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EXTENSIONS AND APPLICATIONS OF THE DE GIORGI-NASH RESULTS

BY

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1. **Introduction.** The results of De Giorgi [3] and Nash [14] which are referred to in the title of this lecture are their a priori estimates for the Hölder continuity of the solutions of equations of the form

$$\frac{\partial}{\partial x^\alpha}(a^{\alpha\beta}u_{,\beta}) = 0 \tag{1.1}$$

on some domain G. Here $u_{,\beta}$ means $\partial u/\partial x^\beta$, $x = (x^1, \cdots, x^\nu)$, and repeated Greek indices are summed from 1 to ν, as will always be done in this paper, and the $a^{\alpha\beta}(x)$ are supposed to satisfy

$$m|\lambda|^2 \leqq a^{\alpha\beta}(x)\lambda_\alpha\lambda_\beta \leqq M|\lambda|^2 \quad (|\lambda|^2 = \sum_{\alpha=1}^{\nu}\lambda_\alpha^2), \qquad 0 < m \leqq M \tag{1.2}$$

for all x on G and all λ. They showed by completely unrelated methods that any solution of (1.1) satisfies a uniform Hölder condition on any compact set $F \subset G$ which depends only on F, G, m, M, and bounds for $|u|$ on G in the case of Nash and the L_2 norm of u on G in the case of De Giorgi. Since the bounds and Hölder conditions on u do not depend on any continuity properties of the $a^{\alpha\beta}$, the results carry over to cases where the $a^{\alpha\beta}$ are merely bounded and measurable, in which case the equations (1.1) must be written in the integrated form

$$\int_G \zeta_{,\alpha}a^{\alpha\beta}u_{,\beta}dx = 0, \qquad \zeta \in C_c^1(G) \tag{1.3}$$

where $C_c^1(G)$ denotes all functions of class C^1 on G with compact support and the function u belongs to the space $H_2^1(D)$ for each bounded D with $\bar{D} \subset G$; here $\zeta_{,\alpha}, \zeta_{,\alpha\beta}$, etc., denote $\partial\zeta/\partial x^\alpha$, $\partial^2\zeta/\partial x^\alpha\partial x^\beta$, etc.; if $\zeta \in H_p^1(G)$, $\zeta_{,\alpha}$ denotes its strong derivative.

Since the notation $H_p^1(D)$ does not seem to be standard, we identify these spaces with those denoted by $W_p^1(D)$ by Browder and the Russians and $\mathscr{P}_p(D)$ by the writer (see [2; 8]); in the case $p = 2$, functions $\in H_2^1(D)$ if and only if they are strongly differentiable in the sense of Friedrichs [4].

It is important to have such estimates in order to discuss nonlinear equations and it is especially important to be able to have the conclusion for the equation (1.3) when the $a^{\alpha\beta}$ are merely bounded and measurable in order to be able to conclude the differentiability of the solutions of minimum problems. For instance, De Giorgi was able to show, using his results, that any function minimizing an integral

$$I(z, G) = \int_G f(x, z, \nabla z)dx \qquad (\nabla z = (z_{,1}, \cdots, z_{,\nu}) \tag{1.4}$$

among all functions z in $H_2^1(G)$ having the same boundary values in which f is of class C_μ^n (nth derivatives Hölder continuous with exponent μ, $0 < \mu < 1$) $n \geqq 2$, is also of class C_μ^n provided that f satisfies the following conditions: There exist numbers, m, M, and K such that

$$\begin{gathered} f(x, z, p) \equiv f(p), \qquad p = (p_1, \cdots, p_\nu) \\ m|p|^2 - K \leqq f(p) \leqq M|p|^2 + K, \\ m|\lambda|^2 \leqq f_{p_\alpha p_\beta}(p)\lambda_\alpha\lambda_\beta \leqq M|\lambda|^2 \end{gathered} \tag{1.5}$$

for all p and λ. The existence of such a minimizing function was proved in 1943 by the writer (see [9] where much more general existence theorems were proved).

The method of proof is as follows: First of all, it is straightforward to show that if z minimizes $I(z, G)$, then

$$\int_G (\zeta_{,\alpha} f_{p_\alpha} + \zeta f_z)dx = 0, \qquad \zeta \in C_c^1(G), \tag{1.6}$$

provided that f satisfies (1.5) or any of the conditions below; in De Giorgi's case, of course, $f_z = 0$. Next select $D \subset \subset G$ (i.e., $\bar{D}$ compact and $\bar{D} \subset G$) and choose D' with $D \subset \subset D' \subset \subset G$. Since each $f_{p_\alpha} \in L_2(G)$, (1.6) holds for all $\zeta \in H_{20}^1(G)$ and, of course, $z \in H_2^1(G)$ if $I(z, G)$ is finite, on account of (1.5). So let $\zeta \in H_{20}^1(D')$, extend it to be zero in $G - D'$, and for small h, define

$$\zeta_h(x) = h^{-1}[\zeta(x - he_\gamma) - \zeta(x)], \quad z_h(x) = h^{-1}[z(x + he_\gamma) - z(x)] \tag{1.7}$$

where e_γ denotes the unit vector in the x^γ direction. If this ζ_h is substituted for ζ in (1.6) and if the obvious change of variables is made to get rid of terms involving $\zeta(x - he_\gamma)$, and if one uses the integral form of the theorem of the mean, one obtains

$$\int_{D'} \zeta_{,\alpha} a_h^{\alpha\beta}(x) z_{h,\beta} dx = 0, \qquad \zeta \in H_{20}^1(D') \tag{1.8}$$

where the $a_h^{\alpha\beta}$ satisfy (1.2) uniformly in h, $a_h^{\alpha\beta}$ being given by

$$a_h^{\alpha\beta}(x) = \int_0^1 f_{p_\alpha p_\beta}[(1 - t)\nabla z(x) + t\nabla z(x + he_\gamma)]dt \qquad \text{(a.e.)}. \tag{1.9}$$

