# Transcendence Theory: Advances and Applications

Edited by
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and
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# Transcendence Theory: Advances and Applications

*Proceedings of a conference held in Cambridge in 1976*

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and

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# PREFACE

This volume is an account of the proceedings of a conference on transcendence theory and its applications held in the University of Cambridge during January and February, 1976. It consists of sixteen papers, contributed by leading mathematicians in the field and containing detailed expositions of their latest researches. Reflecting the considerable current activity in this area, a wide variety of original results are established, emanating, in several instances, from a long series of earlier developments. The authors have been encouraged, therefore, to explain fully the background to their studies and, in general, to make their memoirs as readable as possible; we hope that they have succeeded.

The papers have been arranged in groups with a common theme, rather than in lexicographical order as is perhaps more usual in works of this nature. The first five consist of an account of the most recent progress in connexion with the theory of linear forms in the logarithms of algebraic numbers and its applications, and they constitute, in fact, an essentially self-contained essay on the subject. A similar remark applies to the next four papers which are devoted to various topics in the transcendence theory of elliptic and Abelian functions. There follow five articles of a more miscellaneous character relating, in particular, to linear and algebraic independence of meromorphic functions and to arithmetical properties of polynomials in several variables. The work concludes with two papers on an old, but recently much revived transcendence method of Mahler.

A useful introduction to this volume is provided by the book *Transcendental Number Theory* by A. Baker (Cambridge University Press, 1975). Other valuable texts are the monographs by M. Waldschmidt (1974), D. W. Masser (1975) and K. Mahler (1976) in the Springer Lecture Note Series. Indeed, we believe that the present work, if read in conjunction with these tracts, will bring the reader to the forefront of knowledge in essentially all major aspects of the subject.

We should like to express our gratitude to the Department of Pure Mathematics and Mathematical Statistics in Cambridge for providing the financial support that made the conference possible, and to the Academic Press for their willing cooperation in the production of these Proceedings.

Cambridge, 1977

A.B.
D.W.M.

# CONTENTS





A.

## CHAPTER 1

# The Theory of Linear Forms in Logarithms

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## 1. INTRODUCTION

We shall begin with a short account of the history of the subject and we shall then establish two further results in the field that include many of the earlier theorems as special cases. We shall write, for brevity,

$$\Lambda = \beta_0 + \beta_1 \log \alpha_1 + \ldots + \beta_n \log \alpha_n,$$

where the $\alpha$'s and $\beta$'s denote algebraic numbers. We shall assume that the $\alpha$'s are not 0 or 1, that the $\beta$'s are not all 0, and that the logarithms have their principal values. The latter assumption involves no essential loss of generality, since the results that we shall describe would apply to any values of the logarithms if the constants were allowed to depend on their determinations.

The first result on the non-vanishing of $\Lambda$ goes back to the famous work of Hermite [26] of 1873 in which he proved that $e$, the natural base for logarithms, is transcendental; this implies that $\Lambda \neq 0$ when $n = 1$, $\beta_0 = 1$, $\beta_1 = -1$. Hermite's work rested on the construction of simultaneous approximations to the exponential series $e^x, \ldots, e^{nx}$ by rational functions, and it can be regarded as the main source of modern transcendence theory. The work was generalized by Lindemann [28] in 1882; in his classic memoir, he proved that $\Lambda \neq 0$ when $n = 1$ for all $\beta_0$, $\beta_1$, not both 0, and this yields, in particular, the transcendence of $\pi = -i \log(-1)$. The next major step was taken by Gel'fond [17] in 1929; he showed that $\Lambda \neq 0$ when $n = 2$, $\beta_0 = 0$ and $\beta_1/\beta_2$ is an imaginary quadratic irrational, whence, in particular, $e^\pi = (-1)^{-i}$ is transcendental. Gel'fond's argument has its origins in earlier studies on integral integer-valued functions (cf. [16]), and it depends on an analysis of an extrapolation formula for the exponential function similar to that occurring in some well-known papers of Pólya and Hardy. Gel'fond's result was extended to real quadratic irrationals $\beta_1/\beta_2$ by Kuzmin [27] in 1930; the work implies, in particular, that $2^{\sqrt{2}}$ is transcendental. The result was further extended by Gel'fond [18] and Schneider [37] independently in 1934; they succeeded in covering all $\beta_1$, $\beta_2$ with $\beta_1/\beta_2$ irrational, and thereby solved the famous seventh problem of Hilbert. One of the main features of the work is the construction by means of Dirichlet's box principle of an auxiliary function that vanishes to a high order at certain extrapolation points; some antecedents of the method can be found in the writings of Siegel [42] and Mahler [29], and it has proved to be remarkably powerful.

