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L. Solymar

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BY

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Preface

THIS book is an expanded version of my lectures given in Michaelmas term 1974 to second year undergraduates in the Department of Engineering Science, Oxford University. I aimed at a text which is readable, covers a wide range from electrostatics to relativity in a short space, and can still offer a quantitative mathematical treatment.

A feature of the book, not usually found in textbooks, is the inclusion of a large number of quotations from the pioneers of electromagnetism. I have always felt that our courses give too little historical insight. I made here an attempt to whet some appetites for delving into the past. In particular, by giving quotations from Young to Keller, I followed through the story of edge diffraction which I hope others will find as fascinating as I do. In connection with the translations I wish to acknowledge here the help I received from Professor J. L. Ackrill, Mr. R. S. Lucas, Dr. P. C. H. Wernberg-Moller, and Mr. R. P. Aizlewood in the Latin, German, Danish, and Russian languages respectively.

I am greatly indebted to Drs. J. E. Allen, D. M. S. Bagguley, and A. M. Howatson for many interesting and stimulating discussions on the subject of Chapter 4.

For great help in preparing the manuscript I wish to thank my wife Marianne.

I hope many of the mistakes in the first printing have now been eliminated. I am particularly grateful to Dr. N. E. Christensen of the Technical University of Denmark, Lyngby, for sending me a long list of corrections.

Oxford L. S.

Contents

Introduction

The history of electricity is a field full of pleasing objects, according to all the genuine and universal principles of taste, deduced from a knowledge of human nature.

But though other kinds of history may, in some respects, vie with that of philosophy, nothing that comes under the denomination of history can exhibit instances of so fine a rise and improvement in things as we see in the progress of the human mind in philosophical investigations. To whatever height we have arrived in natural science, our beginnings were very low, and our advances have been exceeding gradual. And to look down from the eminence, and to see, and compare all those gradual advances in the ascent, cannot but give the greatest pleasure to those who are seated on the eminence, and who feel all the advantages of their elevated situation. And considering that we ourselves are, by no means, at the top of human science; that the mountain still ascends beyond our sight, and that we are, in fact, not much above the foot of it, a view of the manner in which the ascent has been made cannot but animate us in our attempts to advance still higher, and suggest methods and expedients to assist us in our further progress.

These histories are evidently much more necessary in an advanced state of science, than in the infancy of it. At present philosophical discoveries are so many, and the accounts of them are so dispersed, that it is not in the power of any man to come at the knowledge of all that has been done, as a foundation for his own inquiries. And this circumstance appears to me to have very much retarded the progress of discoveries.

I likewise think myself peculiarly happy in my subject itself. Few branches of natural philosophy would, I think, make so good a subject for a history. Few can boast such a number of discoveries, disposed in so fine a series, all comprised in so short a space of time, and all so recent, the principal actors in the scene being still living.

I entitle the work the *History and present state of electricity*; and whether there be any new editions of the whole work or not, care will be taken to preserve the propriety of the title, by occasionally printing additions, in the same size, as new discoveries are made; which will always be sold at a reasonable price to the purchasers of the book; or given gratis, if the bulk be inconsiderable.

JOSEPH PRIESTLEY *The history and present state of electricity with original experiments* London 1767

THE subject of electricity can be broadly divided into three big branches: circuits, electromagnetism, and electric properties of materials. You know what circuits are like. They consist of lumped elements interconnected in various ways. The elements themselves are simple; they can be represented by boxes with a number of terminals into which currents may flow and over which voltages may exist. It is not difficult to conceive what each box stands for. Nevertheless when hundreds, thousands, or may be millions of boxes are interconnected it is well beyond the capacity of ordinary mortals to comprehend what is going on. I never know whom to admire more, the academic engineer who designed the whole complex apparatus or the industrial engineer who comes and repairs the thing when something goes wrong. Anyway the point I wish to make is that complexity in circuits is primarily due to the large number of components.

If we wish to understand the electric properties of materials the situation is a lot worse. Then even the simplest elements display a rather complex behaviour. Take for example the hydrogen atom; it consists of one proton and of one electron—you cannot have anything simpler than that. If you want to unravel the properties of the hydrogen atom you need to do pages and pages of mathematics. And if you venture as far as the helium atom you will be told by mathematicians that no exact solution of the differential equation exists. Imagine now 10^{22} atoms heaped upon one another, each one containing 50 electrons, and try to answer the question about the electrical conductivity of the material. It is a miracle that such questions can sometimes be answered.

