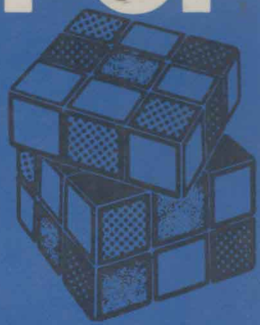
Handbook of CUBIK MATH



By all odds the best book yet written, or likely to be written, on the wonders and the dark unsolved mysteries of the cube.
—*Martin Gardner*

ALEXANDER H. FREY, JR.
+ DAVID SINGMASTER

Handbook of
Cubik Math

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ACKNOWLEDGMENTS

The works of many people over the last decade have contributed to this book. Not only did Ernö Rubik devise the Magic Cube, but he also has contributed substantially to an understanding of the possible movements of cube pieces. Both John Conway and Tamás Varga are among the early pioneers in the study of cubik math, and to these and many others like them, the authors are particularly indebted.

Much of the material in this book is derived from David Singmaster's *Notes on Rubik's Magic Cube*. Related to this, there have been literally thousands of letters from many correspondents over the last several years that have contributed to both books. To facilitate the further exchange of new ideas, there is now a cubik circular. Persons may write to D.S. Ltd., 66 Mount View Rd., London N4 4JR, United Kingdom for particulars. Interesting mathematical puzzles will also be distributed by this company.

The first U.S. importer of the Magic Cube was Logical Games Inc., Haymarket, Virginia. We are most grateful to its founder, Bela Szalai, whose idea for this book brought the authors together.

Many people have contributed to the preparation of the book. Lowell Smith and Alice H. Frey both read the entire manuscript, eliminating many errors. Cathy Schrott of Jantron Inc. did an outstanding job of page layout and typesetting. Special appreciation is extended to Helen Koehler Frey who not only read the entire manuscript several times, but also supplied the patient encouragement needed to finally complete this project.

Alexander H. Frey, Jr.
and
David Singmaster

PREFACE

Handbook of Cubik Math is a book about problem solving and some of the fundamental techniques used in problem solving throughout mathematics and science. Both the problems and the illustrations of concepts for solving them are drawn from Rubik's Magic Cube.

Ernö Rubik invented the Magic Cube as an aid to developing three-dimensional skills in his students. Little did he realize the impact that this puzzle would come to have. In 1980 alone, approximately five million cubes were sold. Predictions for future years are that sales will continue at more than twice that rate. In almost every neighborhood children — and adults — are playing with the cube.

It certainly enhances three dimensional thinking. However, even greater educational value has been found by mathematicians. For them the cube gives a unique physical embodiment of many abstract concepts which otherwise must be presented with only trivial or theoretical examples. Cube processes are non-commutative — that is, changing the order of movements produces different results. Cube processes generate permutations of the pieces of the cube. Sometimes different processes generate the same permutation so that, by looking at the cube, you cannot tell which process was used. This defines an equivalence between processes. The concepts of an identity process, inverses, the cyclic order of a process, commutators, and conjugates all play a part in solving problems on the cube. By experimenting on the cube, a student learns about these concepts and their relation to problem defining and problem solving without having to rely solely on his faith in the teacher or the text.

Perhaps surprisingly, one of the most fundamental concepts which cubik math illustrates is the use of symbolic notation. It is extremely rare to find anyone who can master the complexities of the cube without writing down what movements he has made or is planning to make. Without a good symbolic notation this is cumbersome at best. For communicating about the cube with others a common notation is mandatory.

The *Handbook of Cubik Math* in the early technical chapters orients the reader to the basic problem of the cube. It introduces a standard notation — one which is internationally accepted. Then it describes a logical method for restoring any scrambled cube to its pristine state where every face is a solid color. No background of complex or sophisticated mathematical concepts is required in these first three chapters. Many good students in their early teens have mastered these ideas. At the end of Chapter 3, several games are introduced. Playing these will enhance the competitor's understanding of the concepts inherent in controlled modification of the state of the cube.

