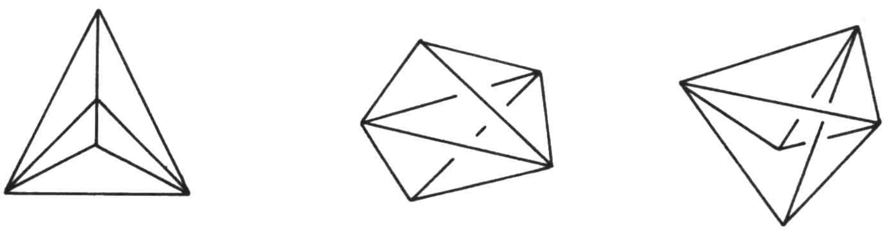


Figure 1

For graphs in the Euclidean plane the 0-tightness just means that every component of the complement of the graph is either convex or the complement of a convex set. Therefore we obtain the following reformulation of a famous theorem in graph theory which in the original formulation characterizes the edge graphs of convex 3-polytopes:

THEOREM (E. Steinitz [Grü;13.1], also attributed to W.T. Tutte [Tu]): *Any 3-connected and planar graph admits a 0-tight embedding into the Euclidean plane.*

The theorem of Steinitz says that any such graph is the edge graph of a certain 3-polytope. The Schlegel diagram of this polytope gives a 0-tight embedding.

2.5 Lemma: *For a polyhedral surface $M \subset E^d$ the following conditions are equivalent:*

(i) M is 0-tight,

(ii) for every open or closed half space $h \subset E^d$ the induced homomorphism

$$H_*(h \cap M; \mathbf{Z}_2) \to H_*(M; \mathbf{Z}_2)$$

is injective where H_* denotes the standard singular homology.

In general, M is called tight if (ii) is satisfied, see 1.3. Lemma 2.5 just says that for compact surfaces without boundary tightness and 0-tightness are equivalent notions. The same holds for 1-dimensional complexes (graphs). In what follows we will often use this equivalence without further comment.

2.5 is a special case of a more general result in 3.18 below. We do not give a proof separately but just mention that it is a simple consequence of elementary Morse theory. (ii) ⇒ (i) is obvious. Assume that (i) is true. For any linear function with finitely many critical points let μ_0, μ_1, μ_2 be the numbers of critical points of index 0, 1, 2, then we have $\mu_0 - \mu_1 + \mu_2 = \chi(M)$. Consequently, if $\mu_0 = \mu_2 = 1$ then $\mu_1 = 2 - \chi(M)$ is completely determined by the Euler characteristic of M. This implies that the inclusion $M \cap h \to M$ can never reduce the rank of the homology.

WARNING:
(a) 2.5 is not true for homology with arbitrary coefficients instead of $\mathbf{Z}_2$. If the surface is nonorientable then it holds for fields only of characteristic 2.
(b) 2.5 holds only for closed surfaces but not for surfaces with boundary. A cone over the boundary of a triangle is 0-tight but not tight.
(c) 2.5 does not hold for surfaces with singularities either, see the remark after 2.28.

2B. Tight surfaces with small codimension

In this section we give some basic results about the existence of tight polyhedral surfaces in the cases of small codimension, that are mainly the cases of codimension 1 and 2. Afterwards we turn to the question how large the essential codimension can be.

2.6 Theorem: *Given an abstract surface M with Euler characteristic $\chi(M)$, the following hold:*

(i) There is a tight polyhedral embedding $M \to E^3$ if M is orientable.

(ii) There is a tight polyhedral immersion $M \to E^3$ if M is nonorientable and $\chi(M) \leq -1$.

(iii) There are tight and substantial polyhedral embeddings $M \to E^4$ and $M \to E^5$ if M is distinct from the 2-sphere.

(iv) There is a tight and substantial polyhedral embedding $M \to E^6$ if M is orientable and distinct from the 2-sphere.

