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Erasmus Landvogt

A Compactification of the Bruhat-Tits Building

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Preface

The aim of this book is the definition and study of the properties of the polyhedral compactification of the Bruhat-Tits building of a reductive group over a local field. For consistency of presentation, I have decided to present comprehensively the construction of the Bruhat-Tits building. I hope that this approach will make this technical work more accessible.

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Introduction

The investigation of locally symmetric spaces can draw on different forms of compactifications. For example, one has the compactification of W.L. Baily and A. Borel (see [BaBo] and [S 2]) and the polyhedral compactification of A. Borel and J.P. Serre (see [BoSe 1] and [S 1]). One can view the Bruhat-Tits building of a reductive group over a local field as the $\wp$-adic analog of a symmetric space (see [Ti 2] 5). Thus a natural question is to find $\wp$-adic compactifications of such buildings which are analogs of various compactifications of symmetric spaces.

A. Borel and J.P. Serre constructed in [BoSe 2] a compactification which differs fundamentally from any other in the classical context. To my knowledge, only P. Gérardin has published results concerning $\wp$-adic versions of polyhedral compactifications (see [Ge]). However, Gérardin considers only the case of a split reductive group G over a $\wp$-adic field K, and he focuses only on the set of special points.

The object of this work is to define compactifications for an arbitrary reductive group G over a local field K. This will be done in two steps:

1. A compactification $\overline{A}$ of an apartment A of the Bruhat-Tits building $X(G)$ will be constructed in complete analogy to the classical case (e.g. see [AMRT]). This part of the construction follows the ideas os Gérardin, though I will focus more closely on the connection of the topology of a corner and combinatorial properties of the Coxeter complex.

2. In the second step, the compactification $\overline{A}$ is used as a "local model" for the compactification of $X(G)$. Starting from the equality $X(G) = G(K) \times A/\sim$, where $\sim$ is a suitable equivalence relation (see [BT 1] §7.4), I will define $\overline{X}(G)$ as $G(K) \times \overline{A}/\sim^*$, where $\sim^*$ is a natural extension of $\sim$. If we equip $G(K)$ with the $\wp$-adic topology $\overline{X}(G)$ will carry the corresponding product-quotient topology. It turns out that the topological space $\overline{X}(G)$ is Hausdorff, compact and contractible, that $\overline{X}(G) = \dot{\bigcup\limits_{P \in \mathfrak{P}}} X(P/R_u(P))$ ($\mathfrak{P}$ denotes the set of K-parabolic subgroups of G and $R_u(P)$ the unipotent radical of P) and that the topology on $X(P/R_u(P))$ induced by $\overline{X}(G)$ coincides with the canonical building-topology.
In this global part this work differs to a high degree from [Ge] as the latter used Chevalley lattices to define $X(G)$, a method which can not be generalized in a suitable way. Furthermore I consider the whole Bruhat-Tits building and not only the subset of special points.

The second purpose of this work is didactic. Many of the properties of $X(G)$ and some of the decomposition theorems for $G(K)$ can only be found in the original works by F. Bruhat and J. Tits, at least in the generality which is needed here. Because of the axiomatic character of these results, I did not think that merely citing their proofs would make the constructions in this paper easier to understand. Therefore I decided to present a nearly complete account of the construction of $X(G)$ and of the necessary theorems when G is a connected reductive K-group and K is complete with respect to a discrete valuation. I have refrained from presenting the axiomatic framework. To reduce the technical difficulties even further, some theorems will only be proved for the case in which the residue class field k of K is finite. This makes it possible to consider only reductive groups, so that one does not need to consider quasi-reductive groups. (In constructing the compactification, the finiteness of k will be assumed anyway.)

Apart from a few exceptions, the reader should be able to follow this text without the aid of the original works [BT 1,2]. Those properties that have not been proved (with the exception of the proof of the commutator-rule for quasi-split groups) are not essential for the understanding of this work.

Even though at some points of the text ad-hoc proofs would have shorted the arguments, I decided to follow on the whole the strategy mapped out in [BT 1,2]. I hope that this approach has made the presentation more clear, and that it will facilitate comparisons with the original texts.

To carry out the approach described above, it is necessary to copy some theorems and proofs word-for-word from [BT 1,2]. At some other points, the ideas of the proofs are the same, but because I have avoided such notions as 'valuated root data' and 'quasi-concave functions' the particular phrases and/or proofs themselves are different. To avoid making reading the quotations more difficult than reading the proofs, quotations are given only at the beginning of each paragraph. At the beginning of each chapter I give a detailed summary of the contents of the chapter.

§0 Definitions and notations

In this paragraph we will give the fundamental definitions and notations used throughout this work. Furthermore, we will present some properties of group schemes and reductive groups.

