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Differential Equations with Small Parameters and Relaxation Oscillations

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Дифференциальные уравнения с малым параметром
и релаксационные колебания

DIFFERENTSIAL'NYE URAVNENIYA S MALYM PARAMETROM
I RELAKSATSIONNYE KOLEBANIYA
E. F. Mishchenko and N. Kh. Rozov

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PREFACE

A large amount of work has been done on ordinary differential equations with small parameters multiplying derivatives. This book investigates questions related to the asymptotic calculation of relaxation oscillations, which are periodic solutions formed of sections of both slow- and fast-motion parts of phase trajectories. A detailed discussion of solutions of differential equations involving small parameters is given for regions near singular points.

The main results examined were obtained by L. S. Pontryagin and the authors. Other works have also been taken into account: A. A. Dorodnitsyn's investigations of Van der Pol's equation, results obtained by N. A. Zheleztsov and L. V. Rodygin concerning relaxation oscillations in electronic devices, and results due to A. N. Tikhonov and A. B. Vasil'eva concerning differential equations with small parameters multiplying certain derivatives.

E. F. Mishchenko
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CHAPTER I

DEPENDENCE OF SOLUTIONS ON SMALL PARAMETERS. APPLICATIONS OF RELAXATION OSCILLATIONS

When the operation of a device or the course of a process is described by differential equations, we are passing from an actual object (process) to an idealized model. Every mathematical idealization involves, to a certain extent, the neglect of small quantities. Hence the question of how much distortion of the original phenomenon is introduced becomes important. We thus arrive at the mathematical problem of the dependence of solutions of differential equations on small parameters.

In this chapter we consider general characteristics of various types of dependence in the case of a normal autonomous system of ordinary differential equations. To simplify the exposition we consider problems involving only one parameter.

1. Smooth Dependence. Poincaré's Theorem

We consider the autonomous differential equation system

$$\dot{x}^i = F^i(x^1, \ldots, x^n, \varepsilon), \quad i = 1, \ldots, n, \tag{1.1}$$

or, in vector form, the equation

$$\dot{x} = F(x, \varepsilon), \tag{1.2}$$

where $x = (x^1, \ldots, x^n)$ is an n-vector of euclidean vector space R^n, $F(x, \varepsilon) = (F^1(x, \varepsilon), \ldots, F^n(x, \varepsilon))$ is an n-dimensional vector-valued function of x and ε, and ε is a numerical parameter. We first assume that ε is *small*,

$$0 \leqslant \varepsilon \leqslant \varepsilon_0, \tag{1.3}$$

where ε_0 is a small number.

Let the functions $F^i(x^1, \ldots, x^n)$, $i = 1, \ldots, n$ be defined and *continuous* in some domain G of the variables $x^1, \ldots, x^n$, where ε satisfies (1.3). We write

$$x = \varphi(t, \varepsilon) \tag{1.4}$$

for the solution of (1.2) satisfying the initial condition $x_0 = \varphi(t_0, \varepsilon)$, $(x_0, \varepsilon) \in G$. Together with (1.2) we consider the system

$$x = F(x, 0), \tag{1.5}$$

obtained from (1.2) by putting $\varepsilon = 0$. Let

$$x = \varphi_0(t) \tag{1.6}$$

be the solution of (1.5) with the same initial condition $x_0 = \varphi_0(t_0)$, defined on some *finite* time interval

$$t_0 \leqslant t \leqslant T. \tag{1.7}$$

If ε is small, the right sides in (1.2) and (1.5) differ by only a small quantity. It is natural to ask how the solutions (1.4) and (1.6) differ. In many cases important in practice, this question is answered by the following well-known theorems [45, 30, 23, 3].