Now, equation (1.8) is just (1.3) where all we know is that the $a_h^{\alpha\beta}$ are measurable and satisfy (1.2). Moreover, from theorems on the Lebesgue integral, it follows that the L_2 norms of the z_h on D' are bounded and, in fact, that $\|z_h - z_{,\gamma}\|_{D'} \to 0$. But from the De Giorgi result, we conclude that the z_h are equi-Hölder continuous on $\bar{D}$ and hence tend uniformly to their limits $z_{,\gamma}$ on $\bar{D}$ which are therefore Hölder continuous. Once this is known, the higher differentiability results follow from previously known results (see [**11**], for instance).

The corresponding difference-quotient device was used in 1912 by Lichtenstein [**6**] to show that any C^2 solution of a minimum problem (1.4) in which $\nu = 2$ and f analytic was of class C^3 on interior domains and hence analytic by Bernstein's theorem [**1**]. In 1929, E. Hopf [**5**] was able to obtain the same conclusion by assuming only that $z \in C_\mu^1$ for some μ, $0 < \mu < 1$. In 1938, the writer [**7**] obtained the same conclusion in case z is merely Lipschitz; $\nu = 2$ in these two latter cases. All of these results assume that something is known a priori about z, but only in special cases had it been shown previously that solutions z existed which had these properties. In 1943, the writer employed the spaces, now denoted by H_p^1, to extend the rather meager previous existence theory. These results applied to cases where ν was arbitrary and z could be a vector function; some of these results were extended to more general integrals in [**10**]. For $\nu = 2$, the writer was able to show that these solutions, known to exist, were also differentiable, provided f satisfied the conditions (1.12) below with $\nu = 2$ and $k = 1$, but z was allowed to be a vector function. In 1950, A. G. Sigalov [**15**] proved corresponding results for integrals where f satisfies (1.11) below with any $k > 1/2$ but ν still $= 2$.

All of these results involved consideration of equations of the form

$$\int_G [\zeta_{,\alpha}(a^{\alpha\beta}u_{,\beta} + b^\alpha u + e^\alpha) + \zeta(c^\alpha u_{,\alpha} + du + f)]dx = 0, \quad \zeta \in C_c^1(G) \tag{1.10}$$

with rough coefficients, but the methods used were peculiar to the case $\nu = 2$. The De Giorgi-Nash results, then, represented an important breakthrough in this field. In 1959, the writer [**12**] was able to extend the De Giorgi-Nash results to certain equations of the form (1.10). These results did not lead immediately to further differentiability theorems for minimum problems but have been a useful tool in the recent results on differentiability obtained during this year by the writer and his student E. R. Buley which are the principal concern of this paper.

Last fall (1959), Buley had obtained a priori bounds for the solutions z of minimum problems in which f satisfies conditions (1.11) below with $1/2 < k \leqq 1$. In January (1960), J. Moser kindly communicated his simplification of the proofs of the De Giorgi-Nash results [**13**]. This enabled Buley to extend his a priori results to problems in which f satisfies either (1.11) or (1.11′) below for any $k > 1/2$. With the aid of several lemmas proved by the writer, Buley could then show that any solution z of such a problem is in fact differentiable provided $k \geqq 1$; he was unable to carry through the difference-quotient procedure when $1/2 < k < 1$. The conditions on f required by Buley are the following (f is assumed of class C'' in all cases):

$$\text{(1.11)}\quad \begin{aligned} &mV^k - K \leqq f(x, z, p) \leqq MV^k, && 0 < m \leqq M,\\ &\sum [f_{p_\alpha}^2 + f_{p_\alpha x^\gamma}^2 + f_z^2 + f_{zx^\gamma}^2] \leqq M_1 V^{2k-1}, && V = 1 + z^2 + |p|^2,\\ &\sum [f_{p_\alpha z}^2 + f_{zz}^2] \leqq M_1 V^{2k-2} && (\text{all } (x, z, p)),\\ &mV^{k-1}|\lambda|^2 \leqq f_{p_\alpha p_\beta}(x, z, p)\lambda_\alpha\lambda_\beta \leqq MV^{k-1}|\lambda|^2 && (\text{all } \lambda); \end{aligned}$$

or the alternative conditions

(1.11′) same as (1.11) except $f = f(x, p)$, $V = 1 + |p|^2$.

The results of Buley and their proofs are sketched in §2.