The Gel'fond–Schneider theorem was generalized to arbitrarily many logarithms of algebraic numbers by the author [2: I, II] in 1966; it was shown in fact that $\Lambda \neq 0$ when $\beta_0 = 0$ and either $\beta_1, \ldots, \beta_n$ or $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the rationals, and shortly afterwards the author [2: III] showed further that $\Lambda \neq 0$ when $\beta_0 \neq 0$. We shall subsequently refer to the conditions $\beta_0 = 0$ and $\beta_0 \neq 0$ as the homogeneous and inhomogeneous cases respectively; and we shall speak of the general case when either condition can occur. The author's work depended on the construction of an auxiliary function of several complex variables in place of the function of a single variable as employed by Gel'fond, and it also involved a new extrapolation technique. Here the range of extrapolation was extended and the order of the derivatives reduced, whereas, in previous work, the range was essentially fixed while the differential order increased.

We shall be concerned henceforth with lower bounds for $|\Lambda|$ in terms of the degrees and heights of the $\alpha$'s and $\beta$'s; it will be recalled that the height of an algebraic number is the maximum of the absolute values of the relatively prime integer coefficients in the minimal defining polynomial. We shall suppose that the height of $\alpha_j$ is at most $A_j (\geqslant 4)$, and we put $A = \max A_j$. Further we shall suppose that the height of $\beta_j$ is at most $B(\geqslant 4)$. The field $\mathbb{K}$ generated by the $\alpha$'s and $\beta$'s over the rationals will be assumed to have degree at most $d$. The first theorem giving a positive lower bound for $|\Lambda|$ was obtained by Morduchai–Boltovskoj [32] in 1923. He showed that $|\Lambda| > B^{-C}$ when $n = 1$ and $\beta_0, \beta_1$ are rational integers, where $C$ depends only on $\alpha_1$ (see also [23]). In 1935 Gel'fond [19] proved that if $n = 2$, $\beta_0 = 0$, $\kappa > 5$ and if $\log \alpha_1/\log \alpha_2$ is irrational then

$$|\Lambda| > C e^{-(\log B)^{\kappa}}, \tag{1}$$

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where $C > 0$ is effectively computable in terms of $\alpha_1$, $\alpha_2$, $d$ and $\kappa$. The result was obtained by a refinement of the method he had used to solve Hilbert's seventh problem, a particular feature being the utilization of the Hermite interpolation formula in place of the maximum-modulus principle. In 1939, Gel'fond [20] relaxed the condition $\kappa > 5$ to $\kappa > 3$, and, in 1949, he relaxed it further to $\kappa > 2$ [22]. He also showed, at about the same time [21], that if $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent then, as a consequence of the Thue–Siegel theorem, an inequality of the form $|\Lambda| > C\, e^{-\delta B}$ holds for any $\delta > 0$ where $\beta_0 = 0$ and $\beta_1, \ldots, \beta_n$ are rational integers, and $C > 0$ depends only on $\delta$, $d$ and the $\alpha$'s; but here $C$ cannot be effectively computed. We shall subsequently refer to the conditions $\beta_0 = 0$ and $\beta_1, \ldots, \beta_n$ rational integers as "the rational case". In fact, in this case, it is easy to show by a Liouville-type argument that if $\Lambda \neq 0$ then

$$|\Lambda| > (dA)^{-4ndB}; \tag{2}$$

moreover the dependence of (2) on $A$ and $n$ is essentially best possible. But, from the point of view of applications, it is crucial to have a slightly stronger dependence on $B$, and thus (2) has proved to be of value only as a supplementary estimate.

