Universality in Chaos

a reprint selection
compiled and introduced by

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The classification of the constituents of a chaos, nothing less is here essayed.

Herman Melville
Moby Dick chapter 32

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Preface

This reprint selection presents some of the recent developments in the study of the chaotic behavior of deterministic systems. The problem, posed in its most general form, is old and appears under many guises: Why are clouds the way they are? Is the solar system stable? What determines the structure of turbulence in liquids, the noise in electronic circuits, the stability of plasma in a tokomak? The subject, defined so broadly, could not possibly be covered in a single reprint selection. This selection concentrates on the universal aspects of chaotic motions: those qualitative and quantitative predictions which apply to large classes of (often very different) physical systems. The selection can be divided into roughly four parts. The first part offers a general introduction to deterministic chaos and universality. The second part presents some of the experimental evidence for universality in transitions to turbulence. The third part concentrates on the theoretical investigations of the universality ideas, and the last part gives a glimpse of the further developments stimulated by the success of the one-dimensional universality theory.

This selection originates from a NORDITA reprint selection prepared together with Mogens Høgh Jensen in the fall of 1981. I am grateful to Mogens and to the NORDITA staff, in particular Nils Robert Nilsson, for their help with this project. I thank Harry L Swinney, J Doyne Farmer, David Ruelle, Albert Libchaber, Yves Pomeau, Robert H G Helleman, David Rand, Robert MacKay and Stellan Ostlund for their suggestions and criticisms. And last, but not least, I thank Mitchell J Feigenbaum for teaching me almost all that I know about universality in chaos, and all that I know about Schubert.

P. Cvitanović
NORDITA
August 1983

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Introduction

Universality in Chaos

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The often repeated statement, that given the initial conditions we know what a deterministic system will do far into the future, is false. Poincaré (1892) knew it was false, and we know it is false, in the following sense: given infinitesimally different starting points, we often end up with wildly different outcomes. Even with the simplest conceivable equations of motion, almost any non-linear system will exhibit chaotic behaviour. A familiar example is turbulence.

Turbulence is <u>the</u> unsolved problem of classical physics. However, recent developments have greatly increased our understanding of turbulence, and given us new concepts and modes of thought that we hope will have far reaching repercussions in many different fields (solid state physics, hydrodynamics, plasma physics, chemistry, quantum optics, biology, meteorology, acoustics, mechanical engineering, elementary particle physics, mathematics, fishery[2], astrophysics, cosmology, electrical engineering and so on).

The developments that we shall describe here are one of those rare demonstrations of the unity of physics. The key discovery was made by a physicist not trained to work on problems of turbulence. In the fall of 1975 Mitchell Feigenbaum, an elementary particle theorist, discovered a universality in one-dimensional iterations. At the time the physical implications of the discovery were rather unclear. During the next few years, however, numerical and theoretical studies established this universality in a number of models in various dimensions. Finally, in 1980, the universality theory passed its first test in an actual turbulence experiment.

The discovery was that large classes of non-linear systems exhibit transitions to chaos which are <u>universal</u> and <u>quantitatively</u> measurable. This advance can be compared to past advances in the theory of solid state phase transitions; for the first time we can predict and measure "critical exponents" for turbulence. But the breakthrough consists not so much in discovering a new set of scaling numbers, as in developing a new way to do physics. Traditionally we use regular motions (harmonic oscillators, plane waves, free particles, etc.) as zeroth-order approximations to physical systems, and account for weak non-linearities perturbatively. We think of a dynamical system as a smooth system whose evolution we can follow by integrating

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1. Lectures given at the XXII-nd Cracow School of Theoretical Physics, Zakopane, June 1982. Sections 1 to 4 were written in collaboration with Mogens Høgh Jensen (Cvitanović and Høgh Jensen 1982). Published in <u>Acta Physica Polonica</u>, vol <u>A65</u> (April 1984).