Theorem 1 (concerning the continuous dependence of solutions on a parameter). If the right sides in (1.2) are continuously differentiable with respect to $x^1, \ldots, x^n$ and continuous functions of ε in a region G, then, for sufficiently small ε, the solution (1.4) is defined on the same interval (1.7) as the solution (1.6), and

$$\varphi(t, \varepsilon) = \varphi_0(t) + R_0(t, \varepsilon), \tag{1.8}$$

where $R_0(t, \varepsilon) \to 0$ when $\varepsilon \to 0$, uniformly with respect to t on the interval (1.7).

Theorem 2 (concerning the differentiability of solutions with respect to a parameter). If the right sides in (1.2) have, in G, continuous partial derivatives up to order $m \geq 1$, inclusive, with respect to the totality of all arguments, then, if ε is small enough, the solution (1.4) has the representation

$$\varphi(t, \varepsilon) = \varphi_0(t) + \varepsilon\varphi_1(t) + \ldots + \varepsilon^{m-1}\varphi_{m-1}(t) + R_m(t, \varepsilon), \tag{1.9}$$

where $R_m(t, \varepsilon) \to 0$, when $\varepsilon \to 0$ like ε^m, uniformly with respect to t on the interval (1.7).

Theorem 3 (Poincaré's theorem on the analyticity of solutions as functions of a parameter). If the right sides in (1.2) are analytic functions of each of their arguments in G, then, for sufficiently small ε, a solution (1.4) has the representation

$$\varphi(t, \varepsilon) = \varphi_0(t) + \sum_{m=1}^{\infty} \varepsilon^m \varphi_m(t); \tag{1.10}$$

the series converges uniformly on the interval (1.7).

Theorems 1, 2, and 3 not only confirm that, for small but *finite* ε, the solution (1.4) differs only slightly from the solution (1.6), but also indicate a method of finding this difference with any required degree of accuracy.

2. Dependence of Solutions on a Parameter, on an Infinite Time Interval

Theorems 1, 2, and 3 give no answer to the question of the deviation of the solution (1.4) from the solution (1.6) on an *infinite* time interval. Simple examples show that this deviation is not always small. Moreover, even if the solution (1.6) is defined for all $t \geq t_0$, the solution (1.4) is not always defined for all $t \geq t_0$.

Example 1. The scalar equation

$$\dot{x} = (x + \varepsilon)^2 \tag{2.1}$$

becomes

$$\dot{x} = x^2 \tag{2.2}$$

when $\varepsilon = 0$. The solution of (2.2) with zero initial value for $t = 0$ is $x = \varphi_0(t) \equiv 0$, $0 \leq t < \infty$, while the solution of (2.1) with the same initial values is

$$x = \varphi(t, \varepsilon) \equiv \frac{\varepsilon}{1 - \varepsilon t} - \varepsilon;$$

this solution is defined only for $0 \leq t < 1/\varepsilon$.

Example 2. Consider an electric circuit, formed of a condenser with capacitance C and a coil with inductance L in series (Fig. 1). If we neglect the small resistance of the circuit, the dependence of the current i on the time is described by the following equation [45]:

$$L \frac{d^2 i}{dt^2} + \frac{1}{C} i = 0. \tag{2.3}$$

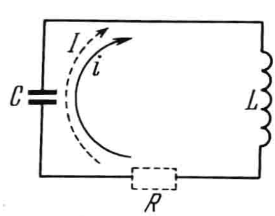

Fig. 1

But is this idealization, with the resistance R neglected, justified? In other words, does the solution of (2.3) differ only slightly from the solution of the equation

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=0 \qquad (2.4)$$

when R is small?

If we are interested only in a finite time interval, solutions of (2.3) and (2.4) with the same initial conditions differ only by a small quantity. On an infinite time interval, however, this is not true; in fact $I(t) \to 0$ when $t \to \infty$, but $i(t)$ performs periodic oscillations with constant amplitude. The phase portraits of Eqs. (2.3) and (2.4) in the $(i, di/dt)$ and the $(I, dI/dt)$ plane differ strikingly; the only equilibrium position for (2.3) is a center, while Eq. (2.4) has a focus (Fig. 2).