As usual $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ denote the sets of natural, integral, rational and real numbers. By $\mathbb{N}_{>0}$ we will denote the set of positive natural numbers and by $\mathbb{R}_+$ the set of positive reals.

In this work all rings are assumed to be commutative and to have an identity element 1. Ring homomorphisms should always preserve the identity elements and all modules are assumed to be unitary. All fields are commutative.
If R is a ring, then a ring R' together with a ring homomorphism $R \to R'$ is called *an R-algebra*. If R is an integral domain, then we will denote the quotient field by $\mathrm{Quot}(R)$.

For a ring R and an R-module M, we denote by M^* the dual module. If $S \subseteq M$ is an arbitrary subset, then we will write $\langle S \rangle$ for the submodule of M generated by S. We abbreviate the group of R-module isomorphisms $M \to M$ as $GL(M)$.

Let G be an abstract group, let $M \subseteq G$ and let X be a G-set. Then we let $X^G = \{x \in X : gx = x \text{ for all } g \in G\}$ and let $\langle M \rangle$ be the subgroup of G generated by M. For $A, B \subseteq G$, we denote the commutator subgroup $\langle \{aba^{-1}b^{-1} : a \in A, b \in B\} \rangle$ by (A, B).

If H is a further group and $\varphi : G \to \mathrm{Aut}(H)$ (= group of automorphisms of H) is a group homomorphism, then we will write $H \rtimes_\varphi G$ for the semi-direct product of G and H with respect to φ. If φ is clear in the context, then we will abbreviate this as $H \rtimes G$.

For a field K and a field extension L/K, let $[L : K]$ be the degree of the field extension. If moreover L/K is Galois, then the Galois group will be denoted by $\mathrm{Gal}(L/K)$. For $\gamma \in \mathrm{Gal}(L/K)$ and $x \in L$, we will also write x^γ instead of $\gamma(x)$.

As usual a local field is a field which is complete with respect to a discrete valuation and locally compact. If K is a field which is complete with respect to the discrete valuation $\omega : K^\times \to \mathbb{R}$, then we let $o_K = \{x \in K : \omega(x) \geq 0\}$ and $\mathfrak{m}_K = \{x \in K : \omega(x) > 0\}$. Here we let $K^\times = K \backslash \{0\}$ and $\omega(0) = \infty > 0$. We will denote the residue class field by the same but small Latin letter as the local field. Finally, we will write K^{sh} for the strict Henselization.

By [CaFr] II 7 we know that K is locally compact if and only if k is finite.

In order to make the notations more clear we will denote schemes over a field (in particular varieties) by capital Latin letters and schemes over arbitrary rings by capital Gothic letters. A separated, reduced scheme of finite type over a field is called *a variety*.

Let R be a ring, let $\mathfrak{X}$ be a scheme over $\mathrm{Spec}(R)$ (also: $\mathfrak{X}$ an R-scheme or $\mathfrak{X}_{/R}$) and let R' be an R-algebra. Then let $\mathfrak{X}(R')$ be the set of morphisms $\mathrm{Spec}(R') \to \mathfrak{X}$ over $\mathrm{Spec}(R)$. We will suppress the base ring in the following as well. For $\mathfrak{X} \times_R R' = \mathfrak{X} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$, we also write $\mathfrak{X}_{R'}$. Following the usual notations from the theory of varieties we will denote $\Gamma(\mathfrak{X}_{R'}, \mathcal{O}_{\mathfrak{X}_{R'}})$ by $R'[\mathfrak{X}]$, if $\mathfrak{X}$ is an affine R-scheme and $\mathcal{O}_{\mathfrak{X}_{R'}}$ denotes the structure sheaf of the scheme $\mathfrak{X}_{R'}$.
If $\mathfrak{Y}$ is a further R-scheme and $f : \mathfrak{X} \to \mathfrak{Y}$ a morphism of R-schemes, then we will write f^* for the R-algebra homomorphism $R[\mathfrak{Y}] \to R[\mathfrak{X}]$ induced by f.
If R is a local ring and $\mathfrak{X}$ an arbitrary R-scheme, then the special fibre of $\mathfrak{X}$ will be denoted by $\overline{\mathfrak{X}}$.

0.1. Let R be a complete, discrete valuation ring with residue class field k and let $\mathfrak{X}$ be a smooth R-scheme. Since, in particular, R is Henselian, we get that the canonical map

$$\mathfrak{X}(R) \to \mathfrak{X}(k)(= \overline{\mathfrak{X}}(k))$$

is surjective (see [BLR] 2.3 Prop. 5).

