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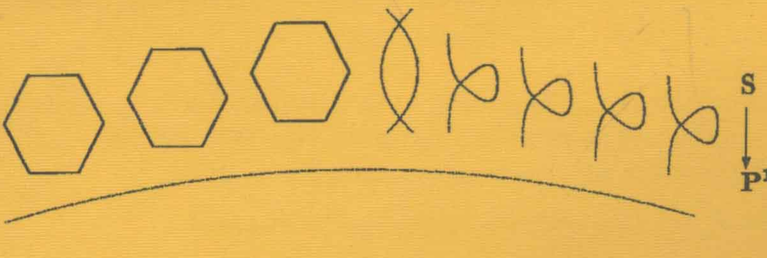
Bruce Hunt

The Geometry of some special Arithmetic Quotients



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The Geometry of some special Arithmetic Quotients

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Author

Bruce Hunt
MPI für Mathematik in den Naturwissenschaften
Inselstr. 22–26
D-04103 Leipzig, Germany

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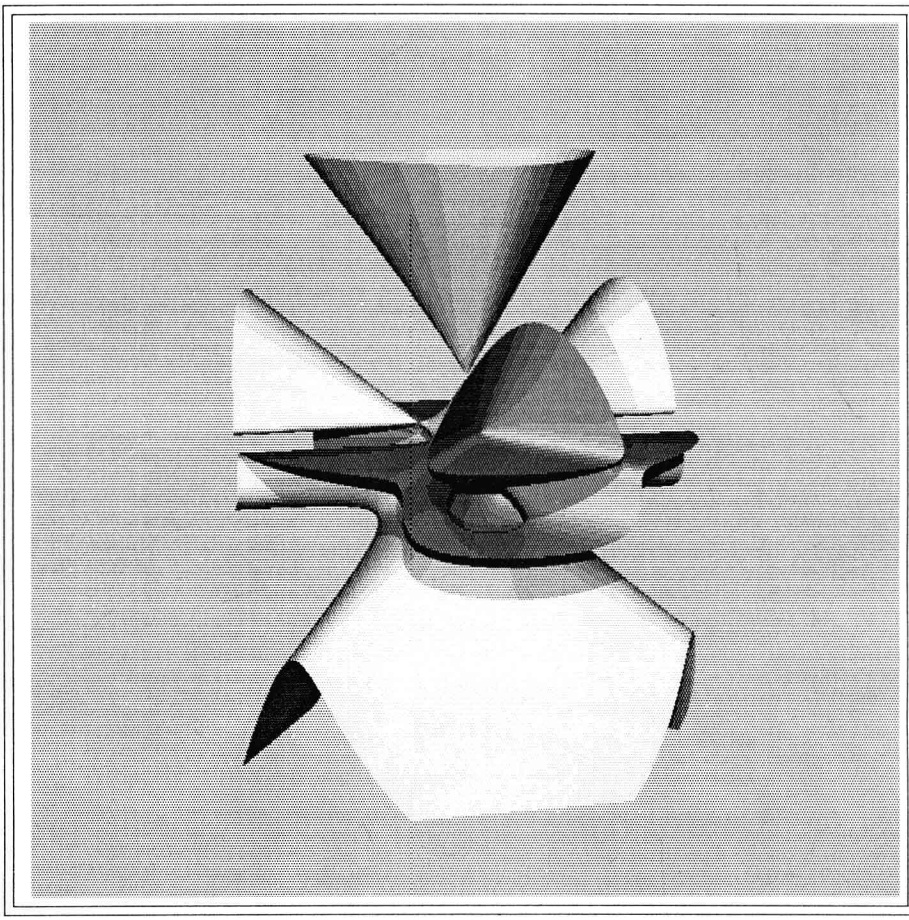

A space section of the invariant quintic $\mathcal{I}_5$

More pictures in living color are available at the WWW site:

`http://www.mathematik.uni-kl.de/~wwwagag/Galerie.html`

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Chapter 0

Introduction

> ... daß man eine ... Gruppe ... an die Spitze stellen und den Stoff nach den Untergruppen ordnen soll, die in der Gesamtgruppe erhalten sind, ... die Gruppentheorie als ordnendes Prinzip im Wirrsal der Erscheinungen zu benutzen.
>
> — Felix Klein