One might think that after learning how to solve the cube — that is, how to restore it to its monochromatic-sided state — a person would lose interest in the cube. We thought so before we had taught many people how to solve it, only to find that with their increased understanding came increased curiosity. They wanted to understand more about how the cube worked, why processes produced the results they do, and what they could do to enhance their mastery of the cube.

Seldom does one realize at this point that the concepts which appeared so logical for solving the cube problem are, in fact, the concepts of identities, inverses, commutators, and conjugates. Chapter 4 defines these generalized concepts with many examples and exercises from the cube. These principles are applied to derive new techniques for manipulating the cube. Then in Chapter 5 these improvements are applied to obtain better ways to restore the cube.

It is in Chapter 6 that the mathematical concepts become more sophisticated. It is here that the concept of a group is introduced. The structure and the size of the cube group and its subgroups are explored in Chapters 6 and 7. This leads finally to a discussion of normal subgroups and the isomorphisms of subgroups and factor groups in Chapter 8.

It is expected that some students of the cube will only be ready to absorb material through Chapter 3. Others will be able to work through Chapter 5. The more advanced students will work all the way through to the end. At all stages it is necessary to have easy access to a cube. The cube is the best teacher and experimentation is the best learning technique.

CONTENTS

CHAPTER 1

INTRODUCTION

The Magic Cube or Rubik's Cube is an ingenious puzzle invented by Ernö Rubik, a sculptor, architect, and teacher of three dimensional design at the Academy of Applied Art in Budapest.

When new, it looks like a cube, about the size of a fist, with each face colored with one of six bright colors. Closer examination shows that the cube is divided in three along each direction so that it appears to be a 3 x 3 x 3 array of little cubes — called *cubies*. Thus each face of the cube is really a 3 x 3 pattern of little faces — called *facelets* — of the small cubies.

One of the first questions about the cube which we usually are asked is, "What is the problem?". We explain that the problem is to devise a method by which, starting with a randomly scrambled cube, you can restore the cube to the position where each face has a single solid color. About half of the people then respond by saying "Oh, so I am supposed to figure out how to take it apart!" We say, "No, the cube does not come apart. At least it is not supposed to."

About half of those people then go away to work out how the cube comes apart, muttering something like, "These fellows are no help. They clearly don't understand the problem." But, for the rest of you who are still here, we can go on to the usual next question. "What movements can the cube make?" One could answer that each of the six faces can be rotated about its central cubie as shown in Figure 1-1. After turning any one of the faces you can now turn any other face. This causes the colored facelets to move about. Sounds simple, doesn't it? If we stop there, we have made the problem about as difficult as one possibly could. Why? Because we may have turned off the most fruitful line of inquiry leading to solutions of the problem. It is important to understand a great deal more about how the cube moves than just that each of the six faces can be rotated.

> *WARNING:* It only takes a few random turns to thoroughly confuse your cube! Each face soon looks like a Mondrian painting. Without a solution, such as that given in Chapter 3, it could take you a long time to restore your cube.

Masochists who insisted on restoring their cubes without any help have taken weeks or months. Several of our friends took nine months to a year! When you understand the basic strategy taught in this book, you will be able to restore any scrambled cube without referring again to the book. The strategy does not require you to memorize any sequences of moves. You are taught the reason for each and every face turn.

CUBE MOVEMENTS

Any of the six faces of the cube can be rotated.

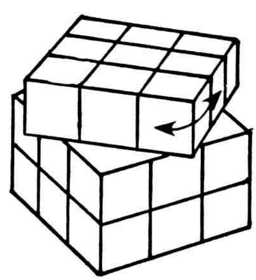


Figure 1-1

CHAPTER 2

A CUBIK ORIENTATION

Before developing a strategy for restoring the cube, it helps to study the cube a little while. What can we observe about the cube that may help us with the solution? What simple terminology and notation will describe the pieces and movements of the cube? Figure 2-1 gives a summary of the terminology and notation to be developed in this chapter.

