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PREFACE

The aim of these notes is to pursue a line of research adopted by many authors (W. Feller, M.G. Krein, I.S. Kac, F.V. Atkinson, W.T. Reid, among others) in order to develop a qualitative and spectral theory of Volterra-Stieltjes integral equations with specific applications to real ordinary differential and difference equations of the second order.

We begin by an extension of the classical results of Sturm (comparison theorem, separation theorem) to this more general setting. In chapter 2 we study the oscillation theory of such equations and, in Chapters 3,4,5, apply some aspects of it to the study of the spectrum of the operators generated by certain generalized ordinary differential expressions associated with the above-mentioned integral equations.

In order to make these notes self-contained some appendices have been added which include results fundamental to the main text. Care has been taken to give due credit to those researchers who have contributed to the development of the theory presented herein - any omissions or errors are the author's sole responsibility.

I am greatly indebted to Professor F.V. Atkinson at whose hands I learned the subject and I also take this opportunity to acknowledge with thanks the assistance of the Natural Sciences and Engineering Research Council of Canada for continued financial support. My sincere thanks go to Mrs. Frances Mitchell for her expert typing of the manuscript.

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Angelo B. Mingarelli
Ottawa, April 1980.

INTRODUCTION

Let $p, q: I \to \mathbb{R}$, $p(t) > 0$ a.e. (in the sense of Lebesgue measure) and $1/p$, $q \in L(I)$ where $I = [a,b] \subset \mathbb{R}$. Consider the formally symmetric differential equation

$$(p(t)y')' - q(t)y = 0, \qquad t \in I. \tag{1}$$

By a solution of (1) we will mean a function $y: I \to \mathbf{C}$, $y \in AC(I)$, (i.e., absolutely continuous on I) such that $py' \in AC(I)$ and $y(t)$ satisfies (1) a.e. on I. Let $\gamma \in I$. Then a quadrature gives, for $t \in I$,

$$p(t)y'(t) = \beta + \int_\gamma^t y(s)q(s)ds$$

where $\beta = (py')(\gamma)$. Since $q \in L(I)$ its indefinite integral $\sigma(t) = \int_a^t q(s)ds$ exists for $t \in I$ and $\sigma \in AC(I)$. Hence y will be a solution of (1) if and only if $y(t)$ satisfies a Stieltjes integro-differential equation of the form

$$p(t)y'(t) = \beta + \int_\gamma^t y(s)d\sigma(s), \qquad t \in I, \tag{2}$$

where the integral may be interpreted, say, in the Riemann-Stieltjes sense. On the other hand (2) also has a meaning whenever $\sigma \in BV(I)$ (i.e., bounded variation on I) and y is continuous on I. Hence equations of the form (2) may be used to deal with differential equations (1). Moreover σ need not be continuous on I (as long as we require a solution of (2) to be continuous on I) and so (2) can be used to treat discrete problems, e.g., difference equations (or three-term recurrence relations) as well as continuous problems as we have seen. More precisely let $t_{-1} = a < t_0 < t_1 \ldots < t_{m-1} < t_m = b$ be a fixed partition of I. Define step-functions $p, \sigma \in BV(I)$ as follows: p, σ will be right-continuous on I and their only jumps, if any, will be at the points $\{t_i\}$ defined above with the saltus of σ being given by

$$\sigma(t_n) - \sigma(t_n - 0) = b_n,$$

where b_n, $n = 0,1,\ldots,m-1$, is a given real finite sequence, and p is defined on $[a,b]$ upon setting

$$p(t) = C_{n-1}(t_n - t_{n-1}), \qquad t \in [t_{n-1}, t_n)$$

where C_{n-1}, $n = 0,1,\ldots, m$ is a given positive real finite sequence. With these identifications one finds that the corresponding real solutions of (2) will be continuous polygonal curves whose vertices $(t_n, y(t_n)) \equiv (t_n, y_n)$ have their ordinates, y_n, satisfying the formally symmetric second-order linear difference equation

$$\Delta(C_{n-1}\Delta y_{n-1}) - b_n y_n = 0, \tag{3}$$

for $n = 0,1,\ldots, m-1$, and Δ is the forward difference operator, $\Delta y_n \equiv y_{n+1} - y_n$. So use of (2) now leads one to understand that solutions of (3) should perhaps be interpreted as continuous functions defined on I and not just as the finite sequence $y_{-1}, y_0, \ldots, y_m$ as one may at first sight suspect. That for (3) solutions are to be interpreted as continuous functions, has its historical precedents. For example, M. Bôcher noted in his survey article [6] that a Sturmian theory could be naturally developed for (3) if "solutions" were treated as continuous functions (in fact, the same polygonal curves that were mentioned above). The advantage in using (2) is that a Sturmian theory can be developed for (2) thus simultaneously yielding such a theory for each of (1) and (3).