Electromagnetism is somewhere half-way between circuits and electric properties of materials. Some of the phenomena are fairly simple, so simple in fact that they are taught in secondary schools: you must have studied Coulomb's law, Biot–Savart's law, the law of induction, and a few similar laws concerned with charges, forces, currents, and voltages. You must also have heard about Snell's laws of reflection and refraction, about lenses and mirrors, about diffraction effects, etc. All these phenomena were treated separately, making only very faint attempts to relate them to each other. The reason is that it is not easy to connect them. Electromagnetic phenomena have such a wide range, display such varied properties, that explanation from any single angle is bound to fail. One may accept that a compass will always point towards the North. Every child knows that. One may equally accept that a light ray will refract when entering water. This is nothing other than putting personal experience into scientific jargon. But to say that the two things are just different manifestations of the same phenomenon is stretching one's credulity a little too far. There is no one—certainly no one I know—to whom such a connection would seem obvious. The links that exist are not physical, they are purely mathematical. The similarity between the magnetic field created by an electric current and that present in a light beam is not tangible. The sole reason why we may call both of them a 'magnetic field' is that they obey the same set of equations.

The conclusion is inescapable. We are in for a lot of mathematics. Having done the mathematics the situation will start to improve, traces of physical intuition will gradually emerge. And the more mathematics you do, the more robust your physical intuition will become. To my mind the greatest difference between electromagnetism and electric properties of materials is that the former may be understood. Not without a great deal of study, not without some sweat and toil, but eventually one gets there. After some steadfast climbing one breaks through the clouds and arrives at a pinnacle from where the whole mountain range may be clearly seen. Well, some of the peaks may be under some light fog but that is of little consequence. Even if we do not know the exact shape of one particular mountain and have no reports on the weather conditions we do know that it is only up to us to set forth on an expedition and conquer the peak. Electromagnetism has still some uncharted territory but contains no more surprises. I know most physicists would strongly disagree with this view. I am really looking at the problem with the eye of an engineer. If we do not wish to enquire into such questions as the size of the electron or where the mass of the electron is coming from then we are in the clear.

Now what about materials? Can we study electromagnetism without enquiring into the properties of materials? Are not all engineering constructions made of real materials possessing widely different properties? Yes, of course; I have no intention of restricting the study to phenomena occurring in a vacuum. We have to take into account *some* of the properties of materials but we shall not be too inquisitive. In most cases we shall characterize a material by three macroscopic constants: permittivity, permeability, and conductivity. This is again a view many teachers of electromagnetism would disagree with. One can argue that the microscopic properties are not to be ignored, and whenever one has a chance one should discuss the interaction between matter and electromagnetic waves. I feel that electromagnetism as I define it is difficult enough. First try to understand it in that limited context; you can break down the self-imposed barrier later when you will be concerned with properties of materials. For example, in the study of solids and plasmas it is regularly assumed that charged-particle pairs can disappear by recombination. We shall have none of that here; all our particles will be considered indestructible.

A further question I would like to touch upon is the necessary compromise between depth and breadth, rigour and intuition. Traditionally, this is a subject where rigour is highly valued. For a long time it has been the playing ground of mathematically minded engineers and practically minded mathematicians. The trouble with rigour is that besides being a bore it is also very elusive merchandise. However rigorous you might attempt to be it is always possible to find a man to whom it will appear contemptuously sloppy. It is not my intention to decry the importance of rigour. It can certainly have value in the solution of engineering problems. A 'uniqueness theorem', for example, will assure you that having found one solution there is no point searching for another one. But by and large engineers can survive without rigour, and in my opinion this should be reflected in the teaching of the subject as well. Accordingly, I shall depart somewhat from the line followed by many standard introductory texts and will attempt to cover a wider range of phenomena at the expense of mathematical rigour.

1. Maxwell's equations

One of the chief peculiarities of this treatise is the doctrine which asserts, that the true electric current on which the electromagnetic phenomena depend, is not the same thing as the current of conduction, but that the time variation of the electric displacement must be taken into account in estimating the total movement of electricity.

JAMES CLARK MAXWELL *A treatise on electricity and magnetism* Oxford 1873

Auf die Frage „Was ist die Maxwell'sche Theorie?" wüsste ich also keine kürzere und bestimmtere Antwort als diese: Die Maxwell'sche Theorie ist das System der Maxwell'schen Gleichungen.