The writer was able to extend (essentially) the results of Buley to the cases where $1/2 < k < 1$ by considering a sequence of auxiliary problems in which is considered a function z_K which minimizes $I(z, G)$ among all z in the appropriate space for which another integral $J(z, G) \leqq K$. This method is considered in some generality and applied to these cases in §3. In §4, the writer applies that method and some more refined estimates to extend Buley's differentiability results to cases where f satisfies the conditions

$$\text{(1.12)}\quad \begin{aligned} &mV^k - K \leqq f(x, z, p) \leqq MV^k, && 0 < m \leqq M,\\ &\sum (f_z^2 + f_{zx^\gamma}^2 + f_{zz}^2) \leqq M_1 V^{2k}, && V = 1 + |p|^2,\\ &\sum (f_{p_\alpha}^2 + f_{p_\alpha z}^2 + f_{p_\alpha x^\gamma})^2 \leqq M_1 V^{2k-1} && (\text{all } (x, z, p, \lambda)),\\ &mV^{k-1}|\lambda|^2 \leqq f_{p_\alpha p_\beta}\lambda_\alpha\lambda_\beta \leqq MV^{k-1}|\lambda|^2, && k \geqq \nu/2. \end{aligned}$$

For example, if the $a^{\alpha\beta}(x, z)$ satisfy (1.2) for all (x, y, λ) and

$$f(x, z, p) = a^{\alpha\beta}(x, z)p_\alpha p_\beta,$$

then f satisfies (1.12) with $k = 1$ but not either (1.11) or (1.11′). But $f = V^k$ satisfies either set of conditions with the appropriate definition of V.

We use the following notations and make the following additional conventions: All integrals are Lebesgue integrals. All domains are bounded and if G is a domain, ∂G denotes its boundary. A domain G is of class C^1 if and only if each point P of ∂G is in a neighborhood $\mathcal{N}$ which is the image under a 1–1 map of class C^1 of a sphere $B(x_0, R)$ (center x_0, radius R) in which P corresponds to x_0 and $\mathcal{N} \cap \partial G$ corresponds to the part of $B(x_0, R)$ where $x^\nu = x_0^\nu$; usually we take $x_0 = 0$. The classes $C^m(G)$, $C^m(\bar{G})$, Lip (G) have their usual significance and $\text{Lip}_c(G)$, for instance, denotes those Lipschitz functions with support (closed) in G. The space $H_{p0}^1(G)$ is the closure with respect to the norm in $H^1_p(G)$ of the space $C^1_c(G)$. If φ is a vector, $\nabla\varphi$ denotes its gradient and $|\varphi|$ its Euclidean length. If G is a domain G_ρ denotes the set of points x such that $B(x, \rho) \subset G$. We use the notation $\rightharpoonup$ for weak convergence. If S is a set $|S|$ denotes its measure. We shall denote any constant C which depends only on the bounds m, M, K, M_1 k, and ν by C; it is not assumed that such constants are all the same. These results will be presented in more detail in the Proceedings of the International Conference on Partial Differential Equations and Continuum Mechanics held in Madison, Wisconsin, in June 1960.

2. **The results of Buley.** We shall treat only the case (1.11) and shall assume $\nu > 2$; the modifications necessary to handle the cases (1.11′) and/or $\nu = 2$ will be clear. On account of (1.11), it is clearly sufficient to restrict ourselves to functions $z \in H^1_{2k}(G)$. So we suppose that $z^* \in H^1_{2k}(G)$ and let z be a function in $H^1_{2k}(G)$ such that $z - z^* \in H^1_{2k,0}(G)$ and z minimizes I among all such functions. We indicate how to prove that z is differentiable on domains $D \subset \subset G$.

FIRST. (1.6) *holds for all* $\zeta \in H^1_{2k,0}(G)$. This is easily seen by approximating in $H^1_{2k}(G)$ by $\zeta \in C^1_c(G)$ and noting from (1.11) that the f_{p_α} and $f_z \in L_r(G)$ for $r = 2k/(2k - 1)$.

Next, we apply the difference-quotient procedure described in the introduction to obtain the equation

$$(2.1)\quad \int_{D'} A_h[\zeta_{,\alpha}(a^{\alpha\beta}_h z_{h,\beta} + b^\alpha_h z_h + P^q_h e^{\alpha\gamma}_h) + \zeta(b^\alpha_h z_{h,\alpha} + c_h z_h + P^q_h f^\gamma_h)]dx = 0,$$
$$\zeta \in H^1_{2k,0}(D'),\quad q = 1/2,$$

where A_h and the other coefficients are given by (a.e.)

$$A_h(x) = \int_0^1 \{1 + [z(x) + t\Delta z]^2 + |p(x) + t\Delta p|^2\}dt,$$
$$\Delta z = z(x + he_\gamma) - z(x), \text{ etc.},$$

$$(2.2)\quad \begin{aligned} A_h a^{\alpha\beta}_h &= \int_0^1 f_{p_\alpha p_\beta}[x + the_\gamma, z(x) + t\Delta z, p(x) + t\Delta p]dt, \qquad p(x) = \nabla z(x), \\ A_h b^\alpha_h &= \int_0^1 f_{p_\alpha z}[\text{same}]dt, \quad A_h P^q_h e^{\alpha\gamma}_h = \int_0^1 f_{p_\alpha x_\gamma}[\text{same}]dt, \end{aligned}$$