In 1966, by means of the many variable techniques referred to earlier, the author [2: I, II] showed that, for any $\kappa > 2n + 1$, the inequality (1) holds in the homogeneous case, where $C > 0$ depends only on $n$, $d$, $\kappa$ and $A$. Further like all constants mentioned subsequently, $C$ is effectively computable. The work was extended to cover the inhomogeneous case in 1968 [2: III]; here the condition on $\kappa$ was relaxed to $\kappa > n + 1$ in general and to $\kappa > n$ when $\beta_0 = 0$. Shortly afterwards, Fel'dman [11], [12], using a rather more complicated auxiliary function, succeeded first in reducing the condition to $\kappa > 1$, and then proved that indeed, if $\Lambda \neq 0$, we have $|\Lambda| > B^{-C}$, where $C$ depends only on $n$, $d$ and $A$; and here the dependence on $B$ is best possible. The value for $C$, as calculated by Fel'dman ,takes the form $C'(\log A)^\kappa$, where $\kappa$ depends only on $n$, and $C'$ depends only on $n$ and $d$.

In another direction, the author [2: IV] proved in 1968 that if, in the rational case, we have $0 < |\Lambda| < e^{-\delta B}$ for some $\delta$ with $0 < \delta \leqslant 1$, and if $d \geqslant 4$, then

$$B < (4^{n^2} \delta^{-1} d^{2n} \log A)^{(2n+1)^2}; \tag{3}$$

and it was implicit in the same paper that if $\Lambda = 0$ but the $\beta$'s are not all 0 then in fact $\Lambda = 0$ for some $\beta_1, \ldots, \beta_n$, not all 0, with absolute values at most $B$, where $B$ satisfies (3) with $\delta = 1$. The estimate (3) was derived mainly for computational purposes and it remains the most useful result of its kind established to date. Frequently in applications, however, it is known that one of the $\alpha$'s has a large height relative to the remainder, and, in this connexion, it was proved by the author [3] in 1968 that (3) could be replaced by $B < C(\log A)^{\kappa}$ for any $\kappa > n + 1$, where $C$ depends on only $n, d, \kappa$ and the maximum $A'$, say, of $A_1, \ldots, A_{n-1}$. The condition on $\kappa$ could readily be relaxed to $\kappa > n$, and a further relaxation to $\kappa > n - 1$ was obtained by Fel'dman [13], [14]. Moreover, in 1971, Stark and the author [7], in a joint study motivated by certain class number problems, reduced the condition to $\kappa > 1$, and here Kummer theory played an important rôle.

Several of the preceding theorems were combined by the author in a general result established in 1972 [4: I]. It was shown, namely, that if, in the rational case, we have $\Lambda \neq 0$ then

$$|\Lambda| > C^{-\log A \log B}, \tag{4}$$

where $C > 0$ depends only on $n$, $d$ and $A'$. The result is plainly best possible with respect to $B$ when $A$ is fixed and with respect to $A$ when $B$ is fixed. It was generalized in 1973 to yield

$$|\Lambda| > (\delta/B')^{C \log A}\, e^{-\delta B} \tag{5}$$

for any $\delta$ with $0 < \delta < \frac{1}{2}$, where $B' = |b_n|\,(>0)$, and $C > 0$ depends only on $n$, $d$ and $A'$ [4: II]. Plainly (5) gives (4) on taking $\delta = 1/B$, and furthermore if $b_n = -1$ and $0 < |\Lambda| < e^{-\varepsilon B}$, where $0 < \varepsilon < \frac{1}{4}$, then (5) gives $B < C \log A$, where $C$ depends only on $n$, $d$, $A'$ and $\varepsilon$. The latter result furnishes, as a corollary, an effective improvement upon Liouville's theorem concerning rational approximations to algebraic numbers (cf. [3] and [15]). An inequality of the same kind as (4) but uniform with respect to each of the parameters $A_1, \ldots, A_n$ was derived by the author [4: III] in 1975; it was proved, namely, that

