Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

Subseries: Nankai Institute of Mathematics, Tianjin, P.R. China
vol. 5

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Boju Jiang Chia-Kuei Peng
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Differential Geometry and Topology

Proceedings, Tianjin 1986–87

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Differential Geometry and Topology

Proceedings of the Special Year at Nankai Institute of Mathematics, Tianjin, PR China, 1986–87

Springer-Verlag

Berlin Heidelberg New York London Paris Tokyo

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Mathematics Subject Classification (1980): 22E, 53A, 53C, 57M, 57R, 58G

ISBN 3-540-51037-0 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-51037-0 Springer-Verlag New York Berlin Heidelberg

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

FOREWORD

The Nankai Institute of Mathematics held a Special Year in Geometry and Topology during the academic year 1986-1987. The program centered around invited series of lectures, listed on the next page. This volume contains several sets of notes from these lectures, along with articles submitted by the participants.

We would like to thank all the participants for their enthusiasm and cooperation. Our thanks are also due to those who offered courses in Fall 1986 which prepared the graduate students for the lectures. Finally, we wish to thank Mr. Zhang Shu-dong for smoothing the English of many articles.

For the editors,

Boju Jiang
Chia-Kuei Peng
Zixin Hou

June 1988

SERIES OF LECTURES

(in chronological order)

Fall 1986

S.S. Chern	Ten Lectures in Differential Geometry
R.J. Stern	Yang-Mills and 4-Manifolds
R.O. Wells, Jr.	Supermanifolds

Spring 1987

U. Simon	A Course on Affine Differential Geometry
R.L. Cohen	Immersions of Manifolds; Algebraic K-Theory and Groups of Diffeomorphisms of Manifolds
S. Murakami	Exceptional Simple Lie Groups and Related Topics in Recent Differential Geometry
W.S. Cheung	Exterial Differential Systems and Calculus of Variations
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R. Kirby	Topology of 4-Manifolds
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DUPIN SUBMANIFOLDS IN LIE SPHERE GEOMETRY

Thomas E. Cecil and Shiing-Shen Chern

1. **Introduction**.

Consider a piece of surface immersed in three-dimensional Euclidean space E^3. Its normal lines are the common tangent lines of two surfaces, the focal surfaces. These focal surfaces may have singularities, and a classical theorem says that if the focal surfaces both degenerate to curves, then the curves are conics, and the surface is a cyclide of Dupin. (See, for example, [CR, pp. 151-166].) Equivalently, the cyclides can be characterized as those surfaces in E^3 whose two distinct principal curvatures are both constant along their corresponding lines of curvature.

The cyclides have been generalized to an interesting class of hypersurfaces in E^n, the Dupin hypersurfaces. Initially, a hypersurface M in E^n was said to be Dupin if the number of distinct principal curvatures (or focal points) is constant on M and if each principal curvature is constant along the leaves of its corresponding principal foliation. (See [CR], [Th], [GH].) More recently, this has been generalized to include cases where the number of distinct principal curvatures is not constant. (See [P3], [CC].)

The study of Dupin hypersurfaces in E^n is naturally situated in the context of Lie sphere geometry, developed by Lie [LS] as part of his work on contact transformations. The projectivized cotangent bundle PT^*E^n of E^n has a contact structure. In fact, if $x^1,\ldots,x^n$ are the coordinates in E^n, the contact structure is defined by the linear differential form $dx^n-p_1dx^1-\ldots-p_{n-1}dx^{n-1}$. Lie proved that the pseudo-group of all contact transformations carrying (oriented) hyperspheres in the generalized sense (i.e., including points and oriented hyperplanes) into hyperspheres is a Lie group, called the Lie sphere group, isomorphic to $O(n+1,2)/\pm I$, where $O(n+1,2)$ is the orthogonal group for an indefinite inner product on $\mathbb{R}^{n+3}$ with signature $(n+1,2)$. The Lie sphere group contains as a subgroup the Moebius group of

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The first author was supported by NSF Grant No. DMS 87-06015, the second author by NSF Grant No. DMS 87-01609.