HEINRICH HERTZ *Untersuchungen über die Ausbreitung der elektrischen Kraft* Leipzig 1894

ALL the problems we shall be concerned with may be solved by calling upon one or more of the following equations:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{1.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.2}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{1.3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.4}$$

$$\mathbf{D} = \varepsilon \mathbf{E}, \tag{1.5}$$

$$\mathbf{B} = \mu \mathbf{H}, \tag{1.6}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{1.7}$$

Where do the above equations come from? They are contained (though not quite in the same form and not in the same system of units) in Chapter IX of Maxwell's *Treatise on electricity and magnetism* published in 1873.

Are we to conclude that electromagnetic theory has made no advance in the course of a century? That conclusion would essentially be correct. Our technique of solving the above equations has improved, and of course we are in a much better position now to evaluate the material constants, but fundamentally electromagnetic theory stands now as it stood a century ago.

As far as the interrelationship of electromagnetic quantities is concerned Maxwell knew as much as we do today. He did not actually suggest communication between continents with the aid of geostationary satellites, but if he was taken now to a satellite ground-station he would not be numbed with astonishment. If we would give him half an hour to get over the shock of his resurrection he would quietly sit down with a piece of paper (the back of a bigger envelope, I suppose) and would work out the relevant design formulae.

The lack of advance on our part should not be attributed to the idleness of a century, much rather to the genius of Maxwell. The moment he conceived the idea of the displacement current, a new era started in the history of mankind. Events of similar importance did not occur often. Newton's *Principia* and Einstein's first paper on relativity would qualify, and perhaps two or three more learned papers, but that's about all. If we assume that our kind of beings will still be around a few millennia hence, I feel certain that the nineteenth century will mainly be remembered as the century when Maxwell formulated his equations.

What was so extraordinary about Maxwell's contribution? It was the first (and may be the best) example of reaching a synthesis on the basis of experimental evidence, mathematical intuition, and prophetic insight. The term $\partial \mathbf{D}/\partial t$ in eqn (1.1) had no experimental basis at the time. By adding this new term to the known equations he managed to describe *all* macroscopic electromagnetic phenomena. And when relativity came, Newton's equations were found wanting but not Maxwell's; they needed no relativistic correction.

I could go on for a long time in praise of Maxwell. Unfortunately we have little time for digressions however entertaining they might be. But before we get down to the equations I must say a few words in defence of the approach I choose. I know it must be hard for anyone to accept a set of equations without going through the usual routine of presenting the relevant experimental justifications. It might seem a little unreasonable at first sight but believe me this is a possible approach, and under the circumstances it may very well be the best approach. You are already familiar with the mathematical operations curl, div, and grad (I prefer using them in vector-operator form), and you need no introduction to the concepts of electricity. You have heard about electric charge, current, magnetic field, etc. All that eqns (1.1)–(1.7) do is to give a number of relationships between these quantities. So if some of them are known, you can use the equations to work out some of the others.

The notation is fairly standard but still I should better say what is what: **H**, magnetic field strength; **E**, electric field strength; **D**, electric flux density; **B**, magnetic flux density; **J**, current density; ρ, charge density; **F**, force, q, charge; **v**, velocity of moving charge; μ, permeability; ε, permittivity. The two latter quantities are constants depending on the material under study. Using the subscript zero to denote their values in free space, they come to

$$\varepsilon_0 = 8{\cdot}854 \times 10^{-12}\ \mathrm{A\,s\,V^{-1}\,m^{-1}}, \qquad \mu_0 = 4\pi \times 10^{-7}\ \mathrm{V\,s\,A^{-1}\,m^{-1}}. \tag{1.8}$$

In any other medium

$$\varepsilon = \varepsilon_0 \varepsilon_r \quad \text{and} \quad \mu = \mu_0 \mu_r, \tag{1.9}$$

where ε_r is the *relative permittivity* (or dielectric constant) and μ_r is usually referred to as the *relative permeability*. Notice that

$$\varepsilon_0 \mu_0 = 1/c^2, \tag{1.10}$$

where c is the velocity of light in free space.†

The rest of the course will be concerned with the various solutions of eqns (1.1)–(1.7). Isn't this boring for an engineer? Shouldn't this be done by mathematicians or by computer programmers? Not for the time being. Perhaps one day computers will be big enough and numerical analysts clever enough so that the engineer will only have to pose the problem, but not yet, and not, I think, for some time to come.

In the large majority of cases a straightforward mathematical solution is just out of the question. So one has to use that delicate substance known as physical intuition. How can one acquire physical intuition? There is no easy way. One has to start with a simple physical configuration, solve the corresponding mathematical problem, then solve a similar problem and then another problem, and then a little more difficult problem, and so on. The first breakthrough comes when one can predict a solution without actually doing the mathematics.