PROOF: The proof consists in a series of examples. It turns out that the most difficult case is the nonorientable surface in E^3 with $\chi = -1$, settled only recently by D. Cervone [Ce2]. The case $\chi = -3$ is also special although it is a consequence of the case $\chi = -1$ by attaching a handle. In fact, in this case there is a better solution which is fairly symmetric and, in addition, smoothable by a tight surface.

(i) The boundary of any convex 3-polytope provides a tight embedding $S^2 \to E^3$. In particular we can take the boundary of a cube (up to affine transformations). Then it is possible to cut g square-shaped holes into the top and the bottom of the cube and to join them by straight polyhedral cylinders, see Figure 2. It is easy to see that the resulting surface of genus g is tight. After a subdivision into convex polygons the conditions in 2.4 are satisfied by the edge graph. All these examples can easily be smoothed, still preserving the tightness.
A completely different example of a tight polyhedral torus in E^3 is given by Császár's torus with 7 vertices, based on the unique 7-vertex triangulation of the torus, see [Cs], [Bo-Eg]. This triangulation was already known to A. Möbius [Mö]. In this case the 1-skeleton is the complete graph K_7 with 7 vertices, see Figure 2. Therefore any simplexwise linear embedding is tight by 2.3 and 2.4. A polyhedral surface of genus $g \geq 1$ must have vertices at which the surface is not locally convex. For the necessary number of non-convex vertices for tight surfaces see [Gri]. An alternative construction of a tight orientable surface in E^3 was given in [Ba-Kui] as the boundary of the difference set of two convex polytopes which are dual to each other.

(ii) If $\chi(M) = -2$ we get an example by starting with a tight polyhedral torus constructed as in (i) (a cube with a rectangular hole). Then a non-orientable cylindrical handle can be attached, joining the outer part to the inner part, see Figure 2. This construction is due to N.H. Kuiper [Kui2]. As in (i) one can add arbitrarily many handles such that the tightness is preserved. This proves the assertion in the case of an even Euler characteristic. If $\chi(M)$ is odd we have to find a starting example. In [Kü-Pi] a tight polyhedral surface with $\chi = -3$ is given; the inner part, shown in Figure 2, is attached to an outer tetrahedron. The tightness follows from 2.3 and 2.4. Note that the curve of self-intersection with the triple point at its centre does not consist of edges. By attaching handles as above, the assertion follows for any $\chi \leq -2$. All these cases can be smoothed tightly, for an explicit smoothing procedure see [Kü-Pi].
The case $\chi = -1$ is very special because a theorem by F. Haab [Haa] says that there is no smooth tight immersion of this topological type. Supported by Haab's result, it had been conjectured that there is no polyhedral tight immersion either.

Surprisingly, this is not true, and in 1994 D. Cervone [Ce2] came up with an example. In Figure 2 we reproduce his triangulation. The vertices of the convex hull are a, c, d, e, f, g, h (b lies on the edge ac). Figure 2 shows the outer part (= a cylinder) and the inner part (= a Möbius band with hole) of the triangulation. The positions of the vertices are as follows:

$$\begin{array}{lllllllll} a & = & (-2,0,0), & e & = & (-2,-1,2), & i & = & (-\frac{3}{8},0,\frac{1}{2}) \\ b & = & (0,0,0), & f & = & (1,-1,2), & j & = & (\frac{1}{2},\frac{1}{4},1) \\ c & = & (1,0,0), & g & = & (1,1,2), & k & = & (-\frac{1}{4},\frac{7}{12},\frac{7}{6}) \\ d & = & (0,1,0), & h & = & (0,3,2), & l & = & (0,\frac{3}{4},\frac{7}{6}) \\ & & & m & = & (\frac{1}{4},0,\frac{1}{2}). & & & \end{array}$$

Again the tightness follows from the 0-tightness of the edge graph, 2.3 and 2.4.

