

# Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

308

766

Tammo tom Dieck

Transformation Groups  
and Representation Theory



Springer-Verlag  
Berlin Heidelberg New York

# Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

766

---

Tammo tom Dieck

Transformation Groups  
and Representation Theory

---



Springer-Verlag  
Berlin Heidelberg New York 1979

## Author

T. tom Dieck  
Mathematisches Institut  
Bunsenstraße 3-5  
D-3400 Göttingen

AMS Subject Classifications (1970): 20C10, 20C15, 20D15, 55-02,  
55B25, 55E50, 57-02, 57D85, 57E15, 57E25

ISBN 3-540-09720-1 Springer-Verlag Berlin Heidelberg New York  
ISBN 0-387-09720-1 Springer-Verlag New York Heidelberg Berlin

Library of Congress Cataloging in Publication Data

Dieck, Tammo tom.

Transformation groups and representation theory.

(Lecture notes in mathematics; 766)

Bibliography: p.

Includes index.

1. Topological transformation groups. 2. Representations of groups. I. Title. II. Series:  
Lecture notes in mathematics (Berlin); 766.

QA3.L28 no. 766 [QA613.7] 510'.8s [514'.223] 79-24606

ISBN 0-387-09720-1

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically those of translation, reprinting, re-use of illustrations, broadcasting, reproduction by photocopying machine or similar means, and storage in data banks. Under § 54 of the German Copyright Law where copies are made for other than private use, a fee is payable to the publisher, the amount of the fee to be determined by agreement with the publisher.

© by Springer-Verlag Berlin Heidelberg 1979

Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.

2141/3140-543210

## Preface

These are extended lecture notes for a course on transformation groups which I gave at the Mathematical Institute at Göttingen during the summer term 1978.

The purpose of these notes is to give an introduction to that part of the theory of transformation groups which centers around the Burnside ring and the topology of group representations. It is assumed that the reader is acquainted with the basic material in algebraic topology, representation theory, and transformation groups. Nevertheless we have presented some elementary topics in detail.

Section 11 contains joint work with Henning Hauschild.

My thanks are due to Christian Okonek who read part of the manuscript and made many valuable suggestions and to Margret Rose Schneider who typed the manuscript.

## Contents

1.	The Burnside ring of finite $G$ -sets	1
1.1.	Finite $G$ -sets	1
1.2.	The Burnside ring $A(G)$	1
1.3.	Congruences between fixed point numbers	4
1.4.	Idempotent elements	7
1.5.	Units	8
1.6.	Prime ideals	9
1.7.	An example: The alternating group $A_5$	10
1.8.	Comments	11
1.9.	Exercises	12
2.	The $J$ -homomorphism and quadratic forms	14
2.1.	The $J$ -homomorphism	14
2.2.	Quadratic forms on torsion groups. Gauß sums	15
2.3.	The quadratic $J$ -homomorphism	22
2.4.	Comments	25
2.5.	Exercises	25
3.	$\lambda$ -rings	27
3.1.	Definitions	27
3.2.	Examples	31
3.3.	$\gamma$ -operations	33
3.4.	Adams operations	35
3.5.	Adams operations on representation rings	38
3.7.	The Bott cannibalistic class $\theta_k$	40
3.8.	$p$ -adic $\gamma$ -rings	41



3.9.	The operation $\mathfrak{S}_k$	47
3.10.	Oriented $\gamma$ -rings	49
3.11.	The action of $\mathfrak{S}_k$ on scalar $\gamma$ -rings	53
3.12.	The connection between $\Theta_k$ and $\mathfrak{S}_k$	57
3.13.	Decomposition of p-adic $\gamma$ -rings	59
3.14.	The exponential isomorphism $\mathfrak{S}_k$	61
3.15.	Thom-isomorphism and the maps $\Theta_k, \Theta_k^{\text{or}}$	67
3.16.	Comments	68
3.17.	Exercises	69
4.	Permutation representations	70
4.1.	p-adic completion	70
4.2.	Permutation representations over $F_q$	71
4.3.	Representations of 2-groups over $F_3$	74
4.4.	Permutation representations over $\mathbb{Q}$	80
4.5.	Comments	81
5.	The Burnside ring of a compact Lie group	82
5.1.	Euler Characteristics	82
5.2.	Euclidean neighbourhood retracts	86
5.3.	Equivariant Euler-Characteristic	91
5.4.	Universal Euler-Characteristic for G-spaces	98
5.5.	The Burnside ring of a compact Lie group	103
5.6.	The space of subgroups	107
5.7.	The prime ideal spectrum of $A(G)$	111
5.8.	Relations between Euler-Characteristics	118
5.9.	Finiteness theorems	121
5.10.	Finite extensions of the torus	131
5.11.	Idempotent elements	137
5.12.	Fundamental properties	143
5.13.	Multiplicative induction and symmetric powers	149