$$|\Lambda| > B^{-C\Omega \log \Omega}, \tag{6}$$

where

$$\Omega = \log A_1 \ldots \log A_n, \tag{7}$$

and $C > 0$ depends only on $n$ and $d$. Further, van der Poorten [33] recently noted that $\log \Omega$ in (6) can be replaced by $\log \Omega'$, where $\Omega' = \Omega/\log A_n$, and then (6) includes (4). The refinement depends on a strengthened version of one of the preliminary lemmas in [4: I], as obtained by Tijdeman [45] in connexion with his well-known work on Catalan's equation (see Lemma 1, below); in fact Tijdeman used the lemma to give an expression for $C$ in (4) of the form $\exp\{C'(\log A')^{\kappa}\}$.

An analogue of (4) in the general case was established by the author [5] in 1973, viz:

$$|\Lambda| > (B \log A)^{-C \log A}, \tag{8}$$

where $C > 0$ depends only on $n$, $d$ and $A'$. Further, Stark [44], using certain techniques from the classical theory of algebraic numbers, showed that if, in the homogeneous case, $0 < |\Lambda| < e^{-\delta B}$, where $\delta > 0$, then, for any $\varepsilon > 0$, we have $B < C\Omega^{1+\varepsilon}$, where $C$ depends only on $n$, $d$, $\varepsilon$ and $\delta$. Furthermore, Shorey [41], developing Stark's work, proved that if $\beta_n = -1$ and $\Lambda \neq 0$ then

$$|\Lambda| > \exp\{-C\Omega(\log \Omega)^2 (\log \Omega B)^2 (\log(\Omega \log B))^{2n+2+\varepsilon}\},$$

where $C = (nd)^{cn}$ and $c > 0$ depends only on $\varepsilon$. The expression for $C$ is better than any given hitherto, and the improvement rests on a new idea concerning the size of the inductive steps. The result has been of particular value in connexion with problems concerning the distribution of the primes (cf. [36]).

This brings our discussion essentially up to date but, before closing, it should be said that we have by no means covered all the papers on the subject. For instance, Ramachandra, Shorey and others have obtained still sharper results when the $\alpha$'s are near to 1 (cf. [1], [10], [36] and [40]), several precise estimates have been given by Mahler, Fel'dman and others in the case $n = 1$ (cf. [8], [9], [30] and [31]), and there is much related work (cf. [6], [23], [24], [25], [34], [35], [38], [39] and [43]). There is moreover an extensive $p$-adic theory, and this will be the theme of the following memoir in these Proceedings.

## 2. RESULTS

We shall adopt the notation of Section 1 without change. Thus $\Lambda$ is the linear form specified at the beginning of Section 1, where $\alpha_j$ and $\beta_j$ are algebraic numbers with heights at most $A_j(\geqslant 4)$ and $B(\geqslant 4)$ respectively. The field $\mathbb{K}$ generated by the $\alpha$'s and $\beta$'s over the rationals has degree at most $d$, $\Omega$ is defined by (7), and $\Omega' = \Omega/\log A_n$. We prove:

THEOREM 1. *If* $\Lambda \neq 0$ *then* $|\Lambda| > (B\Omega)^{-C\Omega \log \Omega'}$, *where* $C = (16nd)^{200n}$.

When $\beta_0 = 0$ and $\beta_1, \ldots, \beta_n$ are rational integers, the bracketed factor $\Omega$ can be eliminated to yield:

THEOREM 2. *If, in the rational case,* $\Lambda \neq 0$, *then* $|\Lambda| > B^{-C\Omega \log \Omega'}$, *where* $C = (16nd)^{200n}$.