0.2. Let R be an integral domain, $K = \mathrm{Quot}(R)$ and let $\mathfrak{X}$ be a flat, affine R-scheme. Then the map $R[\mathfrak{X}] \to K \otimes_R R[\mathfrak{X}]$ is injective. If $\mathfrak{Y}$ is a further affine R-scheme and if $f, g : \mathfrak{X} \to \mathfrak{Y}$ are two morphisms of R-schemes, which coincide on the generic fibre of $\mathfrak{X}$, then $f = g$.

Proposition 0.3. (Extension principle, see [BT 2] 1.7)

Let R be a complete, discrete valuation ring with separably closed residue class field k and let $\mathfrak{X}, \mathfrak{Y}$ be two affine R-schemes of finite type, where $\mathfrak{X}$ is smooth over R. If $g : \mathfrak{X} \times_R K \to \mathfrak{Y} \times_R K$ is a morphism of K-schemes with $g(\mathfrak{X}(R)) \subseteq \mathfrak{Y}(R)$ where $K = \mathrm{Quot}(R)$, then g can be extended uniquely to a morphism $\tilde{g} : \mathfrak{X} \to \mathfrak{Y}$.

Proof. Since $\mathfrak{X}$ is smooth over R, in particular flat, it follows $\mathfrak{X}(R) \subseteq \mathfrak{X}(K)$.

Now the uniqueness follows immediately from (0.2). In order to prove the existence, it obviously suffices to show that $R[\mathfrak{X}] = \{f \in K[\mathfrak{X}] : f(\mathfrak{X}(R)) \subseteq R\}$. Here the inclusion $R[\mathfrak{X}] \subseteq \{f \in K[\mathfrak{X}] : f(\mathfrak{X}(R)) \subseteq R\}$ is clear.

Thus let $f \in K[\mathfrak{X}]$ with $f(\mathfrak{X}(R)) \subseteq R$ and suppose that $f \notin R[\mathfrak{X}]$. Let $\mathfrak{m}$ be the maximal ideal of R and let π be a uniformizer. Further let $k = R/\mathfrak{m}$ be the residue class field and choose the minimal $n \in \mathbb{N}$ such that $\pi^n f \in R[\mathfrak{X}]$. Obviously, we have $n \geq 1$ and therefore $\pi^n f(\mathfrak{X}(R)) \subseteq \mathfrak{m}$ but $\pi^n f \notin \mathfrak{m}$. Hence the image of $\pi^n f$ in $k[\overline{\mathfrak{X}}]$ does not vanish.
Since $\mathfrak{X}$ is smooth, it follows from (0.1) that $\mathfrak{X}(R) \to \mathfrak{X}(k)$ is surjective. Hence the image of $\pi^n f$ vanishes on $\mathfrak{X}(k)$, which has a dense image in $\overline{\mathfrak{X}}$ by [Bo] AG 13.3 (k is separably closed). Contradiction. □

Supplement:

By (0.3) we obtain the following uniqueness statement:
Let R be a complete discrete valuation ring with separably closed residue class field, let $K = \text{Quot}(R)$, let X be an affine K-scheme and let $M \subseteq X(K)$ be an arbitrary subset. If there exists a smooth, affine R-scheme $\mathfrak{X}$ of finite type with generic fibre X and $\mathfrak{X}(R) = M$, then this is up to a unique isomorphism uniquely determined by this data.

The distribution module of an affine scheme (see [Ja] I.7):

Let R be a ring, let $\mathfrak{X}$ be an affine R-scheme and let $x \in \mathfrak{X}(R)$. If I_x denotes the ideal in $R[\mathfrak{X}]$ defining the closed immersion $x : \text{Spec}(R) \to \mathfrak{X}$, then

$$\text{Dist}(\mathfrak{X}, x) := \{\mu \in R[\mathfrak{X}]^* : \text{ there exists a number } n \in \mathbb{N} \text{ with } \mu(I_x^{n+1}) = 0\}$$

is called *the R-module of distributions on $\mathfrak{X}$ with support in x.*
Obviously, this construction is functorial, i.e. a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of affine R-schemes $\mathfrak{X}, \mathfrak{Y}$ induces an R-module homomorphism

$$\begin{array}{ccc} \text{Dist}(\mathfrak{X}, x) & \to & \text{Dist}(\mathfrak{Y}, f(x)) \\ \mu & \mapsto & \mu \circ f^* \end{array}$$

for all $x \in \mathfrak{X}(R)$ (see [Ja] I.7.2).

Proposition 0.4.