5.14.	An example: The group $SO(3)$ .	155
5.15.	Comments	156
5.16.	Exercises	157
6.	Induction theory	159
6.1.	Mackey functors	159
6.2.	Frobenius functors and Green functors	165
6.3.	Hyperelementary induction	168
6.4.	Comments	171
6.5.	Exercises	171
7.	Equivariant homology and cohomology	172
7.1.	A general localization theorem	172
7.2.	Classifying spaces for families of isotropy groups	175
7.3.	Adjacent families	177
7.4.	Localization and orbit families	180
7.5.	Localization and splitting of equivariant homology	185
7.6.	Transfer and Mackey structure	188
7.7.	Localization of equivariant K-theory	193
7.8.	Localization of the Burnside ring	198
7.9.	Comments	200
8.	Equivariant homotopy theory	201
8.1.	Generalities	201
8.2.	Homotopy equivalences	205
8.3.	Obstruction theory	210
8.4.	The equivariant Hopf theorem	212
8.5.	Geometric modules over the Burnside ring	214
8.6.	Prime ideals of equivariant cohomotopy rings	221
8.7.	Comments	226
8.8.	Exercises	227

9.	Homotopy equivalent group representations	228
9.1.	Notations and results	228
9.2.	Dimension of fixed point sets	230
9.3.	The Schur index	237
9.4.	The groups $i(G)$ and $iO(G)$	241
9.5.	Construction of homotopy equivalences	249
9.6.	Homotopy equivalences for $p$ -groups	252
9.7.	Equivariant $K$ -theory and fixed point degrees	254
9.8.	Exercises	259
10.	Geometric modules over the Burnside ring	260
10.1.	Local $J$ -groups	260
10.2.	Projective modules	261
10.3.	The Picard group and invertible modules	267
10.4.	Comments	277
11.	Homotopy equivalent stable $G$ -vector bundles	278
11.1.	Introduction and results about local $J$ -groups	278
11.2.	Mapping degrees. Orientations.	281
11.3.	Maps between representations and vector bundles	283
11.4.	Local $J$ -groups at $p$	286
11.5.	Local $J$ -groups away from $p$	291
11.6.	Projective modules	293
	References	296
	Notation	309



## 1. The Burnside ring of finite G-sets.

In this section let  $G$  denote a finite group. In order to motivate some of the subsequent investigations we give an introduction to the Burnside ring of a finite group. Later we generalize this to compact Lie groups by geometric methods which in case of a finite group are not always suitable for the applications of the Burnside ring in representation theory. The material in this section is mainly due to Andreas Dress.

### 1.1. Finite G-sets.

A finite G-set  $S$  is a finite set together with a left action of  $G$  on this set. A finite G-set is the disjoint union of its orbits. The orbits are transitive G-sets and are G-isomorphic to homogeneous G-sets  $G/H = \{gH \mid g \in G\}$ . The G-sets  $G/H$  and  $G/K$  are isomorphic if and only if  $H$  is conjugate to  $K$  in  $G$ . The set of G-isomorphism classes of finite G-sets becomes a commutative semi-ring  $A^+(G)$  with identity with addition induced by disjoint union and multiplication induced by cartesian product with diagonal action. The non-triviality of the multiplication arises from the decomposition of  $G/H \times G/K$  into orbits. These orbits correspond to the double cosets  $HgK$ ,  $g \in G$ , which can be identified with the orbit space of  $G/K$  under the left  $H$ -action. This correspondence can be described as follows: If  $X$  is an  $H$ -space the  $H$ -orbits of  $X$  correspond to the  $G$ -orbits of  $Gx_H X$ . If moreover  $X$  is a  $G$ -space then we have the G-isomorphism  $G/H \times X \rightarrow Gx_H X : (g, x) \mapsto (g, g^{-1}x)$ . We apply this to  $X = G/K$ . Explicitly, the double coset  $HgK$  corresponds to the orbit through  $(1, g)$ .