3. Equations with Small Parameters Multiplying Derivatives

Another reason for Theorems 1, 2, and 3 to be inapplicable for estimating the deviation of the solution (1.4) from the solution (1.6), even on a finite time interval, is a *discontinuity* (or lack of smoothness) in the dependence of the right sides in (1.1) on ε. This occurs in normal systems in which a small positive parameter ε occurs as a coefficient of some derivatives, for example in the system

$$\begin{cases} \varepsilon\dot{x}^i=f^i(x^1,\ \dots,\ x^k,\ y^1,\ \dots,\ y^l), & i=1,\ \dots,\ k,\\ \dot{y}^j=g^j(x^1,\ \dots,\ x^k,\ y^1,\ \dots,\ y^l), & j=1,\ \dots,\ l,\end{cases} \qquad (3.1)$$

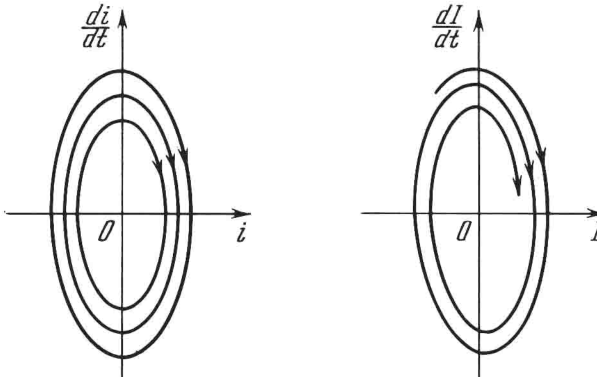

Fig. 2

where f^i and g^j are smooth functions of all their $k + l = n$ arguments. It is clear that, if (3.1) is written in the form (1.1), the right sides contain functions $(1/\varepsilon)f^i$, which increase without limit when $\varepsilon \to 0$.

We can reduce (3.1) to the form (1.1), so that the right side is a smooth function of ε. To this end, we put $t = \varepsilon\theta$ to obtain

$$\begin{cases} \dfrac{dx^i}{d\theta} = f^i(x^1, \ldots, x^k, y^1, \ldots, y^l), & i = 1, \ldots, k, \\ \dfrac{dy^j}{d\theta} = \varepsilon g^j(x^1, \ldots, x^k, y^1, \ldots, y^l), & j = 1, \ldots, l. \end{cases} \tag{3.2}$$

Theorems 1, 2, and 3 can be applied to (3.2), but this is not of great practical interest. We can ensure that solutions of (3.2) and solutions of the system obtained from it by putting $\varepsilon = 0$ differ by a small amount only on a finite interval of values of θ, i.e., on a time interval *whose length tends to zero* with ε.

We next consider two physical situations described by differential equation systems of the form (3.1). These examples will also be useful later in the illustration of certain phenomena.

Example 3 (Van der Pol's equation). Consider a vacuum tube oscillator consisting of a triode with an oscillating anode circuit; the circuit diagram is shown in Fig. 3. If I is the current passing through the resistance ba or, equivalently, through the inductance kb, then I, as a function of t, is described by the following differential equation

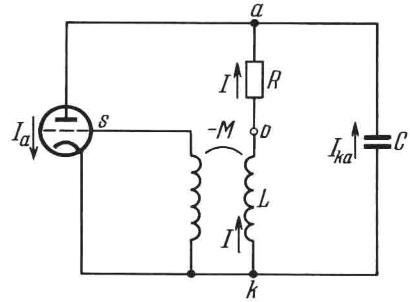

Fig. 3

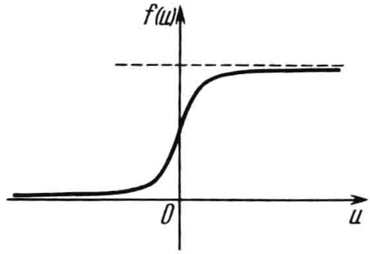

Fig. 4

[45, 7, 3]:

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=\frac{1}{C}f\left(M\frac{dI}{dt}\right); \tag{3.3}$$

here M is a positive constant (the mutual inductance) and $f(u)$, a characteristic of the vacuum tube, is a smooth monotonically increasing function of u with a graph as shown in Fig. 4. We can assume that $f'(u)$ has its maximum value for $u = 0$, i.e., $f''(0) = 0$ and $f'''(0) < 0$.

It is known [45] that a vacuum tube oscillator generates undamped periodic oscillations if its parameters satisfy the condition

$$R<\frac{M}{C}f'(0); \tag{3.4}$$

in this case, Eq. (3.3) has a single stable limit cycle in the $(I, dI/dt)$ plane. If $I(t)$ is replaced by the unknown function $i(t) = I(t) - f(0)$, Eq. (3.3) becomes *Rayleigh's equation*

$$LC\frac{d^2i}{dt^2}+F\left(\frac{di}{dt}\right)+i=0, \tag{3.5}$$

where $F(v) \equiv RCv - f(Mv) + F(0)$.

We now consider the idealized case in which $f(u)$ can (at least when the absolute value of u is not too large) be replaced by a cubic polynomial

$$f(u)=f(0)+f'(0)u+\frac{1}{6}f'''(0)u^3;$$

then

$$F(v)=(RC-f'(0)M)v-\frac{1}{6}f'''(0)M^3v^3,$$

$f'''(0)M^3 < 0$, $RC - f'(0)M < 0$ by virtue of condition (3.4), and (3.5) becomes

$$LC\frac{d^2i}{dt^2} + \left[(RC - f'(0)M) - \frac{1}{6}f'''(0)M^3\left(\frac{di}{dt}\right)^2\right]\frac{di}{dt} + i = 0.$$

If we transform to the new time $t = t/(LC)^{\frac{1}{2}}$ and the new unknown

$$z = \alpha i, \text{ where } \alpha^2 = \frac{RC - f'(0)M}{\frac{1}{2}f'''(0)M^3}LC,$$

then we obtain the equation

$$\frac{d^2z}{dt^2} + \lambda\left[-\frac{dz}{dt} + \frac{1}{3}\left(\frac{dz}{dt}\right)^3\right] + z = 0,$$

where $\lambda = \dfrac{f'(0)M - RC}{\sqrt{LC}} > 0.$

Finally, differentiating this last equation with respect to t and putting $x = dz/dt$, we obtain *Van der Pol's equation* [11]

$$\frac{d^2x}{dt^2} + \lambda[-1 + x^2]\frac{dx}{dt} + x = 0. \tag{3.6}$$

Equation (3.6) describes the operation of a vacuum tube oscillator in our idealization. The parameters of the generator are characterized by the single parameter λ. We have already noted that, if (3.4) holds, we have self-excited periodic oscillations (auto-oscillations); mathematically, this corresponds to the fact that Van der Pol's equation with any $\lambda > 0$ has a stable limit cycle in the $(x, dx/dt)$ plane.

For small λ, Eq. (3.6) differs only slightly from the equation for a linear oscillator, and the auto-oscillations of the generator are close to simple harmonic oscillations. As λ increases, the auto-oscillations differ more and more from harmonic oscillations, and for large λ they take the essentially different form of *relaxation oscillations* (see Sec. 4).