Let R be a discrete valuation ring, $K = \text{Quot}(R)$ and let $\mathfrak{X}$ be an irreducible, smooth affine R-scheme with irreducible generic fibre. Furthermore, let $x \in \mathfrak{X}(R)$. Then $R[\mathfrak{X}] = \{f \in K[\mathfrak{X}] : \mu(f) \in R$ for all $\mu \in \text{Dist}(\mathfrak{X}, x)\}$.

Proof. The proof can be copied word for word from [Ja] I 10.12, since the property of being a group scheme is not needed there. □

The Weil restriction (see [BLR] 7.6):

Let R be a ring and let R' be an R-algebra which is projective and of finite type as an R-module. If $\mathfrak{X}$ is an affine R'-scheme, then the functor

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{affine} \\ R\text{-schemes} \end{array} \right\} & \longrightarrow & Ens \text{ (= category of sets)} \\ \mathfrak{Y} & \longmapsto & \mathfrak{X}(\mathfrak{Y} \times_R R') \end{array}$$

can be represented by an affine R-scheme $\mathcal{R}_R^{R'}(\mathfrak{X})$ (see [BLR] 7.6 Theorem 4). So we obtain the *Weil restriction functor*

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{affine}\\ R'\text{-schemes}\end{array}\right\} & \longrightarrow & \left\{\begin{array}{c}\text{affine}\\ R\text{-schemes}\end{array}\right\} \\ \mathfrak{X} & \longmapsto & \mathcal{R}_R^{R'}(\mathfrak{X}) \end{array}.$$

A simple exercise in universal algebra and [BLR] 7.6 Prop. 5 show that the Weil restriction has the following properties:

0.5. If $\mathfrak{X}$ is an affine R'-scheme, then:

(i) If $\mathfrak{X}$ is of finite type over R', then $\mathcal{R}_R^{R'}(\mathfrak{X})$ is of finite type over R.

(ii) If $\mathfrak{X}$ is smooth over R', then $\mathcal{R}_R^{R'}(\mathfrak{X})$ is smooth over R.

(iii) If S is an arbitrary R-algebra and $S' := S \otimes_R R'$, then

$$\mathcal{R}_S^{S'}(\mathfrak{X} \times_{R'} S') = \mathcal{R}_R^{R'}(\mathfrak{X}) \times_R S$$

Let K be a topological field. We can define on K-varieties next to the Zariski topology a finer (in general strictly finer) topology. We will call this *the K-analytic topology*, following the terminology in the cases $K = \mathbb{R}$ and $K = \mathbb{C}$.

This topology is defined exactly in [We] App. III. Only the most important properties will be recalled here.

0.6. For each K-variety X, there exists a unique topology on $X(K)$ such that the following conditions are valid:

(i) If $X \hookrightarrow \mathbf{A}^N$ (= N-dimensional affine space) is a closed immersion, then the topology on $X(K)$ is induced by the product topology on K^N via $X(K) \hookrightarrow \mathbf{A}^N(K) = K^N$.

(ii) If $(U_i)_{i\in I}$ is a covering of X by open, affine subvarieties, then the inclusion $U_i(K) \to X(K)$ induces the K-analytic topology on $U_i(K)$.

(iii) If $f : X \to Y$ is a morphism of K-varieties, then the induced map $f : X(K) \to Y(K)$ is continuous with respect to the K-analytic topologies.

(iv) The K-analytic topology on $X(K)$ is finer (in general strictly finer) than the Zariski topology.

(v) If Y is a further K-variety, then the canonical bijection $X(K) \times Y(K) \to (X \times_K Y)(K)$ is a homeomorphism, if $X(K) \times Y(K)$ is equipped with the product topology.

In addition, if K is a local field, then we have:

(vi) $X(K)$ is locally compact and

(vii) $X(K)$ is compact, if X is complete.

Let R be an integral domain. Then statements like "$\mathfrak{G}$ is an R-group scheme with generic fibre G ..." should always mean that the group law on G is induced by the group law on $\mathfrak{G}$ for the case that G is a Quot(R)-group scheme.
If $\mathfrak{G}$ and $\mathfrak{H}$ are two R-group schemes, then a homomorphism of R-group schemes $f : \mathfrak{G} \to \mathfrak{H}$ will be called *an R-group homomorphism.*
As usual we let $\mathbb{G}_a = \mathrm{Spec}(\mathbb{Z}[T])$ and $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[T, \frac{1}{T}])$. If R is a ring, then we also write $\mathbb{G}_{a/R}$ and $\mathbb{G}_{m/R}$ for $\mathbb{G}_a \times_{\mathbb{Z}} R$ and $\mathbb{G}_m \times_{\mathbb{Z}} R$, respectively. Furthermore, we let

$$GL_{n/R} = \mathrm{Spec}(R[T_{11}, \dots, T_{nn}, \tfrac{1}{det(T_{ij})}])$$

and

$$SL_{n/R} = \mathrm{Spec}(R[T_{11}, \dots, T_{nn}]/(1 - det(T_{ij}))) \ .$$

In case that the group schemes are defined over an integral domain the following three lemmas intend to make clear, in how far the group law, homomorphisms etc. can be extended from the generic fibre to the whole scheme. Let R be an integral domain and let $K = \mathrm{Quot}(R)$.

