I. V. Proskuryakov

Problems in Linear Algebra

Mir Publishers Moscow

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Translated from the Russian
by
GEORGE YANKOVSKY

Mir
Publishers
Moscow

First published 1978
Revised from the 1974 Russian edition

На английском языке

OTHER MIR TITLES

ELEMENTARY MATHEMATICS (A REVIEW COURSE)

by V. ZAITSEV, Cand. Sc., V. RYZHKOV, D. Sc., and M. SKANAVI, Cand. Sc.

This text constitutes a complete review course of secondary-school mathematics.

Contents. **Arithmetic, Algebra and Elementary Functions.** Real and Complex Numbers. Identical Transformations. Logarithms. Functions and Graphs. Equations. Inequalities. Sequences. Trigonometric Functions of an Angile (ARS). Trigonometric Functions of a Numerical Argument and Their Graphs. Transforming Trigonometric Expressions. Inverse Trigonometric Functions and Their Graphs. Trigonometric Equations and Inequalities. **Geometry.** Basic Concepts. Perpendiculars and Parallel Lines. Construction Problems. Triangles and Quadrilaterals. Similarity of Geometrical Figures. Metric Relations in a Triangle and a Circle. Regular Polygons. The Circumference and Area of a Circle. Straight Lines and Planes in Space. Polyhedrons and Circular Solids. Answers to Exercises. Appendices. Index.

THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

by A. SVESHNIKOV, D. Sc., and A. TIKHONOV, Mem. USSR Acad. Sc.

This textbook is intended for students of physico-mathematical departments of colleges and universities; can be used as a reference book by postgraduate students and research workers.

Contents. The Complex Variable and Functions of a Complex Variable. Series of Analytic Functions. Analytic Continuation. Elementary Functions of a Complex Variable. The Lauret Series and Isolated Singular Points. Conformal Mapping. Analytic Functions in the Solution of Boundary-Value Problems. Fundamentals of Operational Calculus. Saddle-Point Method. The Wiener-Hopf Method. Bibliography. Name Index. Subject Index.

Preface

In preparing this book of problems the author attempted, firstly, to give a sufficient number of exercises for developing skills in the solution of typical problems (for example, the computing of determinants with numerical elements, the solution of systems of linear equations with numerical coefficients, and the like), secondly, to provide problems that will help to clarify basic concepts and their interrelations (for example, the connection between the properties of matrices and those of quadratic forms, on the one hand, and those of linear transformations, on the other), thirdly, to provide for a set of problems that might supplement the course of lectures and help to expand the mathematical horizon of the student (instances are the properties of the Pfaffian of the skew-symmetric determinant, the properties of associated matrices, and so on).

A number of problems involving the proof of theorems that can be found in textbooks are also given. These problems were included because the instructor often (for lack of time) gives part of the material as homework for the student on the basis of the textbook, and this can be done on the basis of the problem book where hints are given for working out proofs. The author feels this can help to develop habits of scientific investigation.

Compared with other problem books, this one has a few new basic features. They include problems dealing with polynomial matrices (Sec. 13), linear transformations of affine and metric spaces (Secs. 18 and 19), and a supplement devoted to groups, rings, and fields. The problems of the supplement deal with the most elementary portions of the theory. Still and all, I think it can be used in pre-seminar discussions in the first and second years of study.

The contents and the sequence of presentation of material in a lecture course depend largely on the lecturer. The author has tried to take into account this diversity of presentation, and the result has been a certain amount of duplication. For example, the same facts are given first in the section devoted to quadratic forms and then again in the chapter on linear transformations; some of the problems are stated so that they can be worked out in the case of a real Euclidean space and also in that of a complex unitary space. I believe that this makes for a certain flexibility of use of the problem book.

Some sections contain an introduction with a few definitions and a brief discussion of terminology and notation when the available textbooks do not exhibit complete unity in this respect. An exception is the introduction to Sec. 5, where basic methods presented for computing determinants of any order and examples of each method are given. This was done because such information is usually not given in standard textbooks, and students encounter considerable difficulties.

Starred numbers indicate problems that have been worked out or provided with hints. Solutions are given for a small number of problems. These are either problems involving a general method that is then applied to a series of other problems (for instance, problem 1151 which offers a method for computing the function of a matrix, problem 1529 which contains the construction of a basis in which the matrix of a linear transformation has Jordan form) or problems that are very difficult (say, problems 1433, 1614, 1617). As a rule, the hints contain only a suggestion of the idea or method of solution and leave to the student the actual solving. Only in the case of more difficult problems is a general plan of solution provided (see problems 546, 1492, 1632).