conformal transformations of E^n and, of course, the Euclidean group. Lie exhibited a bijective correspondence between the set of oriented hyperspheres in E^n and the points on the quadric hypersurface Q^{n+1} in real projective space P^{n+2} given by the equation $\langle x,x\rangle = 0$, where $\langle , \rangle$ is the inner product on $\mathbb{R}^{n+3}$ mentioned above. The manifold Q^{n+1} contains projective lines but no linear subspaces of P^{n+2} of higher dimension. The 1-parameter family of oriented spheres corresponding to the points of a projective line lying on Q^{n+1} consists of all oriented hyperspheres which are in oriented contact at a certain contact element on E^n. Thus, Lie constructed a local diffeomorphism between PT^*E^n and the manifold Λ^{2n-1} of projective lines which lie on Q^{n+1}.

An immersed submanifold $f:M^k \to E^n$ naturally induces a Legendre submanifold $\lambda:B^{n-1} \to \Lambda^{2n-1}$, where B^{n-1} is the bundle of unit normal vectors to f (take $B^{n-1} = M^{n-1}$ in the case $k = n-1$). This Legendre map λ has similarities with the familiar Gauss map, and like the Gauss map, it can be a powerful tool in the study of submanifolds of Euclidean space. In particular, the Dupin property for hypersurfaces in E^n is easily formulated in terms of the Legendre map, and it is immediately seen to be invariant under Lie sphere transformations.

The study of Dupin submanifolds has both local and global aspects. Thorbergsson [Th] showed that a Dupin hypersurface M with g distinct principal curvatures at each point must be taut, i.e., every nondegenerate Euclidean distance function $L_p(x) = |p-x|^2$, $p \in E^n$, must have the minimum number of critical points on M. Tautness was shown to be invariant under Lie transformations in our earlier paper [CC]. Using tautness and the work of Münzner [Mu], Thorbergsson was then able to conclude that the number g must be 1,2,3,4 or 6, as with an isoparametric hypersurface in the sphere S^n. The case $g = 1$ is, of course, handled by the well-known classification of umbilic hypersurfaces. Compact Dupin hypersurfaces with $g=2$ and $g=3$ were classified by Cecil and Ryan (see [CR, p. 168]) and Miyaoka [M1] respectively. In two recent preprints, Miyaoka [M2], [M3] has made further progress on the classification of compact Dupin hypersurfaces in the cases $g=4$ and $g=6$. Meanwhile, Grove and Halperin [GH] have determined several important topological invariants of compact Dupin hypersurfaces in the cases $g=4$ and $g=6$.

In this paper, we study Dupin hypersurfaces in the setting of Lie sphere geometry using local techniques. In Section 2, we give a brief introduction to Lie sphere geometry. In Section 3, we introduce the basic differential geometric notions: the Legendre map and the Dupin property. The case of E^3 is handled in Section 4, where we handle the case of g=2 distinct focal points for E^n. This was first done for n > 3 by Pinkall [P3]. Our main contribution lies in Section 5, where we treat the case E^4 by the method of moving frames. This case was also studied by Pinkall [P2], but our treatment seems to be more direct and differs from his in several essential points. It is our hope that this method will provide a framework and give some direction for the study of Dupin hypersurfaces in E^n for n > 4.

2. Lie Sphere Geometry.

We first present a brief outline of the main ideas in Lie's geometry of spheres in $\mathbb{R}^n$. This is given in more detail in Lie's original treatment [LS], in the book of Blaschke [B], and in our paper [CC].

The basic construction in Lie sphere geometry associates each oriented sphere, oriented plane and point sphere in $\mathbb{R}^n \cup \{\infty\} = S^n$ with a point on the quadric Q^{n+1} in projective space P^{n+2} given in homogeneous coordinates $(x_1,\dots,x_{n+3})$ by the equation

$$\langle x,x\rangle = -x_1^2 + x_2^2 + \dots + x_{n+2}^2 - x_{n+3}^2 = 0. \tag{2.1}$$

We will denote real (n+3)-space endowed with the metric (2.1) of signature (n+1,2) by $\mathbb{R}^{n+3}_2$.