Lemma 0.7.

Let $\mathfrak{G}$ and $\mathfrak{H}$ be two flat, affine R-group schemes. If $f : \mathfrak{G} \to \mathfrak{H}$ is a morphism of R-schemes which induces a K-group homomorphism $\mathfrak{G}_K \to \mathfrak{H}_K$, then f is an R-group homomorphism.

Proof. Let $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ and $m' : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H}$ be the group laws and let $e : \mathrm{Spec}(R) \to \mathfrak{G}$ and $e' : \mathrm{Spec}(R) \to \mathfrak{H}$ be the 1-sections. Then the assertion follows by applying (0.2) to the following diagrams:

$$\begin{array}{ccc} \mathfrak{G} \times \mathfrak{G} & \stackrel{f \times f}{\longrightarrow} & \mathfrak{H} \times \mathfrak{H} \\ m \downarrow & & \downarrow m' \\ \mathfrak{G} & \xrightarrow[f]{} & \mathfrak{H} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Spec}(R) & = & \mathrm{Spec}(R) \\ e \downarrow & & \downarrow e' \\ \mathfrak{G} & \xrightarrow[f]{} & \mathfrak{H} \end{array} \ .$$

□

Lemma 0.8.

Let $\mathfrak{G}$ be a flat, affine R-scheme and let $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$, $s : \mathfrak{G} \to \mathfrak{G}$ and $e : \mathrm{Spec}(R) \to \mathfrak{G}$ be morphisms of R-schemes which make $\mathfrak{G}_K$ a K-group scheme. Then $\mathfrak{G}$ is an R-group scheme.

Proof. This follows immediately by applying (0.2) to the diagrams which will be used in the definition of a group scheme (see [DG] II §1 1.1). □

Lemma 0.9.

Let $\mathfrak{G}$ be an affine R-group scheme. If $\mathfrak{H}$ is a flat, closed subscheme of $\mathfrak{G}$ such that $\mathfrak{H}_K$ is a K-subgroup scheme of $\mathfrak{G}_K$, then $\mathfrak{H}$ is an R-subgroup scheme.

Proof. Let $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ be the group law and let $s : \mathfrak{G} \to \mathfrak{G}$ be the "inverse-morphism". By the assumption these induce morphisms $\mathfrak{H}_K \times \mathfrak{H}_K \to \mathfrak{H}_K$ and $\mathfrak{H}_K \to \mathfrak{H}_K$. Since $\mathfrak{H}$ is flat, the canonical map $R[\mathfrak{H}] \to K[\mathfrak{H} \times_R K]$ is injective, hence $\mathfrak{H}_K$ has a dense image in $\mathfrak{H}$. Therefore m and s induce morphisms $\mathfrak{H} \times \mathfrak{H} \to \mathfrak{H}$ and $\mathfrak{H} \to \mathfrak{H}$. □

Proposition 0.10.

Let R be a complete discrete valuation ring with separably closed residue class field k and let $K = \mathrm{Quot}(R)$. Let $\mathfrak{G}$ and $\mathfrak{H}$ be two affine R-group schemes, suppose $\mathfrak{G}$ has connected fibres and is smooth over R, and let $f_K : \mathfrak{G}_K \to \mathfrak{H}_K$ be a K-group homomorphism. If there exists an open neighbourhood $\mathfrak{U}$ of the 1-section in $\mathfrak{G}$ and an extension of f_K to an R-morphism $f : \mathfrak{U} \to \mathfrak{H}$, then we can extend f_K uniquely to an R-group homomorphism $\mathfrak{G} \to \mathfrak{H}$.