Geometry is one of the oldest and most basic branches of mathematics, as is algebra. Nowhere is the interplay between the two more pronounced than in group theory, and that interplay, with group theory acting as a mediator between geometry and algebra, is the theme of this book. Group theory had its genesis in a decidedly algebraic context, solving algebraic equations (Galois theory). It was Felix Klein in his "Erlanger Programm"[1] who put group theory at the basis of geometry. At that time (1872) he had been pursuing studies with Sophius Lie on one-parameter families of algebraic curves[2], whose invariants, as they had noticed, were correlated. This was the advent of continuous groups, here essentially one-parameter subgroups of the automorphism group of the projective plane, $PGL(3, \mathbb{C})$. Klein advocated considering geometry as "invariance properties under a group of automorphisms", and using groups as a basis of classifications of objects, which otherwise do not seem easily related with one another (like formulas for elliptic functions). We shall take this standpoint and consider the geometry of a very special kind of object, namely that of arithmetic quotients of bounded symmetric domains. We shall require groups at several different levels.

Level 1: *Real Lie groups and symmetric spaces.* A Riemannian manifold X is said to be symmetric, if at any point $x \in X$ there is a symmetry σ_x. A symmetry is an automorphism of X (i.e., a diffeomorphism preserving the Riemannian

[1] "Vergleichende Betrachtungen über neuere geometrische Forschungen" was the title of the talk

[2] Über diejenigen ebenen Kurven, welche durch ein geschloßenes System von einfach unendlich vielen vertauschbaren Transformationen in sich übergehen

structure) which is involutive and has x as an isolated fixed point. The automorphism group of a symmetric space X, $\mathrm{Aut}(X)$, is a real Lie group, and the symmetry at each point of X defines an involution of $\mathrm{Aut}(X)$ by $g \mapsto \sigma_x^{-1} \circ g \circ \sigma_x$. X decomposes into a product $X = X_1 \times \cdots \times X_s$, where each X_i is irreducible; on each irreducible component the curvature is negative, zero or positive. This occurs when $\mathrm{Aut}(X)$ is non-compact simple, abelian or compact simple, respectively. The abelian case yields the Euclidean geometry, and the other cases yield a correspondence:

$$\{\text{real simple Lie groups}\} \longleftrightarrow \{\text{irreducible symmetric spaces of negative or positive curvature}\}.$$

Classifying the symmetric spaces amounts to a classification of involutary automorphisms of compact, simple Lie groups, and was first accomplished by É. Cartan in 1926.

Level 2: *Discrete subgroups of Lie groups and locally symmetric spaces.* Let $G = \mathrm{Aut}(X)$ be as in Level 1, and let $\Gamma \subset G$ be a discrete subgroup, which acts properly discontinuously on X. This assures that the quotient $\Gamma\backslash X$ is a Hausdorf space, and we assume, henceforth, it is of finite volume; Γ is then called a lattice in G. The notion of symmetric spaces of Level 1 can be expressed locally by the condition: the curvature tensor is parallel with respect to the Levi-Cevita connection, and a Riemannian manifold Y is called locally symmetric if this condition is satisfied. The universal cover $\widetilde{Y}$ of Y is a symmetric space as in Level 1, and of course $Y = \pi_1(Y)\backslash\widetilde{Y}$, with the fundamental group $\pi_1(Y)$ acting properly discontinuously. If $\widetilde{Y}$ is compact, then $\pi_1(Y)$ is finite. Interesting things occur if $\widetilde{Y}$ is non-compact, so we get a correspondence, specializing that of Level 1,

$$\left\{\begin{array}{l}\text{pairs } (G,\Gamma),\ \Gamma \text{ a lattice in } G,\ G \text{ a non-compact}\\ \text{semisimple real Lie group}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{locally symmetric spaces of non-positive cur-}\\ \text{vature and finite volume}\end{array}\right\}.$$

Remark: We must formulate this in terms of semisimple groups, since the quotient $\Gamma\backslash X$ may be irreducible, even if the domain X is reducible.