### 1.2. The Burnside ring $A(G)$ .

The Grothendieck ring constructed from the semi-ring  $A^+(G)$  is denoted  $A(G)$  and will be called the Burnside ring of  $G$ . If  $S$  is a finite G-set

let  $[S]$  or  $S$  be its image in  $A(G)$ . Additively,  $A(G)$  is the free abelian group on isomorphism classes of transitive  $G$ -sets. Equivalently, an additive  $\mathbb{Z}$ -basis is given by the  $[G/H]$  where  $(H)$  runs through the set  $C(G)$  of conjugacy classes of subgroups of  $G$ . The multiplication comes from the decomposition of  $G/H \times G/K$  into orbits. The ring  $A(G)$  is commutative with unit  $[G/G]$ .

Example 1.2.1.

Let  $G$  be abelian. Then, since generally the isotropy group of  $G/H \times G/K$  at  $(g_1H, g_2K)$  is  $g_1Hg_1^{-1} \cap g_2Kg_2^{-1}$ , all isotropy groups are  $H \cap K$  in the abelian case. Therefore  $[G/H] \cdot [G/K] = a [G/H \cap K]$  where  $a \in \mathbb{Z}$  is obtained by counting the number of elements on both sides. In particular  $[G/H]^2 = |G/H| [G/H]$ , where  $|S|$  is the cardinality of  $S$ . We see that for abelian  $G$  the  $[G/H]$  are almost idempotent.

If  $H < G$  and  $S, T$  are finite  $G$ -sets then we have for the cardinality of the  $H$ -fixed point sets  $|S^H + T^H| = |S^H| + |T^H|$  and  $|(S \times T)^H| = |S^H| |T^H|$ . Hence  $S \mapsto |S^H|$  extends to a ring homomorphism

$$\varphi_H : A(G) \longrightarrow \mathbb{Z}.$$

Conjugate subgroups give the same homomorphism so that we have one  $\varphi_H$  for each  $(H) \in C(G)$ . We let

$$\varphi = (\varphi_H) : A(G) \longrightarrow \prod_{(H) \in C(G)} \mathbb{Z}$$

be the product of the  $\varphi_H$ .

Proposition 1.2.2.

$\varphi$  is an injective ring homomorphism.

Proof.

By definition  $\varphi$  is a ring homomorphism. Suppose  $x \neq 0$  is in the kernel of  $\varphi$ . We write  $x$  in terms of the basis  $x = \sum a_H [G/H]$ . We have a partial ordering on the  $[G/H]$ , namely  $[G/H] \leq [G/K]$  if and only if  $H$  is sub-conjugate to  $K$ . Let  $[G/H]$  be maximal among the basis elements with  $a_H \neq 0$ . Then  $G/K^H \neq \emptyset$  implies  $[G/H] \leq [G/K]$ . Hence  $0 = \varphi_H x = a_H [G/H^H] = a_H |NH/H| \neq 0$ , a contradiction.

Since  $\varphi$  is an injection of a subgroup of maximal rank the cokernel is a finite group. We want to compute its order. We consider the diagram of injective ring homomorphisms

$$\begin{array}{ccc}
 A(G) & \xrightarrow{\quad \varphi \quad} & \pi Z \\
 \downarrow & & \downarrow \\
 A(G) \otimes \mathbb{Q} & \xrightarrow{\quad \varphi_Q \quad} & \pi \mathbb{Q}
 \end{array}$$

where the lower  $\varphi_Q$  is the rational extension of the upper  $\varphi$ .