The author expresses his deep gratitude to the staff of the Department of higher algebra at the Moscow State University for their helpful comments which the author made use of in the preparation of the book.

I. Proskuryakov

Moscow
May 20, 1978

Contents

SUPPLEMENT

ANSWERS

Chapter I

Determinants

Sec. 1. Second- and Third-Order Determinants

Compute the following determinants:

1. $\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix}$. **2.** $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$. **3.** $\begin{vmatrix} 3 & 2 \\ 8 & 5 \end{vmatrix}$. **4.** $\begin{vmatrix} 6 & 9 \\ 8 & 12 \end{vmatrix}$.

5. $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$. **6.** $\begin{vmatrix} n+1 & n \\ n & n-1 \end{vmatrix}$. **7.** $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$.

8. $\begin{vmatrix} a^2+ab+b^2 & a^2-ab+b^2 \\ a+b & a-b \end{vmatrix}$. **9.** $\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$.

10. $\begin{vmatrix} \sin\alpha & \cos\alpha \\ \sin\beta & \cos\beta \end{vmatrix}$. **11.** $\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta \end{vmatrix}$.

12. $\begin{vmatrix} \sin\alpha+\sin\beta & \cos\beta+\cos\alpha \\ \cos\beta-\cos\alpha & \sin\alpha-\sin\beta \end{vmatrix}$.

13. $\begin{vmatrix} 2\sin\varphi\cos\varphi & 2\sin^2\varphi-1 \\ 2\cos^2\varphi-1 & 2\sin\varphi\cos\varphi \end{vmatrix}$.

14. $\begin{vmatrix} \frac{1-t^2}{1+t^2} & \frac{2t}{1+t^2} \\ \frac{-2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{vmatrix}$. **15.** $\begin{vmatrix} \frac{(1-t)^2}{1+t^2} & \frac{2t}{1+t^2} \\ \frac{2t}{1+t^2} & -\frac{(1+t)^2}{1+t^2} \end{vmatrix}$.

16. $\begin{vmatrix} \frac{1+t^2}{1-t^2} & \frac{2t}{1-t^2} \\ \frac{2t}{1-t^2} & \frac{1+t^2}{1-t^2} \end{vmatrix}$. **17.** $\begin{vmatrix} 1 & \log_b a \\ \log_a b & 1 \end{vmatrix}$.

Compute the following determinants ($i=\sqrt{-1}$):

18. $\begin{vmatrix} a & c+di \\ c-di & b \end{vmatrix}$. **19.** $\begin{vmatrix} a+bi & b \\ 2a & a-bi \end{vmatrix}$.

20. $\begin{vmatrix} \cos\alpha + i\sin\alpha & 1 \\ 1 & \cos\alpha - i\sin\alpha \end{vmatrix}$. 21. $\begin{vmatrix} a+bi & c+di \\ -c+di & a-bi \end{vmatrix}$.

Use determinants to solve the following systems of equations:

22. $2x+5y=1,$ $3x+7y=2.$

23. $2x-3y=4,$ $4x-5y=10.$

24. $5x-7y=1,$ $x-2y=0.$

25. $4x+7y+13=0,$ $5x+8y+14=0.$

26. $x\cos\alpha - y\sin\alpha = \cos\beta,$ $x\sin\alpha + y\cos\alpha = \sin\beta.$

27. $x\tan\alpha + y = \sin(\alpha+\beta),$ $x - y\tan\alpha = \cos(\alpha+\beta),$

where $\alpha \neq \frac{\pi}{2} + k\pi$ (k an integer).

Investigate to see whether the given system of equations is evendetermined (has a unique solution), indeterminate (has an infinity of solutions) or is inconsistent (no solution):

28. $4x+6y=2,$ $6x+9y=3.$ (Do Cramer's formulas yield a correct answer?)