$$P_h(x) = \max_{0 \leq t \leq 1} 1 + [z(x) + t\Delta z]^2 + |p(x) + t\Delta p|^2$$

and the other coefficients are defined correspondingly. We note that the coefficients $a^{\alpha\beta}_h$, b^α_h, c_h, $e^{\alpha\gamma}_h$, and f^γ_h are uniformly bounded and

$$A_h(x) \equiv 1 \text{ if } k = 1,\ z_h \to p_\gamma \text{ in } L_{2k}(D'),$$

$$A_h \to V^{k-1} \text{ in } L_{k/(k-1)}(D') \quad (k > 1),\ P_h \to V \text{ in } L_k(D'), \tag{2.3}$$
$$A_h^q a_h^{\alpha\beta} \to V^{(k-1)/2} a^{\alpha\beta} = V^{-(k-1)/2} f_{p_\alpha p_\beta} \text{ in } L_{2k/(k-1)}(D') \qquad (k > 1),$$

with corresponding convergence for the other coefficients. If $h \neq 0$ and small, we may set

$$\zeta = \eta^2 z_h, \qquad \eta \in \mathrm{Lip}_c(D') \tag{2.4}$$

in (2.1). If this is done, we obtain the result that

$$\int_{D'} \eta^2 A_h |\nabla z_h|^2 dx \leqq C \int_{D'} (\eta^2 + |\nabla \eta|^2) A_h (z_h^2 + P_h) dx, \tag{2.5}$$
$$C = C(m, M, K, M_1, k)$$

by using the Schwarz inequality, the device $2\,ab \leqq \epsilon a^2 + \epsilon^{-1} b^2$, etc. Since for each $\eta \in \mathrm{Lip}_c\,(D')$ the right side of (2.5) is uniformly bounded for all small h, and since there is such an $\eta \equiv 1$ on D, and since $A_h(x) \geqq 1$, we see that $z_h \rightharpoonup p_\gamma$ in $H_2^1(D)$, $A_h^q a_h^{\alpha\beta} z_{h,\beta} \rightharpoonup$ something which must be $V^{(k-1)/2} a^{\alpha\beta} p_{\gamma,\beta}$ in $L_2(D)$, and for any $\zeta \in H_{2k,0}^1(D)$, we see that $\zeta_{,\alpha} A_h^q \to V^{(k-1)/2} \zeta_{,\alpha}$ in $L_2(D)$. Thus with the aid of Lemma 2.1 below, we conclude the following:

SECOND. *The functions p_γ, $U = V^{k/2}$, and $V^{(k-1)/2} p_\gamma \in H_2^1(D)$ and satisfy the equations*

$$\int_D V^{k-1} [\zeta_{,\alpha} (a^{\alpha\beta} p_{\gamma,\beta} + b^\alpha p_\gamma + V^q e^{\alpha\gamma}) + \zeta (b^\alpha p_{\gamma,\alpha} + c p_\gamma + V^q f \gamma)] dx = 0. \tag{2.6}$$

Moreover

$$\int_D |\nabla U|^2 dx \leqq k^2 \int_D V^{k-1} (|\nabla p|^2 + |p|^2) dx < \infty. \tag{2.7}$$

LEMMA 2.1. *Suppose F is of class C^1 for all $(u^1, \cdots, u^P)$, suppose each $u^p \in H_\lambda^1(G)$ for some $\lambda \geqq 1$, suppose $U = F(u^1, \cdots, u^P)$, suppose U and the $V_\alpha \in L_\mu(G)$ for some $\mu \geqq 1$, where*

$$V_\alpha(x) = \sum_{p=1}^{P} F_{,p}[u(x)] u^p_{,\alpha}(x), \qquad \alpha = 1, \cdots, \nu. \tag{2.8}$$

Then $U \in H_\mu^1(G)$ and $U_{,\alpha}(x) = V_\alpha(x)$ (a.e.). The same conclusion holds if F is convex if, in (2.8), we replace the $F_{,p}$ by the coefficients of any supporting plane to F at any point x where F does not have a unique tangent plane at $[u(x), \cdots, u^P(x)]$

This is proved by choosing representatives $\tilde{u}^p$ of u^p which are absolutely continuous along almost all lines parallel to each coordinates axis (see [**2**]) and noting that $\tilde{U} = F(\tilde{u}^1, \cdots, \tilde{u}^P)$ has the same property.

Next we show:

THIRD. *Suppose the function $U = V^{k/2} \in L_{2\tau}(D')$ for some $D' \subset\subset G$ and some $\tau \geqq 1$. Then $w = U^\tau \in H_2^1(D)$ for each $D \subset\subset D'$ and*

$$(2.9) \qquad \int_D |\nabla w|^2 dx \leqq C\tau^2 a^{-2} \int_{D'} w^2 dx \text{ if } D \subset D'_a, \qquad a > 0,$$

where $C = C(m, M, K, M_1, k, \nu)$.