Recall that  $WH = NH/H$  acts freely on  $G/H$  as the group of  $G$ -automorphisms: The action is given by  $WH \times G/H \rightarrow G/H : (wH, gH) \mapsto gw^{-1}H$ . Hence it acts freely on any fixed point set  $G/H^K$ . In particular  $|G/H^K|$  is divisible by  $|WH|$ . Therefore  $\varphi_Q([G/H] \otimes |WH|^{-1})$  is contained in  $\pi Z$ .

Proposition 1.2.3.

The elements  $\varphi_Q([G/H] \otimes |WH|^{-1}) =: x_H$  form a  $\mathbb{Z}$ -basis of  $\pi Z$ . The order of cokernel  $\varphi$  is  $\pi_{(H)} \in C(G) |WH|$ .

Proof.

The first assertion implies the second one. We view elements in  $\pi Z$  as row vectors. Then the  $x_H$  form (suitably ordered) a triangular matrix with one's on the diagonal. Hence they must be a basis.

Remark 1.2.4.

The homomorphism  $\psi$  may be discovered from the ring structure of  $A(G)$  as follows. An element  $x \in A(G)$  is a non-zero-divisor if and only if  $\psi x$  has no zero component. Therefore  $A(G) \otimes Q$  is the total quotient ring of  $A(G)$  (i.e. all non-zero-divisors made invertible). If  $x \in A(G) \otimes Q$  is integral over  $A(G)$  then the components of  $\psi_Q x$  are integral over  $Z$  hence integers. Conversely  $\pi Z$  is integral over  $\psi A(G)$ , e. g. because  $\pi Z$  is generated by idempotent elements which are integral over any subring. Hence  $\psi$  may be identified with the inclusion of  $A(G)$  into the integral closure in its total quotient ring. (For the notion of integral ring extension see Lang [107], Chapter IX; Bourbaki [33], Ch. 5.)

1.3. Congruences between fixed point numbers.

We have seen in 1.2. that  $\psi A(G)$  is a subgroup of maximal rank in  $\pi Z$ . How can we describe its image? If  $G = Z/pZ$  is the cyclic group of prime order  $p$  then  $|S| \equiv |S^G| \pmod p$  because the orbits of  $S \setminus S^G$  have cardinality  $p$ . Hence this congruence gives a condition for elements to be in the image of  $\psi$ . The reader can easily check that this is the only condition, for  $G = Z/pZ$ . We generalize such congruences.

Let  $S$  be a finite  $G$ -set and let  $V(S)$  be the complex vector space spanned by the elements of  $S$ . The  $G$ -action on the basis  $S$  of  $V(S)$  induces a linear action on  $V(S)$ . The resulting  $G$ -module  $V(S)$  is called the permutation representation associated to  $S$ . The character of  $V(S)$  is a function on  $G$ ; it will be denoted with the same symbol. The

orthogonality relations for characters say in particular that for any complex  $G$ -module  $V$  the number  $|G|^{-1} \sum_{g \in G} V(g)$  is the dimension of  $V^G$ . Hence

$$(1.3.1) \quad \sum_{g \in G} V(S)(g) \equiv 0 \pmod{|G|}.$$

Now note that

$$V(S)(g) = \text{Trace}(l_g : V(S) \longrightarrow V(S) : v \longmapsto gv) = |S^g|$$

(look at the matrix of  $l_g$  with respect to the basis  $S$ ). Therefore 1.3.1 can be rewritten

$$(1.3.2) \quad \sum_{g \in G} \psi_{\langle g \rangle}(x) \equiv 0 \pmod{|G|}$$

for any  $x \in A(G)$ , where  $\langle g \rangle$  denotes the cyclic group generated by  $g$ . If  $H$  is a cyclic subgroup of  $G$  the number of elements  $g$  with  $\langle g \rangle$  conjugate to  $H$  is

$$|H^*| |G/NH|$$

where  $H^*$  is the set of generators of  $H$  and  $|G/NH|$  is the number of groups conjugate to  $H$ . Therefore (1.3.2) can be rewritten

$$(1.3.3) \quad \sum_{(H) \text{ cyclic}} |H^*| |G/NH| \psi_H(x) \equiv 0 \pmod{|G|}$$

where now the summation is taken over conjugacy classes of cyclic subgroups of  $G$ .