2. Of special interest to our Icelandic colleagues.

a set of differential equations. The universality theory seems to tell us that the zeroth-order approximations to strongly non-linear systems should be quite different. They show an amazingly rich structure which is not at all apparent in their formulation in terms of differential equations. However, these systems do show self-similar structures which can be encoded by universality equations of a type which we will describe here. To put it more succinctly, junk your old equations and look for guidance in clouds' repeating patterns.

In these lectures we shall reverse the chronology, describing first an actual turbulence experiment, then a numerical experiment, and finally explain the observations using the universality theory. We will try to be intuitive and concentrate on a few key ideas, referring you to the literature for more detailed expositions[3]. Even though we illustrate it by turbulence, the universality theory is by no means restricted to the problems of fluid dynamics. The key concepts of phase-space trajectories, Poincaré maps, bifurcations, and local universality are common to all non-linear dynamical systems. The essence of this subject is incommunicable in print; intuition is developed by computing. We urge the reader to carry through a few simple numerical experiments on a desktop computer, because that is probably the only way to start perceiving order in chaos.

1. Onset of turbulence

We start by describing schematically the experiment of Libchaber and Maurer (1980) (a nice description has been given by Libchaber and Maurer(1981)[4]). In this type of experiment a liquid contained in a small box is heated from the bottom. The salient points are:

1. There is a controllable parameter, the Rayleigh number, which is proportional to the temperature difference between the bottom and the top of the cell. (Rayleigh number describes the stability of a convective flow (see Velarde and Normand 1980).)

2. The system is dissipative. Whenever the Rayleigh number is increased, one waits for the transients to die out.

For small temperature gradients there is a heat flow across the cell, but the liquid is static. At a critical temperature a convective flow sets in. The hot liquid rises in the middle, the cool liquid flows down at the sides, and two convective rolls appear:

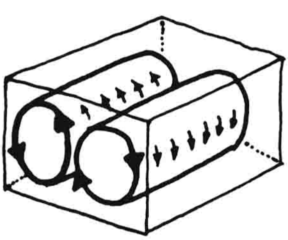

Fig.1.1

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3. The most thorough exposition available is the Collet and Eckmann (1980a) monograph. We also recommend Hu (1982), Crutchfield, Farmer and Huberman (1982), Eckmann (1981) and Ott (1981).

4. See p.109 this selection.

As the temperature difference is increased further, the rolls become unstable in a very specific way - a wave starts running along the roll:

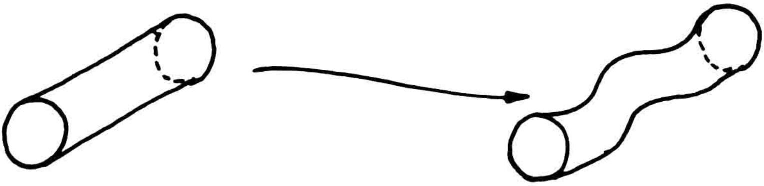

Fig.1.2

As the warm liquid is rising on one side of the roll, while cool liquid is descending down the other side, the position and the sideways velocity of the ridge can be measured with a thermometer:

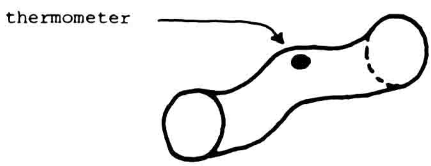

Fig.1.3

One observes a sinusoid:

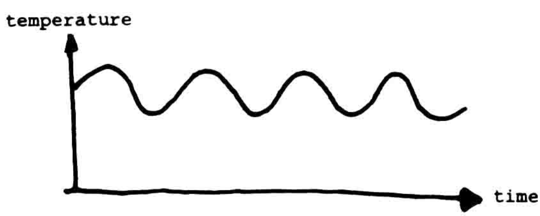

Fig.1.4

The periodicity of this instability suggests two other ways of displaying the measurement:

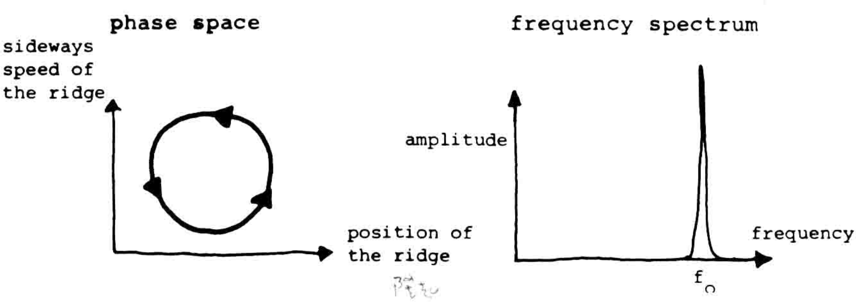

Fig.1.5

Now the temperature difference is increased further. After the stabilisation of the phase-space trajectory, a new wave is observed superimposed on the original sinusoidal instability. The three ways of looking at it (real time, phase space, frequency spectrum) are:

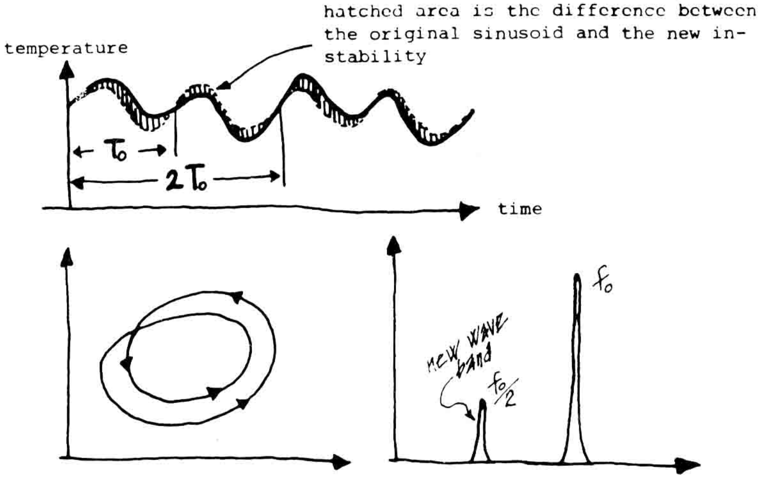

Fig.1.6

A coarse measurement would make us believe that T_o is the periodicity; however, a closer look reveals that the phase-space trajectory misses the starting point at T_o, and closes on itself only after $2T_o$. If we look at the frequency spectrum, a new wave band has appeared at half the original frequency. Its amplitude is small, because the phase-space trajectory is still approximately a circle with periodicity T_o.

Now, as one increases the temperature very slightly, a fascinating thing happens - the phase-space trajectory undergoes a very fine splitting:

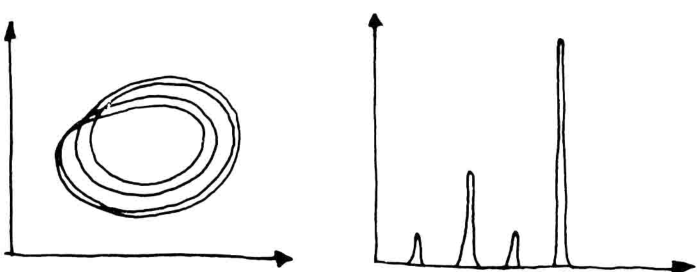

Fig.1.7

We see that there are three scales involved here. Looking casually, we see a circle with period T_o; looking a little closer, we see a pretzel with period $2T_o$; and looking very closely, we see that the trajectory closes on itself only after $4T_o$. The same information can be read off the frequency spectrum; the dominant frequency is f_o (the circle), then $f_o/2$ (the pretzel), and finally, much weaker $f_o/4$ and $3f_o/4$.