We can designate the orientation of a sphere in $\mathbb{R}^n$ by assigning a plus or minus sign to its radius. Positive radius corresponds to the orientation determined by the field of inward normals to the sphere, while a negative radius corresponds to the orientation determined by the outward normal. (See Remark 2.1 below). A plane in $\mathbb{R}^n$ is a sphere which goes through the point ∞. The orientation of the plane can be associated with a choice of unit normal N. The specific correspondence between the points of Q^{n+1} and the set of oriented spheres, oriented planes and points in $\mathbb{R}^n \cup \{\infty\}$ is then given as follows:

Euclidean	Lie
Points: $u \in \mathbb{R}^n$	$\left[\left(\frac{1 + u\cdot u}{2}, \frac{1 - u\cdot u}{2}, u, 0\right)\right]$
∞	$[(1,-1,0,0)]$
(2.2)	
Spheres: Center p, signed radius r	$\left[\left(\frac{1 + p\cdot p - r^2}{2}, \frac{1 - p\cdot p + r^2}{2}, p, r\right)\right]$
Planes: $u\cdot N = h$, unit normal N	$[(h,-h,N,1)]$.

Here the square brackets denote the point in projective space P^{n+2} given by the homogeneous coordinates in the round brackets, and $u\cdot u$ is the standard Euclidean dot product in $\mathbb{R}^n$.

From (2.2), we see that the point spheres correspond to the points in the intersection of Q^{n+1} with the hyperplane in P^{n+2} given by the equation $x_{n+3} = 0$. The manifold of point spheres is called <u>Moebius space</u>.

A fundamental notion in Lie sphere geometry is that of oriented contact of spheres. Two oriented spheres S_1 and S_2 are in <u>oriented contact</u> if they are tangent and their orientations agree at the point of tangency. If p_1 and p_2 are the respective centers of S_1 and S_2, and r_1 and r_2 are the respective signed radii, then the condition of oriented contact can be expressed analytically by

$$|p_1 - p_2| = |r_1 - r_2|. \tag{2.3}$$

If S_1 and S_2 are represented by $[k_1]$ and $[k_2]$ as in (2.2), then (2.3) is equivalent to the condition

$$\langle k_1, k_2\rangle = 0. \tag{2.4}$$

In the case where S_1 and/or S_2 is a plane or a point in $\mathbb{R}^n$, oriented contact has the logical meaning. That is, a sphere S and plane π are in oriented contact if π is tangent to S and their orientations agree at the point of contact. Two oriented planes are in oriented contact if their unit normals are the same. They are in oriented contact at the point ∞. A point sphere is in oriented contact with a sphere or plane S if it lies on S, and ∞ is in oriented contact with each plane. In each case, the analytic condition for oriented contact is equivalent to (2.4) when the two "spheres" in question are represented in Lie coordinates as in (2.2).

<u>Remark 2.1</u>: In the case of a sphere $[k_1]$ and a plane $[k_2]$ as in (2.2), equation (2.4) is equivalent to $p \cdot N = h+r$. In order to make this correspond to the geometric definition of oriented contact, one must adopt the convention that the inward normal orientation of a sphere corresponds to positive signed radius. To get the outward normal orientation to correspond to positive radius, one should represent the plane by $[(-h,h,-N,1)]$ instead of $[(h,-h,N,1)]$. Then (2.4) becomes $p \cdot N = h-r$, which is the geometric formula for oriented contact with the outward normal orientation corresponding to positive signed radius.