We now apply the same argument to  $V(S^H)$  considered as  $NH/H$ -module and obtain

$$\sum_{(K)} |NK/NH \cap NK| |K/H|^* \psi_K(x) \equiv 0 \pmod{|NH/H|}$$

where we sum over  $NH$ -conjugacy classes  $K$  such that  $H$  is normal in  $K$  and  $K/H$  is cyclic. This may also be written in the form

$$(1.3.4) \quad \sum_{(K)} n(H,K) \psi_K(x) \equiv 0 \pmod{|NH/H|}$$

where the  $n(H,K)$  are certain integers with  $n(H,H) = 1$  and the sum is over the  $G$ -conjugacy classes  $(K)$  such that  $H$  is normal in  $K$  and  $K/H$  is cyclic.

For the next Proposition we view elements of  $\pi Z$  as functions  $C(G) \rightarrow Z$ .

Proposition 1.3.5.

The congruences 1.3.4 are a complete set of congruences for image  $\psi$ , i. e.  $x \in \pi Z$  is contained in the image of  $\psi$  if and only if

$$\sum_{(K)} n(H,K) x(K) \equiv 0 \pmod{|NH/H|}$$

for all  $(H) \in C(G)$ .

Proof.

We have already seen that the elements in the image of  $\psi$  satisfy these congruences. The congruences 1.3.6 are independent because they are given by a triangular matrix with one's on the diagonal. Hence they describe a subgroup  $A$  of index  $\pi |NH/H|$ . By Proposition 1.2.3 therefore  $A = \text{im } \psi$ .



Remark 1.3.7.

A slightly different set of congruences is obtained if one considers  $V(S^H)$  as  $N_p H/H$ -module where  $N_p H/H$  is a Sylow  $p$ -group of  $NH/H$ . This yields a set of  $p$ -primary congruences which may be used instead of 1.3.4. These congruences are useful when localizations of  $A(G)$  are considered; e. g. for  $A(G)_{(p)}$ , the Burnside ring localized at  $p$ , only  $p$ -primary congruences are valid.

1.4. Idempotent elements.

Idempotent elements in  $\pi Z$  are the functions with values 0 and 1. We use 1.3 to see when such functions come from  $A(G)$ . We consider  $A(G)$  as subring of  $\pi Z$  via  $\psi$ .

A subgroup  $H$  of  $G$  is called perfect if it is equal to its commutator subgroup. Each  $H < G$  has a smallest normal subgroup  $H_s$  such that  $H/H_s$  is solvable. One has  $(H_s)_s = H_s$ . A subgroup  $H$  is perfect if and only if  $H = H_s$ . Let  $P(G)$  be the subset of  $C(G)$  represented by perfect subgroups.

Proposition 1.4.1.

An idempotent  $e \in \pi Z$  is contained in  $A(G)$  if and only if for all  $(H) \in C(G)$  the equality  $e(H) = e(H_s)$  holds.

Proof.

Suppose  $e \in A(G)$ . Then  $e$  satisfies 1.3.6. Given  $K < G$ . Choose  $K_s = K^n \triangleleft K^{n-1} \triangleleft \dots \triangleleft K^0 = K$  such that  $K^i/K^{i+1}$  is cyclic of prime order  $p(i)$ . Then by 1.3.6 applied to the group  $K^{i+1}$  we have  $e(K^i) \equiv e(K^{i+1}) \pmod{p(i)}$ . Since the values of  $e$  are 0 or 1 we must have  $e(K^i) = e(K^{i+1})$  and therefore  $e(K_s) = e(K)$ . Conversely assume that  $e(K_s) = e(K)$  for all  $K$ . Then we must have  $e(H) = e(K)$  for all  $H \triangleleft K$  with  $K/H$  cyclic so that  $e$  satisfies the congruences 1.3.6.

Corollary 1.4.2.