Proof. Since $\mathfrak{G}$ is flat, the uniqueness follows from (0.2).
Let $x \in \overline{\mathfrak{G}} \subseteq \mathfrak{G}$. Then it suffices to show that we have $f_K^*(g) \in \mathcal{O}_{\mathfrak{G},x}$ for all $g \in R[\mathfrak{H}]$, since $R[\mathfrak{G}] = \bigcap_{x \in \mathfrak{G}} \mathcal{O}_{\mathfrak{G},x}$.
Since $\overline{\mathfrak{G}}$ is connected, we obtain $\overline{\mathfrak{G}} = \mathfrak{U}(k) \cdot \overline{\mathfrak{U}}$ ([Bo] AG 13.3 and 1.3).
$\mathfrak{G}$ is smooth and R is Henselian, hence the canonical map $\mathfrak{U}(R) \to \mathfrak{U}(k)$ is surjective (see (0.1)), i.e. there exists an element y in $\mathfrak{U}(R)$ with $x \in y \cdot \mathfrak{U}$.
Obviously, the morphism $y\mathfrak{U} \to \mathfrak{H}$, $y \cdot u \mapsto f(y) \cdot f(u)$ extends the K-group homomorphism f_K. Hence $f_K^*(g) \in \mathcal{O}_{\mathfrak{G},x}$ for all $g \in R[\mathfrak{H}]$.
Finally, it follows from (0.7) that f is even an R-group homomorphism. □

For a ring R and an affine R-group scheme $\mathfrak{G}$ with 1-section $e : \mathrm{Spec}(R) \to \mathfrak{G}$, we will abbreviate $\mathrm{Dist}(\mathfrak{G}, e)$ as $\mathrm{Dist}(\mathfrak{G})$. By [Ja] I 7.7 we know that $\mathrm{Dist}(\mathfrak{G})$ is an associative (in general not commutative) R-algebra.

If R' is an R-algebra which is projective and of finite type as an R-module, then an affine R'-group scheme $\mathfrak{G}$ becomes an affine R-group scheme $\mathcal{R}_R^{R'}(\mathfrak{G})$ by applying the Weil restriction.

If R is a ring and $\mathfrak{G}$ is a smooth R-scheme, then the identity component of $\mathfrak{G}$ will be denoted by $\mathfrak{G}^\circ$ (see [SGA 3] Exp. VI_B 3.1 ff). By [SGA 3] Exp. VI_B 3.10 we know that $\mathfrak{G}^\circ$ can be represented by a (smooth) open R-subgroup scheme of $\mathfrak{G}$. This will be denoted by $\mathfrak{G}^\circ$, too.

0.11. Let R be a discrete valuation ring and let $\mathfrak{G}$ be a flat, separated R-group scheme of finite type with affine generic fibre. Then according to [SGA 3] Exp. XVIII App. III Prop. 2.1 (iii) we know that $\mathfrak{G}$ is affine.

0.12. Let R be a discrete valuation ring with residue class field k and let $K = \text{Quot}(R)$. Let $\mathfrak{G}, \mathfrak{H}$ be two affine R-group schemes of finite type and let $f : \mathfrak{G} \to \mathfrak{H}$ be an R-group homomorphism. If $\mathfrak{G}, \mathfrak{H}$ are of the multiplicative type (in the sense of [SGA 3] Exp. IX 1), then by [SGA 3] Exp. IX 2.9 the following assertions are equivalent:

(i) f is a monomorphism;

(ii) the K-group homomorphism $\mathfrak{G}_K \to \mathfrak{H}_K$ induced by f is a monomorphism;

(iii) the k-group homomorphism $\mathfrak{G}_k \to \mathfrak{H}_k$ induced by f is a monomorphism.

Proposition 0.13.

Let R be a complete discrete valuation ring with residue class field k and let $\mathfrak{G}, \mathfrak{H}$ be two affine R-group schemes where $\mathfrak{G}$ is of finite type and $\mathfrak{H}$ is smooth. If $\mathfrak{G}$ is of the multiplicative type, then every k-group homomorphism $\overline{f} : \overline{\mathfrak{G}} \to \overline{\mathfrak{H}}$ can be extended to an R-group homomorphism $f : \mathfrak{G} \to \mathfrak{H}$.

Proof. By [SGA 3] Exp. XI 4.2 the functor

$$S \mapsto \text{Hom}_{S-\text{groups}}(\mathfrak{G}_S, \mathfrak{H}_S)$$

can be represented by a smooth R-group scheme $\mathfrak{Hom}_R(\mathfrak{G}, \mathfrak{H})$. Hence by (0.1) the canonical map $\mathfrak{Hom}_R(\mathfrak{G}, \mathfrak{H})(R) \to \mathfrak{Hom}_R(\mathfrak{G}, \mathfrak{H})(k)$ is surjective and therefore there exists an R-group homomorphism $f : \mathfrak{G} \to \mathfrak{H}$ extending $\overline{f}$. □

0.14. Let R be a ring and let M be a free R-module of finite type. Then the functor

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{affine} \\ R\text{-schemes} \end{array} \right\} & \longrightarrow & Ab \; (= \text{category of abelian groups}) \\ \operatorname{Spec} R' & \longmapsto & R' \otimes_R M \end{array}$$

can be represented by a smooth, affine R-group scheme $\mathcal{M}$ of finite type (see [DG] II.1 2.1).