29. $3x-2y=2,$ $6x-4y=3.$

30. $(a-b)x=b-c.$

31. $x\sin\alpha = 1+\sin\alpha.$

32. $x\sin\alpha = 1+\cos\alpha.$

33. $x\sin(\alpha+\beta) = \sin\alpha + \sin\beta.$

34. $a^2x=ab,$ $abx=b^2.$

35. $ax+by=ad,$ $bx+cy=bd.$

36. $ax+4y=2,$ $9y+ay=3.$

37. $ax-9y=6,$ $10x-by=10.$

38. Prove that for a determinant of the second order to be equal to zero it is necessary and sufficient that the rows be proportional. The same holds true for columns as well (if certain elements of the determinant are zero, the proportionality may be understood in the sense that the elements of one row are obtained from the corresponding elements of another row by multiplying by the same number, which may even be zero).

***39.** Prove that when a, b, c are real, the roots of the equation $\begin{vmatrix} a-x & b \\ b & c-x \end{vmatrix}=0$ are real.

***40.** Prove that the quadratic trinomial $ax^2+2bx+c$ with complex coefficients is a perfect square if and only if

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix}=0$$

41. Prove that for real a, b, c, d, the roots of the equation $\begin{vmatrix} a-x & c+di \\ c-di & b-x \end{vmatrix}=0$ are real.

***42.** Show that the value of the fraction $\frac{ax+b}{cx+d}$, where at least one of the numbers c, d is nonzero, is not dependent on the value of x if and only if $\begin{vmatrix} a & b \\ c & d \end{vmatrix}=0$.

Compute the following third-order determinants:

43. $\begin{vmatrix} 2 & 1 & 3 \\ 5 & 3 & 2 \\ 1 & 4 & 3 \end{vmatrix}$. **44.** $\begin{vmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 4 & 2 \end{vmatrix}$. **45.** $\begin{vmatrix} 4 & -3 & 5 \\ 3 & -2 & 8 \\ 1 & -7 & -5 \end{vmatrix}$.

46. $\begin{vmatrix} 3 & 2 & -4 \\ 4 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}$. **47.** $\begin{vmatrix} 3 & 4 & -5 \\ 8 & 7 & -2 \\ 2 & -1 & 8 \end{vmatrix}$. **48.** $\begin{vmatrix} 4 & 2 & -1 \\ 5 & 3 & -2 \\ 3 & 2 & -1 \end{vmatrix}$.

49. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}$. **50.** $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$. **51.** $\begin{vmatrix} 5 & 6 & 3 \\ 0 & 1 & 0 \\ 7 & 4 & 5 \end{vmatrix}$.

52. $\begin{vmatrix} 2 & 0 & 3 \\ 7 & 1 & 6 \\ 6 & 0 & 5 \end{vmatrix}$. **53.** $\begin{vmatrix} 1 & 5 & 25 \\ 1 & 7 & 49 \\ 1 & 8 & 64 \end{vmatrix}$. **54.** $\begin{vmatrix} 1 & 1 & 1 \\ 4 & 5 & 9 \\ 16 & 25 & 81 \end{vmatrix}$.

55. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$. **56.** $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. **57.** $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

58. $\begin{vmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{vmatrix}$. **59.** $\begin{vmatrix} a & x & x \\ x & b & x \\ x & x & c \end{vmatrix}$. **60.** $\begin{vmatrix} a+x & x & x \\ x & b+x & x \\ x & x & c+x \end{vmatrix}$.

61. $\begin{vmatrix} \alpha^2+1 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2+1 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2+1 \end{vmatrix}$.

62. $\begin{vmatrix} \cos\alpha & \sin\alpha\cos\beta & \sin\alpha\sin\beta \\ -\sin\alpha & \cos\alpha\cos\beta & \cos\alpha\sin\beta \\ 0 & -\sin\beta & \cos\beta \end{vmatrix}$.

63. $\begin{vmatrix} \sin\alpha & \cos\alpha & 1 \\ \sin\beta & \cos\beta & 1 \\ \sin\gamma & \cos\gamma & 1 \end{vmatrix}$.

64. Determine under what condition the following equation holds true:

$$\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}.$$

65. Show that the determinant

$$\begin{vmatrix} a^2 & b\sin\alpha & c\sin\alpha \\ b\sin\alpha & 1 & \cos\alpha \\ c\sin\alpha & \cos\alpha & 1 \end{vmatrix}$$

and two other determinants obtained from this one by a circular permutation of the elements a, b, c and α, β, γ are equal to zero if a, b, c are the lengths of the sides of a triangle and α, β, γ are the angles opposite the sides a, b, c, respectively.