If it were possible to substitute

$$(2.10) \qquad \zeta = \eta^2 U^{2\tau-2} p_\gamma, \qquad \eta \in \mathrm{Lip}_c(D'),$$

in equations (2.6), the Schwarz inequality and (2.7) would yield

$$(2.11) \qquad \int_{D'} \eta^2 U^{2\tau-2} |\nabla U|^2 dx \leqq C_3 \int_{D'} (\eta^2 + |\nabla\eta|^2) U^{2\tau} dx,$$

where C_3 is independent of τ. But this implies (2.9). Unfortunately, the ζ's in (2.10) are not known to $\in H^1_{2k,0}(D')$. So, for each L, we define U_L as the "sawed-off" function: $U_L(x) = U(x)$ if $U(x) < L$ and $U_L(x) = L$ if $U(x) \geqq L$; then we define ζ by (2.10) with U replaced by U_L. These ζ are still not known to $\in H^1_{2k,0}(D')$. However, these ζ are, for each L, in $H^1_{20}(D')$ and also $\psi\zeta \in H^1_{20}(D')$ with $\psi\nabla\zeta \in L_2(D')$, where we define

$$(2.12) \qquad \psi = V^{(k-1)/2} \in H^1_2(D').$$

To see that such ζ can be substituted in (2.6), we define

$$(2.13) \qquad A^\alpha = f_{p_\alpha}, \quad B = f_z.$$

It follows easily from our second step that A^α and $B \in H^1_1(D')$, that

$$(2.14) \qquad \begin{aligned} A^\alpha_{,\gamma} &= V^{k-1}(a^{\alpha\beta} p_{\gamma,\beta} + b^\alpha p_\gamma + V^q e^{\alpha\gamma}), \\ B_{,\gamma} &= V^{k-1}(b^\alpha p_{\gamma,\alpha} + c p_\gamma + V^q f^\gamma), \end{aligned}$$

and hence that $\psi^{-1}A^\alpha \in H^1_2(D')$ and $\psi^{-1}\nabla A^\alpha \in L_2(D)$. Then, using (2.14) and a series of lemmas proved by the writer (to appear in the Proceedings of the Madison Conference of June 1960), we conclude that if ζ has compact support in D' with $\zeta \in H^1_2(D')$, $\psi\zeta \in H^1_2(D')$, and $\psi\nabla\zeta \in L_2(D')$, then we may substitute ζ in (2.6) to obtain

$$\begin{aligned} \int_{D'} (\zeta_{,\alpha} A^\alpha_{,\gamma} + \zeta B_{,\gamma}) dx &= \int_{D'} (\zeta_{,\alpha} A^\alpha_{,\gamma} - \zeta_{,\gamma} B) dx \\ &= \int_{D'} (\zeta_{,\alpha} A^\alpha_{,\gamma} - \zeta_{,\gamma} A^\alpha_{,\alpha}) dx = 0, \qquad A^\alpha_{,\alpha} = B, \end{aligned}$$

since (1.6) holds. Making these substitutions leads to

$$(2.15) \qquad \int_{D'} \eta^2 U_L^{2\tau-2} [|\nabla U|^2 + (\tau - 1)|\nabla U_L|^2] dx \leqq C_2 \int_{D'} [\tau\eta^2 + |\nabla\eta|^2] U_L^{2\tau-2} U^2 dx.$$

Since the right side of (2.15) is bounded for all L, we may let $L \to \infty$ to obtain (2.11) (in deriving (2.15), it is convenient to notice that $\nabla U_L = 0$ almost everywhere on the set where $U_L(x) = L$).

FOURTH. *If f is of class C^t_μ with $t \geqq 2$ and $0 < \mu < 1$ (C^∞, analytic), then the solution z is of class C^t_μ(C^∞, analytic) on each domain $D \subset \subset G$.*

In order to prove this, it is sufficient to show that U is bounded on interior domains $D \subset \subset G$. For then the p_γ satisfy (2.6) and the coefficients $V^{k-1}a^{\alpha\beta}$, etc., are all bounded. Then it follows from the writer's extension [**12**] of the De Giorgi-Nash results that the p_γ are Hölder-continuous on such domains. The higher differentiability follows from known results as mentioned in the introduction.

In order to show that U is bounded, we modify Moser's procedure slightly. We suppose that $B_0 = B(x_0, 2R) \subset G$, $B_n = B(x_0, R_n)$ where $R_n = R(1 + 2^{-n})$, and define

$$w_n = U^{s^n} \text{ so that } w_n = w^s_{n-1},\ s = \nu/(\nu - 2).$$

Then, for each n, we apply the Sobolev lemma (see [**12**])

$$\left\{\int_{B_n} w^{2s}_{n-1}dx\right\}^{1/2} \leqq C_0 \int_{B_n} [|\nabla w_{n-1}|^2 + R_n^{-2}w^2_{n-1}]dx, \qquad C_0 = C_0(\nu), \tag{2.16}$$

with the result (2.9) with $D = B_n$, $D' = B_{n-1}$, $a = 2^{-n}R$, which yields

$$\int_{B_n} [|\nabla w_{n-1}|^2 + R_n^{-2}w^2_{n-1}]dx \leqq 2C_1 s^{2n-2} \cdot 4^n R^{-2} \int_{B_{n-1}} w^2_{n-1}dx, \tag{2.17}$$

C_1 being the C of (2.9); note that $R_n \geqq R$. If we let

$$W_n = \int_{B_n} w^2_n dx,$$

(2.16) and (2.17) lead to the recurrence relation

$$W_n \leqq K_0^s K_1^{ns} W^s_{n-1}, \quad K_0 = 2C_1 s^{-2}R^{-2}, \quad K_1 = 4s^2. \tag{2.18}$$

From (2.18), we conclude that U is summable to any power on $B(x_0, R)$ and that

$$|U(x)|^2 \leqq \lim_{n\to\infty} W_n^{1/s^n} = K_0^\alpha K_1^\beta \int_{B_0} U^2 dx, \qquad x \in B(x_0, R),$$

$$\alpha = (1 - s^{-1})^{-1} = \nu/2, \quad \beta = \nu^2/4.$$

3. Extension of the results of Buley to the case $1/2 < k < 1$. We first state an obvious theorem, which will aid in the interpretation of the results of this section, and a convenient definition.