The underlying scheme will be called *the canonical R-scheme associated with M.*

Root systems and Coxeter complexes (see [Bou 1]):

0.15. Let X be a metric space with metric d. For $x \in X$ and $\varepsilon > 0$, we let $B_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\}$.

If A is a real affine space, then we denote by $\mathrm{Aff}(A)$ the group of affine bijections $A \to A$. For $x, y \in A$, we denote by $]x, y[$ and $[x, y]$ the open and closed segment in A with end-points x and y, respectively. For an arbitrary subset $U \subseteq A$, we denote by $\overline{U}$ the topological closure of U in A and by $\overset{\circ}{\widehat{U}}$ the interior of U.

Let Φ be a root system in a finite-dimensional $\mathbb{R}$-vector space V. A root $a \in \Phi$ is called *divisible*, if $\frac{1}{2}a \in \Phi$.
For an arbitrary subset $\Psi \subseteq \Phi$, we let $\Psi^{red} = \{a \in \Psi : \frac{1}{2}a \notin \Psi\}$. If $a, b \in \Phi$, then we let $(a, b) = \{pa + qb : p, q \in \mathbb{N}_{>0}\} \cap \Phi$.
A subset $\Psi \subseteq \Phi$ is called *closed*, if $(a, b) \subseteq \Psi$ for all $a, b \in \Psi$. If in addition Ψ lies in an open half-space of V, then Ψ is called *positively closed.*

Let $\Psi \subseteq \Phi$ be a positively closed subset. A root $\alpha \in \Psi$ is called *extremal*, if the intersection of $\mathbb{R}_+\alpha$ with any system of generators of the convex cone generated by Ψ is non-empty. An arbitrary total ordering of Ψ^{red} will be called simply *an ordering of* Ψ. An ordering of Ψ is called *good*, if every $\alpha \in \Psi^{red}$ is an extremal root for the set of all roots which are greater than α. (See [BT 1] 1.3.15 for the existence of good orderings.) In order to distinguish these (total) orderings from orderings of Φ with respect to a basis, we will call the latter ones *orders on* Φ.

Now let Φ be a root system in V^*. Then Φ defines a Coxeter complex Σ in V such that its faces are the equivalence classes with respect to the following equivalence relation $\sim$:
For $x, y \in V$, we have $x \sim y$ if and only if for all $a \in \Phi$, the following condition is valid: $a(x)$ and $a(y)$ have the same sign or are both equal to zero.

According to [Bou 1] VI 1.5 there exists a canonical bijection between the set of chambers in Σ and the set of bases of Φ. Let $C \in \Sigma$ be a chamber and let $\Delta(C)$ be the basis of Φ defined by it. Obviously, there exists a bijection

$$\begin{array}{rccl} \Delta : & \{F \in \Sigma : F \subseteq \overline{C}\} & \longrightarrow & \mathfrak{P}(\Delta(C)) = \text{set of all subsets of } \Delta(C)). \\ & F & \longmapsto & \Delta(F) = \{a \in \Delta(C) : a|_F > 0\} \end{array}$$

If $\theta \subseteq \Delta$, then $\Delta^{-1}(\theta)$ will also be denoted by F_θ.

Since we will consider finite and infinite (affine) Coxeter complexes simultaneously, the word "vector" (in words like "vector face", "vector chamber" etc.) will indicate in paragraphs in which both types arise that we consider the finite Coxeter complex.

Algebraic and reductive groups (see [Bo], [BoTi] and [Hu]):

Let K be a field and let G be a K-group. As usual we denote by

$X^*(G)$	the group of characters;
$X_*(G)$	the group of 1-parameter subgroups;
$X^*_K(G)$	the group of K-rational characters;
$\mathcal{D}(G)$	the derived group;
$\mathcal{C}(G)$	the connected centre;
$R(G)$	the radical;
$R_u(G)$	the unipotent radical.

If $H \subseteq G$ is a closed subgroup, then we will denote by $\mathcal{N}_G(H)$ and $\mathcal{Z}_G(H)$ the normalizer and the centralizer of H in G, respectively. For closed subgroups $H_1, \ldots, H_n \subseteq G$, we denote by $\langle H_1, \ldots, H_n \rangle$ the closed subgroup of G generated by $H_1, \ldots, H_n$ in G. If $A, B \subseteq G$ are closed subgroups, then we will abbreviate the commutator of A and B as (A, B).

0.16. A reductive K-group is called *split over* K (or K*-split*), if there is a maximal torus which is defined over K and splits over K (see [Bo] 18.7).
A reductive K-group is called *quasi-split over* K (or K*-quasi-split*), if there is a Borel subgroup defined over K. In this case the centralizer of a maximal K-split torus is K-Levi subgroup of a Borel subgroup and therefore equals a maximal torus (see [Bo] 20.5 and 20.6 (iii)).