The experiment now becomes very difficult. A minute increase in the temperature gradient causes the phase-space trajectory to split on an even finer scale, with the periodicity 2^3T_o. If the noise were not killing us, we would expect these splittings to continue, yielding a trajectory with finer and finer detail, and a frequency spectrum

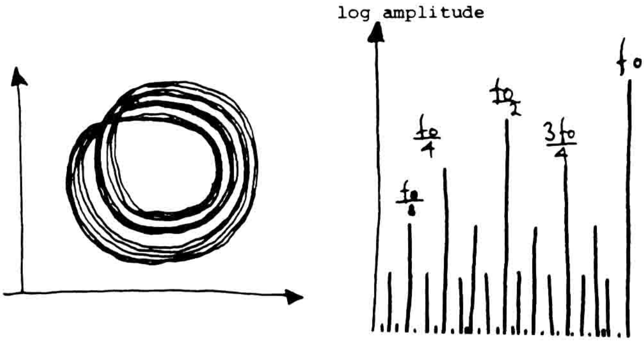

Fig.1.8

with families of ever weaker frequency components. For a critical value of the Rayleigh number, the periodicity of the system is $2^\infty T_O$, and the convective rolls have become turbulent (this is weak turbulence - the rolls persist, wiggling irregularly). The ripples which are running along them show no periodicity, and the spectrum of idealized, noise-free experiment contains infinitely many subharmonics:

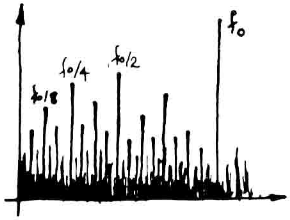

Fig.1.9

If one increases the temperature gradient beyond this critical value, there are further surprises: we refer you to Libchaber and Maurer (1981). We now turn to a numerical simulation of a simple non-linear oscillator in order to start understanding why the phase-space trajectory splits in this peculiar fashion.

2. Onset of chaos in a numerical experiment

In the experiment that we have just described, limited experimental resolution makes it impossible to observe more than a few bifurcations. Much longer sequences can be measured in numerical experiments; the non-linear oscillator studied by Arecchi and Lisi (1982) is a typical example:

$$\ddot{x} + k\dot{x} - x + 4x^3 = A\cos(\omega t) \qquad (2.1)$$

The oscillator is driven by an external force of frequency ω, with amplitude A and the natural time unit $T_O = 2\pi/\omega$. The dissipation is controlled by the friction coefficient k. Given the initial displacement and velocity one can easily follow numerically (by the Runge-Kutta method, for example) the phase-space trajectory of the system. Due to the dissipation it does not matter where one starts in the phase space; for a wide range of initial points the phase-space trajectory converges to a limit cycle (trajectory loops onto itself) which for some $k = k_0$ looks something like this (fig. 12a in Feigenbaum 1980a)[5]:

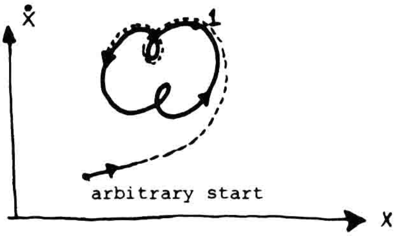

Fig.2.1

If it were not for the external driving force, the oscillator would have simply come to a stop; as it is, it is executing a motion forced on it externally, independent of the initial displacement and velocity. You can easily visualise this non-linear pendulum executing little backward jerks as it swings back and forth. Starting at the point marked 1, the pendulum returns to it after the unit period T_0.