Level 3: *Normal subgroups of finite index in discrete groups and locally symmetric spaces with automorphism groups.* Discrete subgroups $\Gamma \subset G$ as in Level 2 tend to have lots of normal subgroups of finite index, $\Gamma' \subset \Gamma$. Consider the locally symmetric spaces $\Gamma\backslash X$ and $\Gamma'\backslash X$. Since Γ' is normal in Γ, the finite group Γ/Γ' acts on $\Gamma'\backslash X$ with quotient $\Gamma\backslash X$, i.e., the natural morphism $\pi_{\Gamma'|\Gamma} : \Gamma'\backslash X \longrightarrow \Gamma\backslash X$ is a Galois cover. Assuming Γ is torsion free it is étale, while if Γ has torsion this cover will be branched. In this way we get locally symmetric spaces with automorphism groups (here Γ/Γ') and a correspondence

$$\left\{\begin{array}{l}\text{triples } (G,\Gamma,\Gamma'),\ \text{with } (G,\Gamma) \text{ as in}\\ \text{Level 2},\ \Gamma' \lhd \Gamma,\ [\Gamma:\Gamma'] < \infty\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{locally symmetric spaces of non-positive curvature and of}\\ \text{finite volume with automorphism group}\end{array}\right\}.$$

The more interesting the group Γ/Γ' is, the more interesting the automorphism group of $\Gamma'\backslash X$ is.

Our concern in this book is basically with Level 3; we are interested in the geometry of particular locally symmetric spaces with interesting automorphism group. We will be placing two conditions on these data, namely we assume the discrete group Γ is *arithmetic* and the symmetric space X is *hermitian*; these two conditions are logically independent. Consider the first. Assuming the discrete group Γ is arithmetic necessitates introducing a new object into the picture, an algebraic group, which we may assume is defined over $\mathbb{Q}$. This algebraic group, call it $G_{\mathbb{Q}}$, is, compared with the real Lie group, a rather mysterious object: it is an algebraic scheme defined over $\mathbb{Q}$. In fact, the real Lie group is just "a tiny part" of $G_{\mathbb{Q}}$, namely the group of $\mathbb{R}$-valued points of the algebraic group:

$$G_{\mathbb{Q}}(\mathbb{R}) \cong G.$$

This very statement shows that, as far as notation is concerned, things can get very confusing in this business, and we must be very careful in choosing notation and making statements.

A discrete group $\Gamma \subset G$ is arithmetic, if there is a $\mathbb{Q}$-group $G_{\mathbb{Q}}$ with $\Gamma \subset G_{\mathbb{Q}}(\mathbb{Q}) \subset G_{\mathbb{Q}}(\mathbb{R}) = G$, and a rational representation $\varrho : G_{\mathbb{Q}} \longrightarrow GL(V_{\mathbb{Q}})$ such that $\varrho^{-1}(GL(V_{\mathbb{Z}}))$ and Γ are commensurable. This presupposes a choice of lattice $\mathcal{L} = V_{\mathbb{Z}}$ such that $V_{\mathbb{Q}} = \mathcal{L} \otimes \mathbb{Q}$, and this is, or course, not entirely canonical. That is why the notion "preserves a lattice" is only well-defined on a commensurability class of groups. Now, note that since $G_{\mathbb{Q}}$ defines G, it also defines X, and this leads us to refine our levels of groups as follows:

Level 0: start with a semisimple algebraic group $G_{\mathbb{Q}}$ defined over $\mathbb{Q}$;
Level 1: the group of $\mathbb{R}$-points of $G_{\mathbb{Q}}$ is a real semisimple Lie group, $G_{\mathbb{Q}}(\mathbb{R}) = G$, and defines a symmetric space X;
Level 2: choose a lattice $\mathcal{L} \subset V_{\mathbb{Q}}$ and a representation $\varrho : G_{\mathbb{Q}} \longrightarrow GL(V_{\mathbb{Q}})$; this defines an arithmetic group

$$G_{\mathcal{L}} := \{g \in GL(V_{\mathbb{Q}}) \big| g(\mathcal{L}) \subset \mathcal{L}\};$$

Level 3: as above, normal subgroups $\Gamma' \subset G_{\mathcal{L}}$ determine arithmetic quotients with automorphism group.