If in (2) one chooses $\sigma \in C(I)$ (i.e., continuous on I) then (2) is a pure Stieltjes integro-differential equation. If, in addition, $p \in C(I)$ say, then (2) may be integrated once again to yield the Volterra-Stieltjes integral equation

$$y(t) = \alpha + \beta \int_{\gamma}^{t} \frac{ds}{p(s)} + \int_{\gamma}^{t} (t-s)y(s)d\sigma(s) \qquad t \in I$$

Note that (2) also includes equations of mixed type obtained by, say, setting $\sigma \in C^1(I)$ except at a finite number of points or by defining σ to be a C^1-function on a part of I and a step-function elsewhere.

An intensive study of equations of the form (2) was undertaken by F. V. Atkinson [3] in his monograph, (See also the fundamental paper of Krein [39] and the related papers of W.T. Reid [79], [80]).

In order to derive a spectral theory for (2) one needs to use (2) in order to define an operator on some suitable space. To this end, note that if y is a solution of (2) then

$$\frac{d}{dt} \{p(t)y'(t) - \int_{\gamma}^{t} y(s)d\sigma(s)\} = 0 \qquad (4)$$

and conversely if one defines a solution of (4) as a function $y \in AC(I)$ for which $p(t)y'(t) - \int_{\gamma}^{t} y(s)d\sigma(s) \in AC(I)$. We can then recover (2) from (4). On the other hand the left-side of (4) defines a generalized differential expression, viz.

$$\ell[y](t) = -\frac{d}{dt}\{p(t)y'(t) - \int_{\gamma}^{t} y(s)d\sigma(s)\}.$$

and such an expression may then be used to define a linear operator on $L^2(I)$ with due care for domain considerations.

If one wishes to treat boundary problems for Sturm-Liouville equations with a weight-function $r(t) \in L(I)$,

$$-(p(t)y')' + q(t)y = \lambda r(t)y,$$

consideration of the generalized ordinary differential expression

$$\ell[y](t) = -\frac{d}{d\nu(t)}\{p(t)y'(t) - \int_{\gamma}^{t} y(s)d\sigma(s)\}, \quad (5)$$

may be made, where the generalized derivative appearing on the right is, in general, a Radon-Nikodym derivative. The case $r(t) > 0$ corresponds to $\nu(t)$ non-decreasing and the case of unrestricted $r(t)$ corresponds to $\nu(t) \in BV(I)$. In the former case the operator defined by the differential expression is formally symmetric (under suitable domain restrictions) in the weighted Hilbert space $L^2(I,d\nu)$. In the latter case the operator is J-symmetric in a Krein (Pontrjagin) space, since the measure induced by $\nu(t)$ is a signed measure.

Expressions of the form (5) were first considered by W. Feller [68],[69],[70],[71],[72],[73] in the case when $\sigma(t) \equiv$ constant on I, $p(t) \equiv 1$, and ν a given non-decreasing function on I, (cf., also Langer [41]). The more general case $\sigma \in BV(I)$ was treated by I.S. Kac [35],[36],[37] when ν is monotone, cf., [46,p.49].

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION:

In this chapter we shall study the Sturmian theory of Stieltjes integro-differential equations; that is, equations of the form

$$p(t)y'(t) = c + \int_a^t y(s)\,d\sigma(s) \qquad (1.0.0)$$

defined on a finite interval $I = [a, b]$ and p, σ are real valued right-continuous functions of bounded variation on I and $p(t) > 0$ there.