DEFINITION. If f and z are such that f_{p_α} and f_z are summable over each $D \subset \subset G$ and if z satisfies (1.6), we say that z is an *extremal* for the integral $I(z, G)$.

THEOREM 3.1. *If f satisfies the conditions* (1.11′) *for some* $k > 1/2$ *or if f satisfies* (1.11) *and is convex in* (p, z), *and* $z^* \in H^1_{2k}(G)$, *there is a unique extremal z for the integral* $I(z, G)$ *such that* $z \in H^1_{2k}(G)$ *and* $z - z^* \in H^1_{2k,0}(G)$.

For then $I(z, G)$ is a convex functional.

It is clear that (1.6) holds for all $\zeta \in H^1_{2k,0}(G)$. And if we apply the difference quotient procedure, we arrive again at (2.5); but this time, it is not immediately evident that the right side of (2.5) is bounded for all small h, although the result of replacing A_h by its limit V^{k-1}, z_h by p_γ and P_h by V is bounded. The trouble is that $A_h \leqq 1$ and z_h is not uniformly in $L_2(D')$. So we consider a sequence of problems of the type described in the introduction where the finiteness of the second integral guarantees that we may let $h \to 0$ in the difference-quotient procedure. We then study what happens as the second integral is allowed to be arbitrarily large.

We begin with some general remarks about such problems. The second integral will be denoted by $J(z, G)$, where

$$J(z, G) = \int_G F(x, z, \nabla z)dx \tag{3.1}$$

where we shall assume for simplicity that F satisfies (1.11) with k replaced by m.

THEOREM 3.2. *Suppose f satisfies* (1.11) *or* (1.12) *with some* k ($\geqq \nu/2$ *if* (1.12)). *Let m' denote the larger of k and m and suppose that* $z^* \in H^1_{2m'}(G)$ *and that* $J(z^*, G) \leqq L$. *Then there is a function* $z_L \in H^1_{2m'}(G)$ *with* $z_L - z^* \in H^1_{2m',0}(G)$ *which minimizes* $I(z, G)$ *among all such z for which* $J(z, G) \leqq L$. *If z_L is not an extremal for J, there is a unique number* $\mu \geqq 0$ *such that z_L is an extremal for the integral* $I(z, G) + \mu J(z, G)$; *so*

$$\int_G [\zeta_{,\alpha}(f_{p_\alpha} + \mu F_{p_\alpha}) + \zeta(f_z + \mu F_z)]dx = 0, \qquad \zeta \in H^1_{2m'0}(G). \tag{3.2}$$

PROOF. The first statement is obvious from the lower semi-continuity of both integrals (see [9])with respect to weak convergence. If $J(z_L, G) < L$ and $\zeta \in \mathrm{Lip}_c(G)$, it is easily seen that $J(z_L + \lambda\zeta, G) < L$ for all sufficiently small λ; in this case, (3.2) holds with $\mu = 0$. If $J(z_L, G) = L$ and z_L is not an extremal, there is a ζ_1 such that

$$\int_G (\zeta_{1,\alpha}F_{p_\alpha} + \zeta_1 F_z)dx = 1, \qquad \zeta_1 \in \mathrm{Lip}_c(G).$$

It follows by fairly straightforward arguments that

$$\int_G (\zeta_{,\alpha}f_{p_\alpha} + \zeta f_z)dx = 0 \quad \text{whenever} \quad \int_G (\zeta_{,\alpha}F_{p_\alpha} + \zeta F_z)dx = 0, \qquad \zeta \in \mathrm{Lip}_c(G),$$

so that a number μ exists. Since $I(z, G) \geqq I(z_L, G)$ whenever $J(z, G) \leqq L$, it follows easily that $\mu \geqq 0$. It is clear that μ is unique if z_L is not an extremal for J.

THEOREM 3.3. *Assume the hypotheses and notation of Theorem 3.2 and also that $m > k$, G is of class C^1 and J has no extremal with $z - z^* \in H^1_{2m,0}(G)$ for which $J(z, G) > K_0$. Then, if $z_K \to z_0$ as $K \to \infty$ through a sequence of values, z_0 is a minimizing function for $I(z, G)$ with $z_0 - z^* \in H^1_{2k}(G)$. There is a sequence of $K \to \infty$ such that $K\mu(K) \to 0$.*

PROOF. Suppose z_0^* minimizes $I(z, G)$ among all $z \in H^1_{2k}(G)$ such that $z - z^* \in H^1_{2k,0}(G)$. Then, from our hypotheses, it follows that z_0^* is the strong limit in $H^1_{2k}(G)$ of functions in $H^1_{2m}(G)$ and I is continuous with respect to strong convergence. The first statement follows easily. To prove the second, we define

$$\varphi(K) = I(z_K, G).$$

Then, clearly, φ is nonincreasing. Next if $K > K_0$, we have

$$\varphi(K + \Delta K) \leqq I(z_K + \lambda\zeta_1, G) \quad \text{where} \quad \Delta K = J(z_K + \lambda\zeta_1, G) - J(z_K, G)$$

for all λ near 0. Since $\Delta K/\lambda \to 1$ as $\lambda \to 0$, we see that $\varphi'(K) = -\mu(K)$ a.e. Hence $\mu(K)$ is summable for $K \geqq K_1 > K_0$ and the result follows.