However, as one decreases the friction, the same phenomenon is observed[6] as in the turbulence experiment; the limit cycle undergoes a series of period-doublings

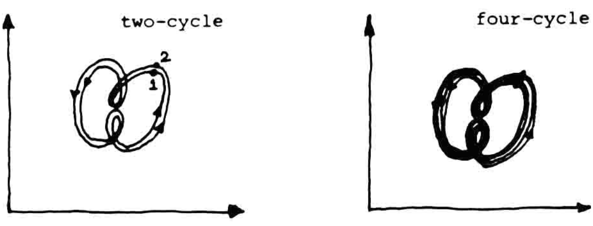

Fig.2.2

The trajectory keeps on nearly missing the starting point, until it hits it after exactly $2^n T_0$. The phase-space trajectory is getting increasingly hard to draw; however, the sequence of points 1, 2, ..., 2^n, which corresponds to the state of the oscillator at times T_0, $2T_0$, ..., $2^n T_0$, sits in a small region of the phase space, so we enlarge it for a closer look:

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5. See p. 49 this selection.

6. If you have a desktop computer with graphics, you can easily do this experiment yourself. For example, if you take $k = 0.154$, $\omega = 1.2199778$ and $A = 0.1, 0.11, 0.114, 0.11437, \ldots$, you will observe bifurcations. There is nothing special about these parameter values; we give them just to help you with finding your first bifurcation sequence.

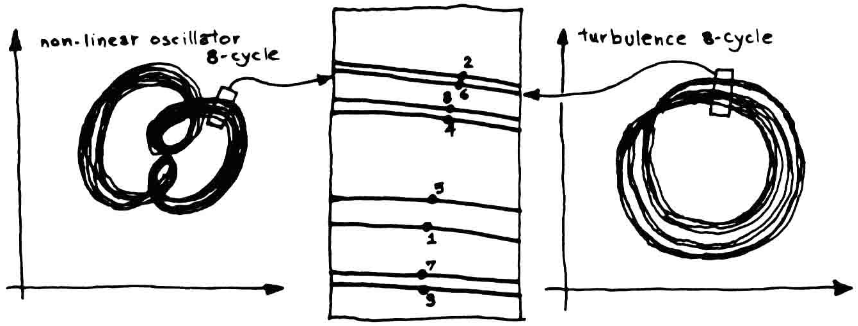

Fig.2.3

Globally the phase-space trajectories of the turbulence experiment and of the non-linear oscillator numerical experiment look very different. However, the above sequence of near misses is local, and looks roughly the same for both systems. Furthermore, this sequence of points lies approximately on a straight line

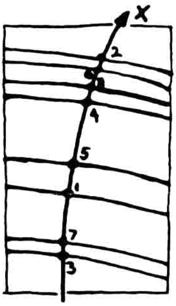

Fig.2.4

Let us concentrate on this line: this way of reducing the dimensionality of the phase space is often called a Poincaré map. Instead of staring at the entire phase-space trajectory, one looks at its points of intersection with a given surface. The Poincaré map contains all the information we need; from it we can read off when an instability occurs, and how large it is. One varies continuously the non-linearity parameter (friction, Rayleigh number, etc.) and plots the location of the intersection points; in the present case, the Poincaré surface is a line, and the result is a bifurcation tree:

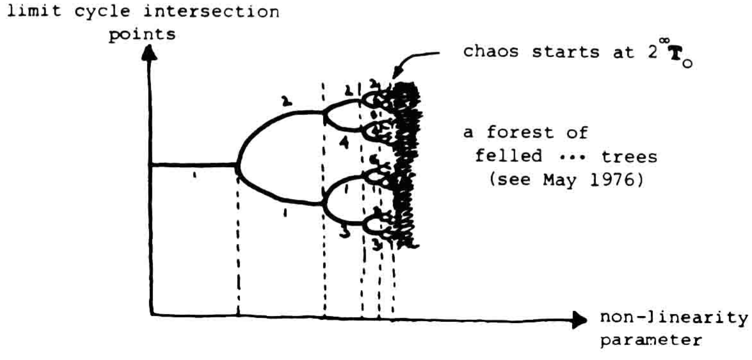

Fig.2.5