In order to have a unified view of the subject we have started with Maxwell's equations. It means a new approach but not a radical departure. The subject is still the same. You will be able to see that the laws you love and cherish (Coulomb's, Biot–Savart's, Snell's, etc.) all follow from our eqns (1.1)–(1.7).

The order of discussion will follow the traditional one: electrostatics first, followed by steady currents, then we shall move on to slowly varying phenomena, and reach finally the most interesting part, fast-varying phenomena, exhibiting the full beauty of Maxwell's wonderful equations.

† It is no coincidence that the product of ε_0 and μ_0 is related to c^2. Strictly speaking we should have said $\mu_0 = 4\pi \times 10^{-7}$ and should have defined ε_0 as $1/\mu_0 c^2$, where c is obtained from measurement. But why should μ_0 be $4\pi \times 10^{-7}$ and why should it ha,e the dimensions volt × second/ampere × metre? This is a problem intimately related to the choice of units. Since we are using SI (Système Internationale, known previously as MKS by electrical engineers) units and since I hope all other units will go out of fashion, it does not appear to be worthwhile to waste much time upon discussing their interrelationship.

2. Electrostatics

The English philosophers, and perhaps the greater part of foreigners too, have now generally adopted the theory of *positive* and *negative* electricity. As this theory has been extended to almost all the phenomena, and is the most probable of any that have been hitherto proposed to the world, I shall give a pretty full account of it.

According to this theory, all the operations of electricity depend upon one fluid *sui generis*, extremely subtile and elastic, dispersed through the pores of all bodies; by which the particles of it are as strongly attracted, as they are repelled by one another.

When the equilibrium of this fluid in any body is not disturbed; that is, when there is in any body neither more nor less of it than its natural share, or than that quantity which it is capable of retaining by its own attraction, it does not discover itself to our senses by any effect. The action of the rubber upon an electric disturbs this equilibrium, occasioning a deficiency of the fluid in one place, and a redundancy of it in another.

I shall close the account of my experiments with a small set, in which, as well as in the last, I have little to boast besides the honour of following the instructions of Dr. Franklin. He informed me, that he had found cork balls to be wholly unaffected by the electricity of a metal cup, within which they were held; and he desired me to repeat and ascertain the fact, giving me leave to make it public.

Accordingly, December the 21st. I electrified a tin quart vessel, standing upon a stool of baked wood; and observed, that a pair of pith balls, insulated by being fastened to the end of a stick of glass, and hanging entirely within the cup, so that no part of the threads were above the mouth of it, remained just where they were placed, without being in the least affected by the electricity;

May we not infer from this experiment, that the attraction of electricity is subject to the same laws with that of gravitation, and is therefore according to the squares of the distances; since it is easily demonstrated, that were the earth in the form of a shell, a body in the inside of it would not be attracted to one side more than another?

JOSEPH PRIESTLEY *The history and present state of electricity with original experiments* London 1767

Loi fondamentale de l'Électricité.

La force répulsive de deux petits globes électrisés de la méme nature d'électricité, est en raison inverse du carré de la distance du centre des deux globes.

L'électricité des deux balles diminue un peu pendant le temps que dure l'expérience; j'ai éprouvé que, le jour où j'ai fait les essais qui précèdent, les balles électrisées se trouvant par leur répulsion à 30 degrés de distance l'une de l'autre, sous un angle de torsion de 50 degrés, elles se sont rapprochées d'un degré dans trois minutes; mais comme je n'ai employé que deux minutes à faire les trois essais qui précèdent, l'on peut, dans ces expériences, négliger l'erreur qui résulte de la perte de l'électricité. Si l'on desire une plus grande précision, où lorsque l'air est humide, & que l'électricité se perd rapidement, l'on doit, par une première observation, déterminer la doit ou la diminution de l'action électrique des deux balles dans chaque minute, & se servir ensuite de cette première observation, pour corriger les résultats des expériences que l'on voudra faire ce jour-là.

CHARLES AUGUSTIN COULOMB *Premier mémoire sur l'electricité et le magnetisme, Histoire de l'Académie Royale des Sciences* Paris 1785

Dans un Corps conducteur chargé d'Électricité, le fluide électrique se répand sur la surface du corps, mais ne pénètre pas dans l'interieur du corps.