There is a subtle point about algebraic groups which we mention here. Even if the group $G_{\mathbb{Q}}$ is simple over $\mathbb{Q}$ (that is, has no normal $\mathbb{Q}$-subgroups), there will, in general, be a finite field extension $k|\mathbb{Q}$ for which the lifted group G_k is no longer simple, but rather only semisimple, a product of simple groups. Consequently, the real group $G_{\mathbb{R}}$, and hence the symmetric space X, is also a product. In particular, it can and does happen that some factors of $G_{\mathbb{R}}$ may be *compact*, so this goes beyond the description we originally started with. It was proven quite early by Borel and Harish-Chandra that the existence of a compact factor of $G_{\mathbb{R}}$ implies that $G_{\mathbb{Q}}$ is anisotropic, and this in turn implies that any quotient $\Gamma\backslash X$ for an arithmetic group $\Gamma \subset G_{\mathbb{Q}}(\mathbb{Q})$ is *compact*. A deep theorem of Margulis states that if the $\mathbb{R}$-rank of $G_{\mathbb{R}}$ is ≥ 2, then *any* discrete subgroup

$\Gamma \subset G_{\mathbb{R}}$ is *automatically* arithmetic. So the condition "Γ is arithmetic" is only a condition for the rank 1 real Lie groups.

The second assumption we will be making, which is much more serious, is that X is *hermitian* symmetric. By definition, this means X has a complex structure compatible with the symmetric Riemannian structure, in other words that X is Kähler, with the Riemannian part of the hermitian metric being symmetric. In particular, X is Kähler homogenous, and this implies that the compact dual $\check{X}$ of X is an *algebraic* variety, and in fact it is a rational variety. There is a natural, group theoretic embedding $X \subset \check{X}$, displaying X as an (homogenous) open (in the Euclidean topology) subset of an (homogenous) algebraic variety. But it turns out that this assumption implies much more. First, taking the topological closure (in the Euclidean topology) of X in $\check{X}$ defines the *boundary* of X; this decomposes into irreducible pieces (holomorphic arc components), defining the *boundary components* of X. Then the following facts hold:

i) each maximal parabolic of $G_{\mathbb{Q}}$ is the normalizer of a boundary component;

ii) for $\Gamma \subset G_{\mathbb{Q}}$ arithmetic, the quotient $\Gamma\backslash X$ can be compactified to a normal, analytic space $(\Gamma\backslash X)^*$, which is projective algebraic $(\Gamma\backslash X)^* \subset \mathbb{P}^N$;

iii) the embedding $(\Gamma\backslash X)^* \subset \mathbb{P}^N$ is given by modular forms (with the usual exception of dimension 1 factors);

iv) there is a smooth compactification $\overline{\Gamma\backslash X}$ which resolves the singularities of $\Gamma\backslash X^*$, and for which $\Delta = \overline{\Gamma\backslash X} - \Gamma\backslash X$ is a normal crossings divisor.

Items ii) and iii) are the Baily-Borel embedding, iv) the toroidal compactification. These results display the fact that in this case the locally symmetric space $\Gamma\backslash X$ is an *algebraic* object, or more precisely, an object of algebraic geometry. And so we arrive at one of the main themes of this book: the geometry of arithmetic quotients of bounded symmetric domains is geometry in the sense of *algebraic geometry*. Let us pause for a moment to explain this statement. Generally speaking, a locally symmetric space is an object in the category of Riemannian manifolds, so geometry of them is clearly geometry in the sense of Riemannian geometry. Everything is expressed in terms of curvature, and the geometry *is* the geometry of that curvature tensor. On the other hand, in algebraic geometry, there is no curvature tensor to consider. Rather, one considers embeddings in projective space (like $\Gamma\backslash X^* \subset \mathbb{P}^N$) and their properties: singular locus, hyperplane sections (subvarieties), invariants (of the isomorphism class under $PGL(N+1, \mathbb{C})$), inflection points and the like. If a variety V has an interesting automorphism group, this usually induces a projective representation of the group, and there will be some "invariant configuration" in the ambient $\mathbb{P}^N$, of which V is only one aspect. This is what we understand by geometry when we speak of algebraic geometry of arithmetic quotients of bounded symmetric domains.