We now apply these results to extend Buley's results as indicated:

FIRST. *We suppose that f satisfies* (1.11) *with $1/2 < k < 1$, we define $F = V/2$, $V = 1 + z^2 + |p|^2$, we assume $z^* \in H^1_2(G)$ and is the unique* (*Theorem* 3.1) *extremal for J with those boundary values, $K_0 = J(z^*, G)$, and G is of class C'. Then, for each K, the functions $p_{K\gamma}$, $U_K = V_K^{k/2}$, $\psi_K = V_K^{(1-k)/2}$, and $\psi_K p_{K\gamma} \in H^1_2(D)$ for each $D \subset\subset G$ with*

$$(3.3) \qquad \int_D (\mu_K + V_K^{k-1})|\nabla p_K|^2 dx \leqq 2Ca^{-2}K\mu(K) + Ca^{-2}\int_{D'} V_K^k dx,$$
$$D \subset D'_a, \quad D' \subset\subset G.$$

To prove this, we apply the difference quotient procedure to equation (3.2) to obtain

$$(3.4) \quad \int_{D'} \{\zeta_{,\alpha}[\mu z_{h,\alpha} + A_h(a_h^{\alpha\beta} z_{h,\beta} + b_h^\alpha z_h + e_h^\alpha P_h^\gamma)] + \zeta A_h(b_h^\alpha z_{h,\alpha} + c_h z_h + f_h^\gamma)\}dx = 0,$$
$$\zeta \in H^1_{20}(D'), \ \mu = \mu(K), \ z_h = z_{Kh}, \text{ etc.}$$

and the coefficients are given by their formulas in (2.2) with z replaced by z_K. This time, the A_h are bounded and

$$A_h a_h^{\alpha\beta} \to V^{k-1}a^{\alpha\beta}, \quad A_h b_h^\alpha \to V^{k-1}b^\alpha, \quad A_h c_h \to V^{k-1}c \qquad \text{(a.e.)},$$

$$A_h e_h^{\alpha\gamma} P_h^q = \int_0^1 f_{p_\alpha x_\gamma}[x + the_\gamma, \cdots]dt \to V^{k-q} e^{\alpha\gamma} \quad \text{in } L_r(D'), \tag{3.5}$$

$$A_h f_h^\gamma P_h^q \to V^{k-q} f^\gamma \quad \text{in } L_r(D'), \qquad r = 2k/(2k-1) > 2,\ q = 1/2.$$

Setting $\zeta = \eta^2 z_h$ and proceeding as before leads to

$$\int_D (\mu + A_h)|\nabla z_h|^2 dx \leqq Ca^{-2} \int_{D'} (\mu + A_h)(z_h^2 + P_h)dx. \tag{3.6}$$

For each fixed K, the right side is bounded and tends to a limit and so we may conclude as before that the $p_{K\gamma} \in H_2^1(D)$ and $z_h \rightharpoonup p_{K\gamma}$ in $H_2^1(D)$ and we may let $h \to 0$ in (3.6) and sum on γ to obtain (3.3) and the other conclusions, remembering the definition of J.

SECOND. *For a subsequence of $K \to \infty$, $K\mu(K) \to 0$ and $z_K \to z_0$ in $H_{2K}^1(G)$, z_0 being a minimizing function for I with $z_0 - z^* \in H_{2k,0}^1(G)$, and on each domain $D \subset\subset G$, $\psi_K \rightharpoonup \psi_0$, $U_k \rightharpoonup U_0$, $\psi_K p_{K\gamma} \rightharpoonup \psi_0 p_{0\gamma}$ in $H_2^1(D)$, $p_{K\gamma} \rightharpoonup p_{0\gamma}$ in $H_{2k}^1(D)$, and (3.3) holds in the limit.*

The first statements follow from Theorems 3.2 and 3.3. Then, since $K\mu(K) \to 0$ and $V_K \to V_0$ in $L_{2k}(G)$, we may let $K \to \infty$ on the right in (3.3). From (3.3), we conclude that the $H_2^1(D)$ norms of ψ_K, U_K, and $\psi_K p_{K\gamma}$ are uniformly bounded. Also

$$\int_D |\nabla p_K|^{2k} dx = \int_D V_K^h \cdot V_K^{-h} |\nabla p_K|^{2k} dx$$
$$\leqq \left(\int_D V_K^k dx\right)^{1-k} \cdot \left(\int_D V_K^{k-1} |\nabla p_K|^2 dx\right)^k \qquad (h = k(1-k)).$$

Accordingly the results follow.