It will be seen that no particular significance attaches to the constant 200 occurring in the expression for $C$, and it could in fact be substantially reduced if, for instance, one imposed minor restrictions on $n$ or $d$. The theorems include (4), (6), (8) and the results of Stark and Shorey referred to above; and they could certainly be generalized to include also (5). Further, as noted by van der Poorten [33], Theorem 2 yields at once a result announced by Chudnovsky to the effect that $|\Lambda| > B^{-C\Omega \log B}$; for if $\Omega' < B$ then the assertion is obvious, and if $\Omega' \geqslant B$ then it follows from (2). But the theorems do not include (2), and nor do they furnish numerical bounds as precise as those given by (3). In fact, the analogue of (3), applicable when $\Lambda = 0$, is utilized in the proof of Theorem 1, and both (2) and (3) occur in the proof of Theorem 2. Apart from this, however, and also some preliminary lemmas relating mainly to Kummer theory, the proof of Theorem 1 will be essentially self-contained. Theorem 2, on the other hand, is new only with respect to the value of $C$ and it will suffice therefore to give the demonstration in outline.

In the course of this conference I learnt that Cijsouw and Waldschmidt had obtained a result similar to Theorem 1 but with $\Omega$ in place of $\Omega'$ and with an unspecified value of $C$; and Loxton and van der Poorten had derived Theorem 2, with in fact rather better numerical values, but subject to a restrictive condition concerning the $q$th roots of $\alpha_1, \ldots, \alpha_n$ (cf. Section 6).

## 3. PRELIMINARIES

For any integer $h \geqslant 1$, let $v(h)$ denote the least common multiple of $1, \ldots, h$. Let

$$\Delta(x; h) = (x+1)(x+2)\ldots(x+h)/h!$$

and let $\Delta(x; 0) = 1$. Further, for any integers $l \geqslant 0, m \geqslant 0$, write

$$\Delta(x; h, l, m) = \frac{1}{m!}\frac{d^m}{dx^m}(\Delta(x; h))^l$$

The following lemma, due to Tijdeman [45], improves upon an earlier version given in [4: I].

LEMMA 1. *Let $q$ and $qx$ be positive integers and let $\Delta = \Delta(x; h, l, m)$. Then $q^{2hl}(v(h))^m\Delta$ is a positive integer and we have $\Delta \leqslant 4^{l(x+h)}$, $v(h) \leqslant 4^h$.*

*Proof.* For the last estimate we have merely to note that the number $\pi(h)$ of primes $p \leqslant h$ is at most $\frac{4}{3}h/\log h$, whence

$$v(h) = \prod_{p \leqslant h} p^{[\log h/\log p]} \leqslant \prod_{p \leqslant h} h = h^{\pi(h)} \leqslant 4^h.$$

To estimate $\Delta$ we observe that

$$\Delta = (\Delta(x; h))^l \sum ((x + j_1) \ldots (x + j_m))^{-1},$$

where the summation is over all selections $j_1, \ldots, j_m$ of $m$ integers from the set $1, \ldots, h$ repeated $l$ times, and the right-hand side is read as 0 if $m > hl$. Thus for $x > 0$ we have

$$\Delta \leqslant \binom{[x] + h + 1}{h}^l \binom{hl}{m} \leqslant 2^{(2x+h)l+hl} = 4^{l(x+h)}.$$

Now from the above summation formula it is plain that $q^{hl-m}(h!)^l\Delta$ is a positive integer. Further, by a well-known counting argument, we see that if $p^r$ is the highest power of a prime $p$ that divides $h!$ then also $p^r$ divides $h!q^h\Delta(x; h)$, provided that $(p, q) = 1$. Furthermore, the same holds if any individual factor $qx + qj$ in $q^h\Delta(x; h)$ is replaced by $v(h)$. Thus the denominator of $q^{hl-m}(v(h))^m\Delta$, when expressed in lowest terms, is free of all primes that do not divide $q$. To complete the proof we recall that $r \leqslant h$, whence, if $p$ divides $q$, then $q^{hl}$ contains at least as many factors $p$ as $(h!)^l$. The desired result clearly follows. It improves upon the original version to the extent that $v(h)$ now replaces the lowest common multiple of $qx + q, \ldots, qx + qh$.