Now let G be a reductive K-group, let S be a maximal K-split torus in G and let $\Phi = \Phi(G, S, K)$ be the root system of G with respect to S (see [Bo] V 21.1, there denoted by ${}_K\Phi(G)$). If $\mathfrak{g}$ is the Lie algebra of G, Ad : $G \to GL(\mathfrak{g})$ the adjoint representation and $a \in \Phi$, then we let

$$\mathfrak{g}_a = \{X \in \mathfrak{g} : (\mathrm{Ad}\ s)(X) = a(s)X \text{ for all } s \in S\}.$$

0.17. By use of [Bo] 21.9 and 14.5 (∗) we obtain the following characterization of the root groups:

(i) If $a \in \Phi$, then there exists a unique closed, connected, unipotent K-subgroup U_a of G which is normalized by $\mathcal{Z}_G(S)$ and has Lie algebra $\mathfrak{g}_a + \mathfrak{g}_{2a}$ (if $2a \notin \Phi$, then we let $\mathfrak{g}_{2a} = 0$).

(ii) If $\Psi \subseteq \Phi$ is positively closed, then there exists a unique closed, connected, unipotent K-subgroup U_Ψ of G which is normalized by $\mathcal{Z}_G(S)$ and has Lie algebra $\sum_{a \in \Psi} \mathfrak{g}_a$.

(iii) If $\Psi \subseteq \Phi$ is positively closed, then the product morphism

$$\prod_{a\in\Psi^{red}} U_a \to U_\Psi$$

is an isomorphism of K-varieties for each ordering of Ψ.

(iv) Let $a, b \in \Phi$ and suppose that a and b are linear independent. Then (a, b) is positively closed and we have

$$(U_a, U_b) \subseteq U_{(a,b)}.$$

Here $U_\emptyset = \{1\}$ (see (ii)).

If $\Psi \subseteq \Phi$ is positively closed, then we let $G_\Psi = \langle U_\Psi, U_{-\Psi}, \mathcal{Z}_G(S)\rangle$. For an order on Φ, the groups U_{Φ^+} and U_{Φ^-} will also be denoted by U^+ and U^-, respectively. If Σ denotes the Coxeter complex in $X_*(S)\otimes_{\mathbb{Z}}\mathbb{R}$ defined by Φ and if $C \in \Sigma$ is a chamber, then C defines an order on Φ. The groups U_{Φ^+} and U_{Φ^-} will also be denoted by U_C^+ and U_C^-, respectively.

0.18. According to [BoTi] 5.15 we have the following decompositon of $G(K)$:

Let $C, C' \in \Sigma$ be two chambers. Then:

(i) $G(K) = U_C^+(K)\mathcal{N}_G(S)(K)U_{C'}^+(K)$.

(ii) For $n, n' \in \mathcal{N}_G(S)$, we have $n = n'$ if and only if the double cosets $U_C^+ n U_{C'}^+$ and $U_C^+ n' U_{C'}^+$ are equal.

(iii) $U_C^+ U_C^- \cap \mathcal{N}_G(S) = \{1\}$.

Lemma 0.19.

Let $a \in \Phi$ and $u \in U_a(K)$. Then $\mathcal{N}_G(S)(K) \cap U_{-a}(K)uU_{-a}(K) =: \{m(u)\}$ consits of one element. For $u \neq 1$, the element $m(u)$ induces the reflection r_a in $X_*(S)$ and in $X^*(S)$ (see [BoTi] §5).

Proof. There exists at most one $n \in \mathcal{N}_G(S)(K)$ with $u \in U_{-a}(K)nU_{-a}(K)$ by (0.18). On the other hand, by applying (0.18) to the subgroup of G generated by U_{-a}, U_a and $\mathcal{Z}_G(S)$ the existence is clear, and the supplement also holds. □

0.20. Let $\Delta \subseteq \Phi$ be a basis of Φ and let $\theta \subseteq \Delta$. Then $\langle\theta\rangle$ denotes the set of all roots in Φ which are linear combinations of roots in θ. If we let $S_\theta = (\bigcap_{a\in\theta} \ker a)^\circ$, then $\mathcal{Z}_G(S_\theta)$ normalizes the group $U_{\Phi^+\setminus\langle\theta\rangle}$, and $P_\theta := \mathcal{Z}_G(S_\theta)U_{\Phi^+\setminus\langle\theta\rangle}$ is a parabolic K-subgroup of G (see [Bo] 21.11). P_θ is called the *standard parabolic K-subgroup of type θ*.