We first explain the geometry of the boundary. The Satake compactification X_{Γ}^* (we will from now on use the notation of the text: $\mathcal{D}$ denotes the non-compact hermitian symmetric space, $\check{\mathcal{D}}$ its compact dual, $X_\Gamma = \Gamma\backslash\mathcal{D}$ an arithmetic quotient, X_Γ^* (respectively $\overline{X}_\Gamma$) the Satake compactification (respectively a toroidal embedding)) is a disjoint union

$$X_\Gamma^* = X_\Gamma \cup V_1 \cup \ldots \cup V_s,$$

with each V_i an arithmetic quotient of lower rank than that of X_Γ, say $V_i = \Gamma_i\backslash\mathcal{D}_i$, $\Gamma_i \subset \Gamma$. On the other hand, for the Baily-Borel embedding $X_\Gamma^* \subset \mathbb{P}^N$, the *singular locus* (under mild assumptions on Γ) coincides with $X_\Gamma^* - X_\Gamma = V_1 \cup \cdots \cup V_s$, the union of lower-dimensional varieties. If one wishes, these lower-dimensional subvarieties can be turned into divisors, by means of $\overline{\pi}_\Gamma : \overline{X}_\Gamma \longrightarrow X_\Gamma^*$, under which $\overline{\pi}_\Gamma^{-1}(V_i) =: B_i$ is a divisor and a fibre space over V_i. The neighborhood of B_i in $\overline{X}_\Gamma$ is described by the normal bundle $N_{\overline{X}_\Gamma}\overline{B}_i$ of the closure $\overline{B}_i$ of B_i in $\overline{X}_\Gamma$. Now the compactification $\overline{X}_\Gamma$ is not unique, but it will be birationally unique, and if V_i has some group of automorphisms H_i then B_i is unique up to an equivariant birational equivalence.

There is a simplicial complex associated with this situation, the Tits building of Γ, $\mathcal{T}(\Gamma)$, whose vertices correspond to the components V_i and whose j-simplices correspond to j-flags of components in the closures (Satake compactifications) of the others, $V_{i_1} \subset V_{i_2}^* \subset \cdots \subset V_{i_j}^*$. This complex, together with the descriptions of the individual components $V_i = \Gamma_i\backslash\mathcal{D}_i$, completely determines the boundary. But by its very definition (which is in terms of parabolic subgroups), $\mathcal{T}(\Gamma)$ thus relates the (geometric) boundary to a group theoretic problem, that of parabolic subgroups. Indeed, each Γ_i is the intersection of Γ and a factor L_i of the Levi component of a unique parabolic $P_i \subset G_\mathbb{Q}$, $\Gamma_i = L_i \cap \Gamma$. But if we consider the smooth model $\overline{X}_\Gamma$, then in fact, we can get the entire parabolic lattice $\Gamma_{P_i} = P_i \cap \Gamma$ by considering an ε−neighborhood of B_i in $\overline{X}_\Gamma$ (with respect to any smooth Riemannian metric on the Riemannian manifold $\overline{X}_\Gamma$). In a nutshell, we have

$$\{\text{singular locus of } X_\Gamma^*\} \longleftrightarrow \{\text{parabolic subgroups of } \Gamma\}.$$

For an algebraic group (semisimple, say) G, a *symmetric* subgroup is an algebraic subgroup defined by a closed symmetric set of roots. If G is of hermitian type, i.e., if the symmetric space associated with $G(\mathbb{R})$ is hermitian symmetric, then a symmetric subgroup M is *hermitian symmetric*, if $\mathcal{D}_M \subset \mathcal{D}$ is a hermitian symmetric subspace. Finally, if M is hermitian symmetric and defined over $\mathbb{Q}$, we call it $\mathbb{Q}$*-hermitian symmetric*. This is the notion one requires on a subgroup M to be able to conclude that for a lattice $\Gamma \subset G$, the arithmetic subgroup $\Gamma_M = M \cap \Gamma$ determines an algebraic subvariety X_{Γ_M} of X_Γ.