The set of indecomposable idempotents of  $A(G)$  corresponds bijectively to  $P(G)$ . In particular  $G$  is solvable if and only if 0 and 1 are the only idempotents in  $A(G)$ .

Remark 1.4.3.

Let  $P \subset \mathbb{Z}$  be a set of prime numbers. Let  $A(G)_P$  be the localization of  $A(G)$  at  $P$ , i. e. the primes not in  $P$  are made invertible. Then one can show as in the proof of Proposition 1.4.1 that the idempotents of  $A(G)_P$  are the functions  $e$  with  $e(H) = e(H_P)$  where  $H_P$  is the smallest normal subgroup of  $H$  such that  $H/H_P$  is solvable of order involving only primes in  $P$ .

1.5. Units.

If  $A$  is a commutative ring we let  $A^*$  be the multiplicative group of its units.

Let  $e \in A$  be an idempotent. Then  $1-2e = u$  is a unit. Conversely it can happen that for a unit  $u$  the element  $(1-u)/2 = e$  is contained in  $A$ . Then  $e$  is an idempotent, because  $(1-u)^2 = 2(1-u)$  for any unit  $u$ . In case of the Burnside ring  $(1-u)/2$  is contained in  $\pi \mathbb{Z}$  but not in general in  $A(G)$  as we shall see in a moment. But if  $G$  has odd order then  $\text{coker } \psi$  is odd and hence  $1-u \in A(G)$  and  $(1-u)/2 \in \pi \mathbb{Z}$  implies  $(1-u)/2 \in A(G)$ . Since a non-solvable group has non-trivial idempotents, by 1.4.2, we obtain

Proposition 1.5.1.

If  $G$  is non-solvable then  $A(G)^* \neq \{\pm 1\}$ . If  $G$  is solvable of odd order then  $A(G)^* = \{\pm 1\}$ .

Let  $H$  be a subgroup of index 2 in  $G$ . Then  $H \triangleleft G$ ,  $[G/H]^2 = 2 [G/H]$

and therefore  $u(H) := 1 - [G/H] \in A(G)^*$ . Note that  $(1-u(H))/2$  is not in  $A(G)$ . The subgroups of index 2 are precisely the kernels of non-trivial homomorphisms  $G \rightarrow \mathbb{Z}/2\mathbb{Z}$ . Hence we obtain an injective map  $j : \text{Hom}(G, \mathbb{Z}/2\mathbb{Z}) \rightarrow A(G)^*$  given by  $j(f) = 1 - G/\ker(f)$ . The image of  $j$  is in general not a subgroup.

#### Problem 1.5.2.

Determine the structure of  $A(G)^*$  in terms of the structure of  $G$ . (Of course one knows by the famous theorem of Feit - Thompson that groups of odd order are solvable. Therefore the 2-primary structure of  $G$  is relevant. In particular  $A(G)^*$  for 2-groups would be interesting. (See also the next remark.)

#### Remark 1.5.3.

We shall prove later by geometric methods that for a real representation  $V$  the function  $(H) \mapsto (-1)^{\dim V^H}$  is contained in  $A(G)$ . This function is then a unit in  $A(G)$ . It would be interesting to see units which are not of this form (2-groups?).

#### 1.6. Prime ideals.

Since  $\pi \mathbb{Z}$  is integral over  $A(G)$  by the "going-up theorem" of Cohen-Seidenberg (see Atiyah-Mac Donald [11], p. 62) every prime ideal of  $A(G)$  comes from  $\pi \mathbb{Z}$  hence has the form

$$q(H, p) := \{x \in A(G) \mid \psi_H(x) \equiv 0 \pmod{p}\}$$

for a subgroup  $H$  of  $G$  and a prime ideal  $(p)$  of  $\mathbb{Z}$ . The elementary proof of Dress [79] for this fact shall be given later (section 5) in the slightly more general context of compact Lie groups. The prime ideals  $q(H, 0)$  are minimal; the ideals  $q(H, p)$ ,  $p \neq 0$ , are maximal with residue field  $\mathbb{Z}/p\mathbb{Z}$ . If  $q(H, p) = q(K, q)$  then  $p = q$  and