(iii) A tight polyhedral torus in E^4 is given by the cartesian product of two planar convex polygons, e.g., by two squares. This is an analogue of the Clifford torus in S^3. The edge graph of this square torus coincides with the edge graph of a 4-dimensional cube. The tightness follows from 2.2. As in (i), it is possible to attach arbitrarily many orientable handles tightly. A nonorientable handle can be attached in the following way: In the 4-cube $C^4 = [0,1]^4$ take two opposite 2-faces, spanned by the vertices (0,0,0,0), (0,0,0,1), (0,0,1,0), (0,0,1,1), and the opposite face spanned by (1,1,0,0), (1,1,0,1), (1,1,1,0), (1,1,1,1). These two squares lie in a common 3-space. In this 3-space we can attach a cylindrical handle as in (i). The resulting surface is nonorientable and still tight. Its image under a suitable projection is coincides with the tight Klein bottle with handle in E^3 described above. Additional handles can be added. Note that all these examples can be smoothed tightly as in (ii). A tight Klein bottle in E^4 has been constructed by T. Banchoff [Ba5]: Start with two copies of a 5-vertex Möbius band (Figure 2) in two parallel hyperplanes of E^4 and join their boundaries by a straight cylinder.

This covers the case of even Euler characteristic in E^4. In the case of odd Euler characteristic we can start with a simplexwise embedding of the 6-vertex triangulation of the real projective plane (= half of the icosahedron), see Figure 2. The tightness of any simplexwise embedding is guaranteed by the completeness of the edge graph and by 2.3 and 2.4. Then handles can be attached tightly in any 3-space spanned by two adjacent triangles. The same argument shows that any surface with odd Euler charateristic admits a tight and substantial polyhedral embedding into E^5. In this case we start with the 6-vertex real projective plane, regarded as a subcomplex of the 5-dimensional simplex. The remaining case of surfaces with even Euler characteristic in E^5 can be treated in a similar way. We only need a starting example of a torus and of a Klein bottle . For the torus we can use a simplexwise linear embedding of the 7-vertex torus, see Figure 2, for the Klein bottle we can start with two 5-vertex Möbius bands in two parallel hyperplanes of E^5 and join their boundaries by a straight cylinder, a construction first given in [Ba5].

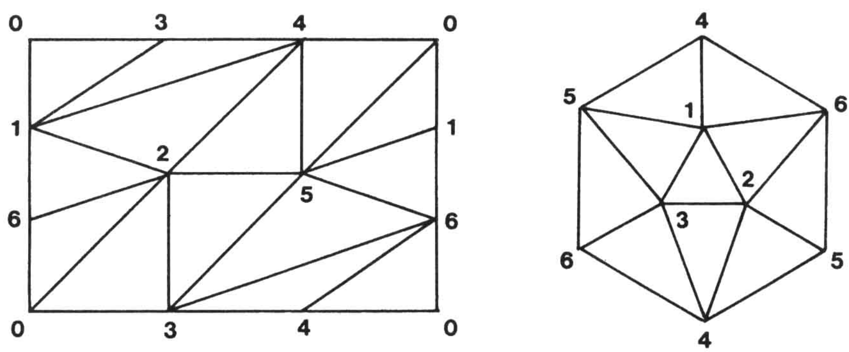
(iv) As in (iii), we start with a tight embedding of the torus: regard the 7-vertex triangulation as a subcomplex of the 6-dimensional simplex. This is tight by 2.3. Then handles can be added as described above.

This completes the proof of 2.6. The cases not covered by 2.6 are the following:

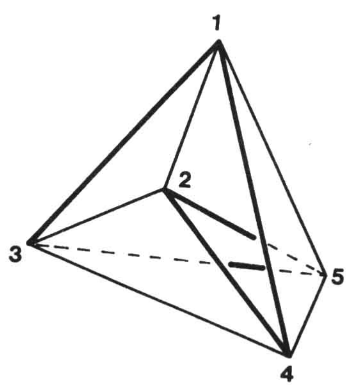
$M \to E^3$ for the real projective plane and for the Klein bottle. No tight immersion of this type exists, not even purely topologically, see [Kui10]. We do not give separate proofs for the polyhedral case.

$S^2 \to E^k$, $k \geq 4$, see 2.9 below,

$M \to E^k$, $k \geq 6$, see 2.14, 2.15 below.



7-vertex torus and 6-vertex real projective plane



5-vertex Möbius band