Having the notion of $\mathbb{Q}$-hermitian symmetric subgroup $M \subset G$, it is canonical to define the modular subgroups. It is a general property that Γ_M is an arithmetic subgroup of M, and given an explicit description of M and Γ, one gets an explicit description of Γ_M. A description of M is well known. A description of possible arithmetic groups is sketched in Appendix A. Generally speaking this is given by a pair $(V, \mathcal{L})$, where V is a D-vector space and $\mathcal{L}$ is a Δ-lattice, where $\Delta \subset D$ is a fixed maximal order in D, where D is a division algebra over an algebraic number field. Hence the description depends only on the classification of maximal orders in divison algebras and is generally not too much more complicated than the classification of the $\mathbb{Q}$-groups themselves (see Theorems A.5.2 and A.5.3).

We then turn to our original object of study, Level 3 above, arithmetic quotients with "nice" automorphism group. More precisely, we wish to apply the general theory of arithmetic quotients to give a conceptual understanding of an incredible set of examples, which are the primary object of interest. To do this the most natural way of viewing things is in terms of moduli spaces. Deligne has shown that the bounded symmetric domains are parameter spaces of a certain representation theoretic problem, which, in particular cases, is a known geometric moduli problem. Indeed, in his intensive studies of moduli spaces associated with the moduli problem of isomorphism classes of abelian varieties with polarization, given endomorphism ring and level structure, Shimura (much earlier, in the 1960's) gave a complete list of domains and groups which occur in this manner. In this list, all domains of types $\mathbf{I_{p,q}}$, $\mathbf{II_n}$, $\mathbf{III_n}$ occur. The domains of type $\mathbf{IV_n}$ are period domains of pure Hodge structures of weight 2 and type $(1, n, 1)$. So in fact, all except the exceptional domains occur in this way. We give a real quick review of Shimura's theory in Chapter 1, and in Chapter 2 we study the split over $\mathbb{R}$ case, in which a maximal $\mathbb{Q}$-split torus is also a maximal $\mathbb{R}$-split one. This is the easiest case, in which the well-known geometry of the domain $\mathcal{D}$ is reflected in the geometry of the quotient. Most of the material of Chapters 1 and 2 is more or less well-known, but we give them a unified treatment, and, for example, Shimura's theory is very easy to formulate. The result in Chapter 2 on Janus-like varieties is recent; proofs have appeared in [J].

In Chapter 3 we come to "real geometry", and study particular algebraic varieties, say $X \subset \mathbb{P}^N$, which turn out to be Baily-Borel embeddings of arithmetic quotients, say $X = X_\Gamma^*$. There are, generally speaking, two general approaches to this kind of problem. The first (and standard) method utilizes *automorphic forms* (usually theta functions) to display explicit embeddings $X_\Gamma^* \subset \mathbb{P}^N$. The second (and non-standard) approach is the question of *uniformization*. That is, we take as given a singular algebraic variety $X \subset \mathbb{P}^N$ with smooth locus $X_{sm} \subset X$ and inquire as to the universal cover $\widetilde{X}_{sm}$ of X_{sm}. Note that this inverts Baily-Borel: $X_\Gamma \subset X_\Gamma^*$ is the smooth locus, and $\widetilde{X}_\Gamma = \mathcal{D}$. Although we review the work of Igusa and Coble as well as recent results of v. Geemen giving explicit embeddings by means of theta functions, we adhere to the second approach and try to get uniformization results "without automorphic forms". Now, it turns out this works with present technology only for ball quotients, which consequently give our most important examples. We now give a brief description of the examples studied in Chapter 3. First of all, we note that all the examples are related with one another and in all examples the automorphism group of the arithmetic quotient is Σ_6, the symmetric group on 6 letters. In all cases we consider the following questions:

i) the geometry of $X \subset \mathbb{P}^N$;

ii) description of X as an arithmetic quotient, $X = X_\Gamma$;