Evaluate the following third-order determinants ($i = \sqrt{-1}$):

66. $\begin{vmatrix} 1 & 0 & 1+i \\ 0 & 1 & i \\ 1-i & -i & 1 \end{vmatrix}$. **67.** $\begin{vmatrix} x & a+bi & c+di \\ a-bi & y & e+fi \\ c-di & e-fi & z \end{vmatrix}$.

68. $\begin{vmatrix} 1 & \varepsilon & \varepsilon^2 \\ \varepsilon^2 & 1 & \varepsilon \\ \varepsilon & \varepsilon^2 & 1 \end{vmatrix}$, where $\varepsilon = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

69. $\begin{vmatrix} 1 & 1 & \varepsilon \\ 1 & 1 & \varepsilon^2 \\ \varepsilon^2 & \varepsilon & 1 \end{vmatrix}$, where $\varepsilon = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi$.

70. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon \end{vmatrix}$, where $\varepsilon = \cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi$.

71. Prove that if all the elements of a third-order determinant are equal to ± 1, then the determinant itself will be an even number.

***72.** Find the maximum value that can be assumed by a third-order determinant provided that all its elements are equal to ± 1.

***73.** Find the largest value of a third-order determinant provided that the elements are equal to $+1$ or 0.

Use determinants to solve the following systems of equations:

74. $2x + 3y + 5z = 10,$
$3x + 7y + 4z = 3,$
$x + 2y + 2z = 3.$

75. $5x - 6y + 4z = 3,$
$3x - 3y + 2z = 1,$
$4x - 5y + 2z = 1.$

76. $4x - 3y + 2z + 4 = 0,$
$6x - 2y + 3z + 1 = 0,$
$5x - 3y + 2z + 3 = 0.$

77. $5x + 2y + 3z + 2 = 0,$
$2x - 2y + 5z = 0,$
$3x + 4y + 2z + 10 = 0.$

***78.** $\frac{x}{a} - \frac{y}{b} + 2 = 0,$
$-\frac{2y}{b} + \frac{3z}{c} - 1 = 0,$
$\frac{x}{a} + \frac{z}{c} = 0.$

79. $2ax - 3by + cz = 0,$
$3ax - 6by + 5cz = 2abc,$
$5ax - 4by + 2cz = 3abc,$
where $abc \neq 0$.

***80.** $4bcx + acy - 2abz = 0,$
$5bcx + 3acy - 4abz + abc = 0,$
$3bcx + 2acy - abz - 4abc = 0$ $(abc \neq 0)$.

***81.** Solve the following system of equations:

$x + y + z = a,$
$x + \varepsilon y + \varepsilon^2 z = b,$
$x + \varepsilon^2 y + \varepsilon z = c,$
(ε is a value of $\sqrt[3]{1}$ different from 1).

Investigate each system of equations to determine whether it is evendetermined, indeterminate, or inconsistent:

82. $2x-3y+z=2,$
$3x-5y+5z=3,$
$5x-8y+6z=5.$

83. $4x+3y+2z=1,$
$x+3y+5z=1,$
$3x+6y+9z=2.$

84. $5x-6y+z=4,$
$3x-5y-2z=3,$
$2x-y+3z=5.$

85. $2x-y+3z=4,$
$3x-2y+2z=3,$
$5x-4y=2.$

86. $2ax-23y+29z=4,$
$7x+ay+4z=7,$
$5x+2y+az=5.$

87. $ax-3y+5z=4,$
$x-ay+3z=2,$
$9x-7y+8az=0.$

88. $ax+4y+z=0,$
$2y+3z-1=0,$
$3x-bz+2=0.$

89. $ax+2z=2,$
$5x+2y=1,$
$x-2y+bz=3.$

Prove, either by direct computation via the triangle rule or by the rule of Sarrus, the following properties of third-order determinants:

90. A determinant of the third order remains unchanged if the rows and columns are interchanged (that is to say, if the matrix of the determinant is transposed).

91. If all elements of some row (or column) are equal to zero, then the determinant itself is equal to zero.

92. If all elements of some row (or column) of a determinant are multiplied by one and the same number, the whole determinant is then multiplied by that number.