For large $\lambda > 0$, (3.6) can easily be reduced to the form (3.1). In fact, if we put

$$y = \int_0^x (x^2 - 1)\,dx + \frac{1}{\lambda}\frac{dx}{dt}, \quad t_1 = \frac{t}{\lambda}, \quad \varepsilon = \frac{1}{\lambda^2}, \tag{3.7}$$

then after the obvious transformations we obtain from (3.6) a second-order system (in which we write t instead of t_1 for simplicity)

$$\begin{cases} \varepsilon \dfrac{dx}{dt} = y - \dfrac{1}{3} x^3 + x, \\ \dfrac{dy}{dt} = -x; \end{cases} \tag{3.8}$$

here $\varepsilon > 0$ is a *small* parameter. We shall also refer to (3.8) as a Van der Pol equation.

Second-order systems of the form (3.1) (i.e., with $k = l = 1$) arise in the study of many electronic devices (for example, vacuum tube multivibrators with a single RC-component), in the description of the operation of which small parasitic capacitances play an important role [3], and in some types of multivibrators using tunnel diodes [54].

Example 4. The operation of several electronic devices (for example, two-tube Frugauer generators and symmetric multivibrators [3]), when small parasitic capacitances and inductances are taken into account, is described by a fourth-order differential system of the form

$$\begin{cases} \varepsilon \dot{x}^1 = -\alpha (y^1 - y^2) + \varphi (x^1) - x^2, \\ \varepsilon \dot{x}^2 = \alpha (y^1 - y^2) + \varphi (x^2) - x^1, \\ \dot{y}^1 = x^1, \\ \dot{y}^2 = x^2; \end{cases} \tag{3.9}$$

here $\alpha > 0$ is a constant and $\varphi(u)$, $-1 < u < 1$, is a function of u whose graph is shown in Fig. 5 (which is obtained by a transformation of a vacuum tube characteristic function; see Fig. 4). If the parasitic parameters are not taken into account, the operation of these devices is described by the equations obtained by putting $\varepsilon = 0$ in (3.9):

$$\begin{cases} -\alpha (y^1 - y^2) + \varphi (x^1) - x^2 = 0, \\ \alpha (y^1 - y^2) + \varphi (x^2) - x^1 = 0, \\ \dot{y}^1 = x^1, \\ \dot{y}^2 = x^2. \end{cases} \tag{3.10}$$

Radiophysicists have recently discovered that such devices can generate periodic oscillations of an unusual nature: At certain times (or for certain current strengths), discontinuous changes can occur between periods of smooth variations.

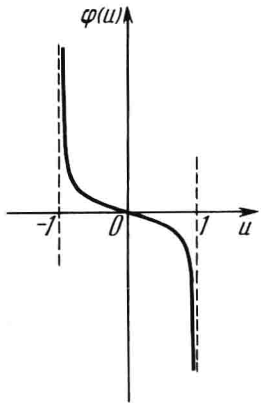

Fig. 5

Oscillations of this type are called *relaxation* oscillations. However all attempts at a theoretical explanation of this phenomenon by means of the system (3.10) failed. It was found necessary to introduce supplementary physical hypotheses (the "discontinuity hypothesis" [27]). The first purely mathematical explanation of relaxation oscillation in systems of the form (3.1), without any extra physical hypotheses, was given in [20] and developed further in [38]. We discuss this explanation in Sec. 5, using the systems (3.9) and (3.10) as examples.

4. Second-Order Systems. Fast and Slow Motion. Relaxation Oscillations

We now consider the second-order system

$$\begin{cases} \varepsilon \dot{x} = f(x, y), \\ \dot{y} = g(x, y), \end{cases} \tag{4.1}$$

where x and y are scalar functions of t and ε is a small positive parameter. Let

$$\begin{cases} f(x, y) = 0, \\ \dot{y} = g(x, y) \end{cases} \tag{4.2}$$

be the *degenerate system* corresponding to (4.1), i.e., the system obtained from (4.1) by putting $\varepsilon = 0$. The system (4.2) is not a normal differential equation system [the first of (4.2) is not a differential equation]. Hence it does not have solutions with arbitrary initial values (x_0, y_0). We must find solutions with initial points on the curve $f(x,y) = 0$,