Historical Background:

The comparison and separation theorems of Sturm comprise what we call the Sturmian theory. Comparison theorems for the scalar equation

$$\big(p(t)y'(t)\big)' - q(t)y(t) = 0 \qquad (1.0.1)$$

were first obtained by Sturm [58, p. 135] in his famous memoir of 1836. In that paper Sturm considered the equations

$$(K_1 y')' - G_1 y = 0 \qquad (1.0.2)$$

$$(K_2 z')' - G_2 z = 0 \qquad (1.0.3)$$

on a finite interval and showed that if $0 < K_2 \leqq K_1$, $G_2 \leqq G_1$, equality not holding everywhere on the interval, then between any two zeros of some solution of (1.0.2) there is at least one zero of any solution of (1.0.3). This is the result usually known as the *Sturm-Comparison Theorem*. Sturm's proof depended upon the introduction of a parameter in the coefficients which allowed him to pass continuously from K_1 to K_2 and from G_1 to G_2, as the parameter was increased, and then he studied the location of the zeros of the solutions as the parameter varied. It also depended upon the identity valid for all $t_1, t_2 \in I$,

$$\left[K_2 yz' - K_1 y'z\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} (G_2 - G_1) yz \, dt + \int_{t_1}^{t_2} (K_2 - K_1) y'z' \, dt \qquad (1.0.4)$$

which can be obtained by an application of Green's theorem [13, p. 291].

It seems [58, p. 186] that Sturm came to the conclusion of the comparison theorem by first having shown it true for the case of a three-term recurrence relation or second order difference equation though the latter result was not published. A discrete analog of the comparison theorem was published by Fort [21, p.] whose method of proof was, in essence, that of Sturm applied to difference equations instead of differential equations.

In 1909 Picone [48, p. 18] gave by far the simplest proof of the comparison theorem in the continuous case. He made use of the formula

$$\left[\frac{y}{z}(K_2 yz' - K_1 y'z)\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} (K_2 - K_1)y'^2\,dt + \int_{t_1}^{t_2} (G_2 - G_1)y^2\,dt - \int_{t_1}^{t_2} K_2\left(y' - \frac{z'y}{z}\right)^2 dt \qquad (1.0.5)$$

commonly known as the Picone Identity. The use of (1.0.5) allows an immediate proof of the Sturm Comparison Theorem [33, p. 226]. (cf., also [74]).

One important extension of the comparison theorem was that of Leighton [42, p. 604] who interpreted the theorem in a variational setting: He made use of a "quadratic functional" $Q[y]$ associated with (1.0.2-3) acting on functions $y \in C^1(a, b)$ and $y(a) = y(b) = 0$ (such functions were termed 'admissible'). For such y,

$$Q[y] \overset{df}{=} \int_a^b (K_2 y'^2 + G_2 y^2)\,dt . \qquad (1.0.6)$$

The main result was that if some non-trivial admissible function y had the property that $Q[y] < 0$ then every real solution of (1.0.3) would have to vanish at some point in (a, b). Swanson [59, p. 3] weakened Leighton's condition $Q[y] < 0$ to $Q[y] \leqq 0$ for $y \not\equiv 0$ reaching the same conclusion provided the solutions were not constant multiples of y.

The *Sturm-Separation theorem* states that the zeros of linearly independent solutions of, say, (1.0.2) interlace or separate one another. A similar result holds for three-term recurrence relations and in fact a more general result is known in the latter case. (See section 2).

In section 1 we shall give an extension of the aforementioned "Leighton-Swanson Theorem" to the class of integral equations (1.0.0) and give, as corollaries, the corresponding continuous and discrete versions of the comparison theorem.

In section 2 we give a proof of the Sturm Separation Theorem for (1.0.0) and give some applications to both differential and difference equations. We conclude this chapter with a study of the Green's function for boundary problems associated with (1.0.0) and its application to the problem of finding an explicit representation for the solution of the non-homogeneous problem. (See section 3).

§1.1 COMPARISON THEOREMS FOR STIELTJES INTEGRO-DIFFERENTIAL EQUATIONS :

Let $p_i(t)$, $\sigma_i(t)$, $i = 1, 2$, be real valued functions of bounded variation over $[a, b]$. We assume that $p_i(t) > 0$, $t \in [a, b]$, $i = 1, 2$, and that all four functions are right-continuous on $[a, b]$ with each possessing a finite number of discontinuities there. (This is for simplicity only. In the following chapters this hypothesis can be omitted, in most theorems, without affecting the conclusions.) We will, in general, assume that all these