93. A determinant changes sign if two rows (or two columns) are interchanged.

94. If two rows (or two columns) of a determinant are the same, the determinant is zero.

95. If all elements of one row are proportional to the corresponding elements of another row, the determinant is equal to zero (the same holds true for the columns).

96. If each element of some row of a determinant is represented as the sum of two terms, then the determinant is equal to the sum of two determinants in which all rows, except the given row, remain the same, while the given row in the first determinant contains the first terms and the given row in the second determinant contains the second terms (the same holds true for the columns).

97. A determinant remains unchanged if elements of one row of the determinant are added to the corresponding elements of another row multiplied by the same number (the same applies to columns).

98. We say that one row of a determinant is a *linear combination* of the other rows if each element of the given row is equal to the sum of the products of the corresponding elements of the other rows into certain numbers that are constant for each row, that is to say, that are independent of the position number of the element in the row. A similar definition applies to a linear combination of columns. For example, the third row of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is a linear combination of the first two if there exist numbers c_1 and c_2 such that $a_{3,\,j} = c_1 a_{1,\,j} + c_2 a_{2,\,j}$ $(j = 1, 2, 3)$.

Prove that if one row (or column) of a third-order determinant is a linear combination of the other rows (or columns), the determinant is equal to zero.

Hint. The converse is also true, but it follows from a further development of the theory of determinants.

***99.** Use the preceding problem to demonstrate with an illustration that, unlike second-order determinants (see problem 38), the proportionality of two rows (or columns) is no longer necessary for a third-order determinant to be equal to zero.

Using the properties of third-order determinants given in problems 91-98, compute the following determinants:

100. $\begin{vmatrix} \sin^2\alpha & 1 & \cos^2\alpha \\ \sin^2\beta & 1 & \cos^2\beta \\ \sin^2\gamma & 1 & \cos^2\gamma \end{vmatrix}.$ **101.** $\begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix}.$

102. $\begin{vmatrix} x & x' & ax+bx' \\ y & y' & ay+by' \\ z & z' & az+bz' \end{vmatrix}.$ **103.** $\begin{vmatrix} (a_1+b_1)^2 & a_1^2+b_1^2 & a_1b_1 \\ (a_2+b_2)^2 & a_2^2+b_2^2 & a_2b_2 \\ (a_3+b_3)^2 & a_3^2+b_3^2 & a_3b_3 \end{vmatrix}.$

104. $\begin{vmatrix} a+b & c & 1 \\ b+c & a & 1 \\ c+a & b & 1 \end{vmatrix}.$ **105.** $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^y+b^{-y})^2 & (b^y-b^{-y})^2 & 1 \\ (c^z+c^{-z})^2 & (c^z-c^{-z})^2 & 1 \end{vmatrix}.$

106. $\begin{vmatrix} 1 & \varepsilon & \varepsilon^2 \\ \varepsilon & \varepsilon^2 & 1 \\ x & y & z \end{vmatrix}$, where ε is a value of $\sqrt[3]{1}$ different from 1.

107. $\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$.

108. $\begin{vmatrix} a_1+b_1i & a_1i-b_1 & c_1 \\ a_2+b_2i & a_2i-b_2 & c_2 \\ a_3+b_3i & a_3i-b_3 & c_3 \end{vmatrix}$, where $i=\sqrt{-1}$.

109. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \frac{x_1+\lambda x_2}{1+\lambda} & \frac{y_1+\lambda y_2}{1+\lambda} & 1 \end{vmatrix}$ (give a geometrical interpretation of the result obtained).

***110.** $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$, where α, β, γ are roots of the equation $x^3+px+q=0$.

Prove the following identities without expanding the determinants:

111. $\begin{vmatrix} a_1 & b_1 & a_1x+b_1y+c_1 \\ a_2 & b_2 & a_2x+b_2y+c_2 \\ a_3 & b_3 & a_3x+b_3y+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

112. $\begin{vmatrix} a_1+b_1x & a_1-b_1x & c_1 \\ a_2+b_2x & a_2-b_2x & c_2 \\ a_3+b_3x & a_3-b_3x & c_3 \end{vmatrix} = -2x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

113. $\begin{vmatrix} a_1+b_1i & a_1i+b_1 & c_1 \\ a_2+b_2i & a_2i+b_2 & c_2 \\ a_3+b_3i & a_3i+b_3 & c_3 \end{vmatrix} = 2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

114. $\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

115. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (b-a)(c-a)(c-b)$.