1. CUBIES AND CUBICLES

Looking at the cube as a whole, at first glance, it appears to be made up of $3 \times 3 \times 3 = 27$ cubies in three layers, each layer being a three-by-three square of small cubies. However, it is only possible to see the outside of the cube, so that only 26 cubies can be seen. The one in the center is only imaginary. Also, all that we can see of each of the 26 visible cubies are the colored facelets which combine to form the six faces of the entire cube. Each face of the cube is made up of nine such facelets. Thus there are $6 \times 9 = 54$ facelets on the cube.

SUMMARY OF TERMINOLOGY AND NOTATION

Terminology	Definition or Abbreviation
Cubies	The small cube pieces which make up the whole cube.
Cubicles	The spaces occupied by cubies.
Facelets	The faces of a cubie.
Types of Cubies: Corner, Edge, and Center	A corner cubie has three facelets. An edge cubie has two facelets. A center cubie has one facelet.
Home Location — of a cubie	The cubicle to which a cubie should be restored.
Home Position — of a cubie	The orientation in the home location to which a cubie should be restored.
Positional Names for Cube Faces	Up Down Right Left Front Back
Notation for Cubicles — shown in *italics*	Lower case initials. For example, *uf* denotes the Up-Front edge cubicle.
Notation for Cubies — shown in *italics*	Upper case initials. For example, *URF* denotes the cubie whose home position is in the Up-Right-Front corner.
Notation for Face Turns — shown in BLOCK CAPITAL LETTERS	The initials, U, F, R, D, B, and L denote clockwise quarter turns. U^{-1}, F^{-1}, R^{-1}, D^{-1}, B^{-1}, and L^{-1} denote counter-clockwise quarter turns. U^2, F^2, R^2, D^2, B^2, and L^2 denote half turns.
Moving the Whole Cube	$\mathcal{U}$, $\mathcal{F}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{B}$, and $\mathcal{L}$ denote clockwise turns of the whole cube behind the indicated face.

Figure 2-1

Look now at the cubies which make up the cube. Notice that the cube has three types of cubies. Some cubies have three visible facelets as indicated in Figure 2-2. These are called *corner pieces.* There are eight corner pieces corresponding to the eight corners of the cube. Other cubies have only two visible facelets as indicated in Figure 2-3. These cubies fill in the space along an edge between two corner pieces. Therefore, they are called *edge pieces.* There are twelve edge pieces, one on each of the twelve edges of the cube. The third type of cubie has only one visible facelet. This facelet, as shown in Figure 2-4 is in the middle of a face. Thus these cubies are called *center pieces.* There are six center pieces corresponding to the six faces of the cube.

By rotating different faces of the cube, the cubies can be moved about. Each cubie moves to the location vacated by another cubie. These locations are called *cubicles.* The locations occupied by corner cubies are corner cubicles and the locations occupies by edge cubies are edge cubicles. Observe that no matter how faces are rotated, the corner pieces always move from one corner cubicle to another corner cubicle and the edge pieces always move from one edge cubicle to another edge cubicle. Rotating a face never moves a center cubie from one face to another. The center pieces have a fixed location relative to the other center pieces. They can only be spun in place. This is a particularly important observation, because it shows the following:

> *The color of the center piece of any face defines the only color to which that face of the cube can be restored.*

For each center piece the color of the opposite center piece never changes. Furthermore, if two opposite center pieces are placed in the positions of north and south poles respectively, then the sequential order of the other four center pieces around the equator is always the same.