CHARLES AUGUSTIN COULOMB *Quatrième mémoire sur l'électricité Histoire de l'Academie Royale des Sciences* Paris 1786

Lorsqu'une science déja fort avancée a fait un pas important, il s'établit des liaisons nouvelles entre les branches qui la composent: on aime alors à porter ses regards en arrière pour mesurer la carrière qui a été parcourue, et voir comment l'esprit humain l'a franchie. Si nous remontons ainsi à la naissance de l'électricité, nous la trouvons, au commencement du dernier siècle, réduite aux seuls phénomènes d'attraction et de répulsion; Dufay, le premier, reconnut les règles constantes auxquelles ils sont assujettis, et expliqua leurs bizarreries apparentes. Sa découverte des deux électricités, résineuse et vitrée, fonda les bases de la science; et Franklin, en la présentant sous un nouveau point de vue, en fit le fondement de sa théorie, à laquelle tous les phénomènes, même celui de la bouteille de Leyde, vinrent naturellement se plier. Epinus acheva de prouver cette théorie, la perfectionna en l'assujettissant au calcul, et parvint, à l'aide de l'analyse, jusqu'à ces phénomènes que le citoyen Volta a si heureusement employés dans le condensateur et dans l'électrophore. La loi rigoureuse des attractions et des répulsions électriques manquoit encore, elle fut établie par des expériences exactes; et, se liant à celle du magnétisme, elle se trouva la même que pour les attractions célestes. On sait que le citoyen Coulomb est l'auteur de cette découverte.

Enfin parurent les phénomènes galvaniques, si singuliers dans leur marche, et si différens en apparence de tout ce que l'on connoissoit déja. On créa d'abord, pour les expliquer, un fluide particulier; mais par une suite d'expériences ingénieuses, conduites avec sagacité, le citoyen Volta se propose de les ramener à une seule cause, le développement de l'électricité métallique; les fait servir à la construction d'un appareil qui permet d'augmenter à volonté leur force, et les lie, par ses résultats, avec des phénomènes importans de la chimie et de l'économie animale.

Rapport sur les expériences du citoyen Volta, par le citoyen Biot, Memoires de l'Institut National des Sciences et Arts Tome V Fructidor An XII

2.1. Introduction

STATIC means not varying as a function of time. So all our quantities ρ, **J**, **E**, **B**, **D**, and **H** will be independent of time. Is there such a thing as time-independent charge? Yes, there is. It means that neither the magnitude nor the position of the charge varies as a function of time. And similarly we can imagine constant electric and magnetic fields. Can we talk of time-independent current? Well, we have to permit the motion of charges to get any current but if the amount of charge crossing a certain cross-section is always the same then the current at that point is independent of time. On this basis constant currents also belong to the static branch of electricity. It is, though, usual to distinguish between electrostatics and magnetostatics; in the former case the variables are ρ, **E**, and **D**, whereas in the latter case they are **J**, **H**, and **B**.

We shall now proceed with the equations of electrostatics, which may be obtained from eqns (1.1)–(1.7) by substituting $\partial/\partial t = 0$ and assuming that **v**, **J**, **H**, and **B** are all zero. We get then

$$\nabla \times \mathbf{E} = 0 \tag{2.1}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{2.2}$$

$$\mathbf{D} = \varepsilon \mathbf{E}, \tag{2.3}$$

$$\mathbf{F} = q\mathbf{E}. \tag{2.4}$$

We shall introduce now a scalar function ϕ by the relationship

$$\mathbf{E} = -\nabla\phi. \tag{2.5}$$

The physical significance of this new function may be recognized by determining the work performed by carrying a charge from point a to point b:

$$W = -\int_a^b \mathbf{F} \cdot \mathrm{d}\mathbf{s}, \tag{2.6}$$

where the negative sign is due to the fact that the work is done against the electrical forces. Substituting eqns (2.4) and (2.5) into eqn (2.6) we get

$$W = -q\int_a^b \mathbf{E} \cdot \mathrm{d}\mathbf{s} = q\int_a^b \nabla\phi \cdot \mathrm{d}\mathbf{s} = q\{\phi(b) - \phi(a)\}, \tag{2.7}$$

where $\phi(b)$ and $\phi(a)$ are the values of the function ϕ at the end-points of the path (Fig. 2.1). We have used here a mathematical theorem stating that the line integral of a gradient depends only on the end-points and not on the connecting path. The potential at point b may be written with the aid of eqn (2.7) in the form

$$\phi(b) = \phi(a) - \int_a^b \mathbf{E} \cdot \mathrm{d}\mathbf{s}. \tag{2.8}$$