THIRD. *Suppose $z^* \in H_{2k}^1(G)$ and f satisfies the hypotheses* (1.11) *with $1/2 < k < 1$, G being any bounded domain. Then there is a minimizing function for $I(z, G)$ with $z - z^* \in H_{2k,0}^1(G)$ which has the differentiability properties stated in* §2.

To prove this, we let $\{G_n\}$ be an expanding sequence of domains of class C' having union G and let z_0 be a minimizing function for $I(z, G)$ with $z_0 - z^* \in H_{2k,0}^1(G)$. On each G_n, we approximate strongly in $H_{2k}^1(G_n)$ by functions $z'_{np} \in C^1(\bar{G}_n)$, and for each n and p, we let $z_{np} = \lim z_{npK}$ as in the second part. Each z_{np} is minimizing for $I(z, G_n)$ with $z_{np} - z'_{np} \in H_{2k,0}^1(G_n)$ and satisfies the interior boundedness conditions. Thus a subsequence of $z_{np} \rightharpoonup z_n$ in $H_{2k}^1(G_n)$ where $z_n - z_0 \in H_{2k,0}^1(G_n)$ and z_n is minimizing. If, for each n, we let $Z_n = z_n$ on G_n and z_0 on $G - G_n$, then $Z_n - z^* \in H_{2k,0}^1(G)$ and each Z_n is minimizing. Thus a subsequence $\rightharpoonup z$ in $H_{2k}^1(G)$, $z - z^* \in H_{2k,0}^1(G)$ and z is minimizing and the limiting bound (3.3) holds for z. The remainder of the development in §2 now goes through except that this time $\psi = V^{(1-k)/2}$, ψA^α, ψB, A^α, B and $\psi^{-1}\zeta \in H_2^1(D)$, so that the former ζ's can be substituted in (2.6) as before.

4. **Extension to the integrands f satisfying** (1.12) **with** $k > \nu/2$. The case $k = \nu/2$ can be treated by first showing that a minimizing function in this case satisfies a "Dirichlet growth" condition

$$\varphi(r) \leqq [\varphi(a) + Ka^{\nu}](r/a)^{\mu}, \qquad \mu > 0,\ 0 \leqq r \leqq a,$$
$$\varphi(r) = \left[\int_{B(x_0,r)} V^{\nu/2}dx\right]^{2/\nu}, \qquad B(x_0, r) \subset G.$$

This is omitted here but will appear in the Madison Proceedings.

If one attempts to carry through the procedure of §2, one finds that the equations (2.1) and (2.6) must be altered by replacing b_h^{α}, c_h, b^{α}, and c by $P_h^q b_h^{\alpha}$, $P_h c_h$, $V^q b^{\alpha}$, and Vc, respectively, if the b_h^{α}, c_h, b^{α}, and c are to be bounded. The argument in the proof of the second part would require that V^{k+1} be summable. In order to carry through the difference quotient procedure, we must use the device of the preceding section and in order to handle the limiting equations, we need the following lemma:

LEMMA 4.1. *Suppose $w \in L_2(B_b)$ $(B_r = B(x_0, r))$, $w \in H_2^1(B_r)$ for $0 < r < b$, $H \in L_{\nu}(B_b)$ and satisfies*

$$\left(\int_{B_r} H^{\nu}dx\right)^{2/\nu} \leqq C_1 r^{\mu}, \quad 0 \leqq r \leqq b,\ \mu > 0,\ H(x) \geqq 0. \tag{4.1}$$

Suppose w satisfies the condition

$$\int_{B_r} |\nabla w|^2 dx \leqq C_2\tau^2 \int_{B_{r+a}} H^2w^2dx + C_3\tau^2a^{-2}\int_{B_{r+a}} w^2dx \tag{4.2}$$
$$0 < a \leqq r, \quad r + a \leqq b, \quad \tau > 1.$$

It is assumed that these conditions hold on any spheres $B(x_0, b)$ and $B(x_0, r + a) \subset G$. Then there is a constant C_4, depending only on μ, ν, C_1, C_2, C_3 and an upper bound for a, such that

$$\int_{B_r} |\nabla w|^2dx \leqq C_4 r^{\lambda} a^{-2}\int_{B_{r+a}} w^2dx, \quad 0 < a \leqq r, \quad r + a \leqq b,$$
$$\lambda = 2 + 4\mu^{-1}, \quad B(x_0, r + a) \subset G.$$

PROOF. Let us assume first that $\bar{B}(x_0, R) \subset G$ so that $w \in H_2^1[B(x_0, R)]$ and there is a constant C_5 such that

$$\int_{B_r} |\nabla w|^2dx \leqq C_5(R - r)^{-2}\int_{B_r} w^2dx, \qquad 0 \leqq r < R.$$

From the Sobolev lemma used in (2.16), we conclude that

$$\begin{aligned}\left\{\int_{B_r} w^{2s}dx\right\}^{1/s} &\leqq C_0(\nu)\int_{B_r}(|\nabla w|^2 + r^{-2}w^2)dx \quad (s = \nu/(\nu - 2)) \\ &\leqq C_0(C_5 + 1)(R - r)^{-2}\int_{B_r} w^2dx, \text{ if } R/2 \leqq r < R.\end{aligned} \tag{4.3}$$