THE CORNER CUBIES, SHADED, HAVE THREE FACELETS

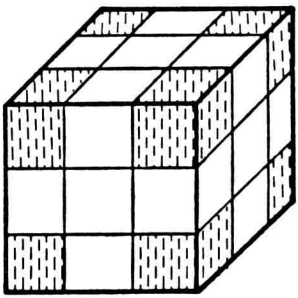


Figure 2-2

THE EDGE CUBIES, SHADED, HAVE TWO FACELETS

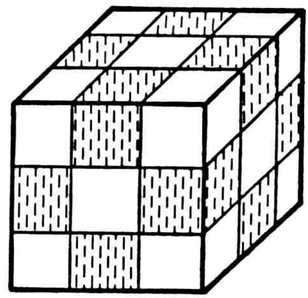


Figure 2-3

THE CENTER CUBIES, SHADED, HAVE ONE FACELET

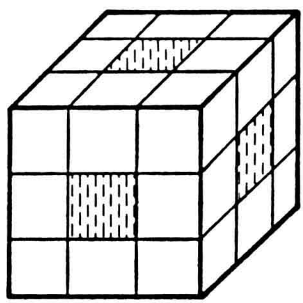


Figure 2-4

Since the center cubie of each face determines the only color to which that face can be restored, we can also define the one and only cubicle in which each cubie can be placed to restore the cube. For example, if the two facelets of an edge cubie are orange and green, then that piece must be placed in the unique edge cubicle between the orange center piece and the green center piece as shown shaded in Figure 2-5. Furthermore, the cubie must be

AN EDGE HOME POSITION

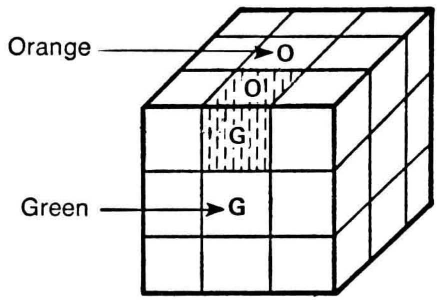


Figure 2-5

placed in that cubicle so that its orange facelet is next to the orange center piece and the green facelet is next to the green center piece.

Similarly, if the three facelets of a corner cubie are orange, green, and white then, to restore that cubie, it must be placed in the corner cubicle where the orange face, the green face, and the white face meet — shaded in Figure 2-6. Furthermore, its orange, green, and white facelets must be on the orange, green, and white faces respectively.

A CORNER HOME POSITION

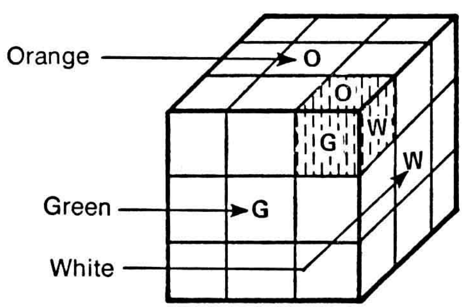
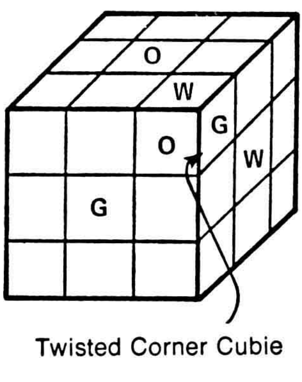


Figure 2-6

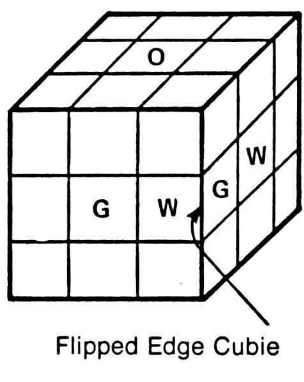
For each edge and corner cubie in the cube, the unique cubicle to which it must be restored is called the *home location* for that cubie. When a cubie is in its home location and its facelet colors match the colors of the center pieces on each face, then the cubie is said to be in its *home position.*

It is possible for a cubie to be in the cubicle of its home location without being in its home position. A corner piece in this condition is said to be *twisted* in its home location. An edge piece in this condition is said to be *flipped* in its home location. Figure 2-7 shows a twisted corner cubie and a flipped edge cubie. Thus, each corner and edge cubie has a unique home location and in that cubicle it has a unique placement which puts it in its home position.

TWISTED AND FLIPPED CUBIES



Twisted Corner Cubie



Flipped Edge Cubie

Figure 2-7

EXERCISES:

2.1-1 How many of the 54 facelets of the cube are
- a. facelets of corner cubies?
- b. facelets of edge cubies?
- c. facelets of center cubies?

2.1-2 At how many locations can an edge cubie be placed so that