We record some further lemmas that will be needed later. The proofs of Lemmas 2 and 3 are given in [6: pp. 13, 26], and the proofs of Lemmas 4 and 5 in [7]. Lemma 2 is obtained as a consequence of Dirichlet's box principle; it is often referred to as Siegel's lemma, though in fact the form we shall give is slightly more precise than Siegel's. Lemma 3 provides a basis for the space of polynomials with bounded degree; a particular case was first utilized in this context by Fel'dman [11]. Lemma 4 arises from Kummer theory and Lemma 5 follows from Gauss' lemma in algebraic number fields. As remarked in Section 1, Lemma 6 is implicit in the work of [2: IV].

LEMMA 2. *Let $M$, $N$ denote integers with $N > M > 0$ and let $u_{ij}(1 \leqslant i \leqslant M,$*

*$1 \leqslant j \leqslant N$) denote integers with absolute values at most $U(\geqslant 1)$. Then there exist integers $x_1, \ldots, x_N$, not all 0, with absolute values at most $(NU)^{M/(N-M)}$ such that*

$$\sum_{j=1}^{N} u_{ij} x_j = 0 \qquad (1 \leqslant i \leqslant M).$$

LEMMA 3. *If $P(x)$ is a polynomial with degree $n > 0$ and with coefficients in a field $K$ then, for any integer $m$ with $0 \leqslant m \leqslant n$, the polynomials $P(x)$, $P(x+1)$, $\ldots$, $P(x+m)$ and $1, x, \ldots, x^{n-m-1}$ are linearly independent over $K$.*

LEMMA 4. *Let $\alpha_1, \ldots, \alpha_n$ be non-zero elements of an algebraic number field $K$ and let $\alpha_1^{1/p}, \ldots, \alpha_n^{1/p}$ denote fixed pth roots for some prime $p$. Further let $K' = K(\alpha_1^{1/p}, \ldots, \alpha_{n-1}^{1/p})$. Then either $K'(\alpha_n^{1/p})$ is an extension of $K'$ of degree $p$ or we have*

$$\alpha_n = \alpha_1^{j_1} \ldots \alpha_{n-1}^{j_{n-1}} \gamma^p$$

*for some $\gamma$ in $K$ and some integers $j_1, \ldots, j_{n-1}$ with $0 \leqslant j_r \leqslant p$.*

LEMMA 5. *Suppose that $\alpha$, $\beta$ are elements of an algebraic number field with degree $d$ and that for some positive integer $p$ we have $\alpha = \beta^p$. If $a\alpha$ is an algebraic integer for some positive rational integer $a$, and if $b$ is the leading coefficient in the minimal defining polynomial of $\beta$ then $b \leqslant a^{d/p}$.*

LEMMA 6. *If, in the rational case, $\Lambda = 0$, then in fact $\Lambda = 0$ for some integers $\beta_1, \ldots, \beta_n$, not all 0, with absolute values at most*

$$(4^{n^2} d^{2n} \log A)^{(2n+1)^2}.$$

## 4. NOTATION

We assume now that the $\alpha$'s and $\beta$'s are elements of a field $\mathbb{K}$ with degree at most $d(\geqslant 8)$, and we put $k = (nd)^{40n}$. We define

$$L = k\Omega \log \Omega', \qquad h = L_{-1} + 1 = [\log(BL)],$$

$$L_j = [k^{-\varepsilon} L/\log A_j] \qquad (0 \leqslant j \leqslant n),$$

where $A_0 = \Omega'$, and $\varepsilon = 1/(3n)$. We introduce the function

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