Because of the signature of the metric (2.1), the quadric Q^{n+1} contains lines in P^{n+2} but no linear subspaces of higher dimension. A line on Q^{n+1} is determined by two points $[x]$, $[y]$ in Q^{n+1} satisfying $\langle x,y \rangle = 0$. The lines on Q^{n+1} form a manifold of dimension 2n-1, to be denoted by Λ^{2n-1}. In $\mathbb{R}^n$, a line on Q^{n+1} corresponds to a 1-parameter family of oriented spheres such that any two of the spheres are in oriented contact, i.e., all the oriented spheres tangent to an oriented plane at a given point, i.e., an oriented contact element. Of course, a contact element can also be represented by an element of T_1S^n, the bundle of unit tangent vectors to the Euclidean sphere S^n in E^{n+1} with its usual metric. This is the starting point for Pinkall's [P3] considerations of Lie geometry.

A <u>Lie sphere transformation</u> is a projective transformation of P^{n+2} which takes Q^{n+1} to itself. Since a projective transformation takes lines to lines, a Lie sphere transformation preserves oriented contact of spheres. The group G of Lie sphere transformations is isomorphic to $O(n+1,2)/\{\pm I\}$, where $O(n+1,2)$ is the group of orthogonal transformations for the inner product (2.1). Moebius transformations are those Lie transformations which take point spheres to point spheres. The group of Moebius transformations is isomorphic to $O(n+1,1)/\{\pm I\}$.

3. <u>**Legendre Submanifolds**</u>.

Here we recall the concept of a Legendre submanifold of the contact manifold $\Lambda^{2n-1}(= \Lambda)$ using the notation of [CC]. In this section, the ranges of the indices are as follows:

$$1 \le A,B,C \le n+3, \qquad 3 \le i,j,k \le n+1. \tag{3.1}$$

Instead of using an orthonormal frame for the metric <,> defined by (2.1), it is useful to consider a <u>Lie frame</u>, that is, an ordered set of vectors Y_A in $\mathbb{R}_2^{n+3}$ satisfying

$$(3.2) \qquad \langle Y_A, Y_B \rangle = g_{AB},$$

with

$$(3.3) \qquad (g_{AB}) = \begin{bmatrix} J & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & J \end{bmatrix},$$

where I_{n-1} is the identity $(n-1) \times (n-1)$ matrix and

$$(3.4) \qquad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The space of all Lie frames can be identified with the orthogonal group $O(n+1,2)$, of which the Lie sphere group, being isomorphic to $O(n+1,2)/\{\pm I\}$, is a quotient group. In this space, we introduce the Maurer-Cartan forms

$$(3.5) \qquad dY_A = \Sigma\, \omega_A^B\, Y_B .$$

Through differentiation of (3.2), we show that the following matrix of 1-forms is skew-symmetric

$$(3.6) \qquad (\omega_{AB}) = \begin{bmatrix} \omega_1^2 & \omega_1^1 & \omega_1^i & \omega_1^{n+3} & \omega_1^{n+2} \\ \omega_2^2 & \omega_2^1 & \omega_2^i & \omega_2^{n+3} & \omega_2^{n+2} \\ \omega_j^2 & \omega_j^1 & \omega_j^i & \omega_j^{n+3} & \omega_j^{n+2} \\ \omega_{n+2}^2 & \omega_{n+2}^1 & \omega_{n+2}^i & \omega_{n+2}^{n+3} & \omega_{n+2}^{n+2} \\ \omega_{n+3}^2 & \omega_{n+3}^1 & \omega_{n+3}^i & \omega_{n+3}^{n+3} & \omega_{n+3}^{n+2} \end{bmatrix}.$$

Next, by taking the exterior derivative of (3.5), we get the Maurer-Cartan equations

$$(3.7) \qquad d\omega_A^B = \Sigma_C\, \omega_A^C \wedge \omega_C^B .$$

In [CC], we then show that the form

$$\omega_1^{n+2} = \langle dY_1, Y_{n+3}\rangle$$

gives a contact structure on the manifold Λ.

Let B^{n-1}(= B) be an (n-1)-dimensional smooth manifold. A Legendre map is a smooth map $\lambda: B \to \Lambda$ which annihilates the contact form on Λ, i.e., $\lambda^*\omega_1^{n+2} = 0$ on B. All of our calculations are local in nature. We use the method of moving frames and consider a smooth family of Lie frames Y_A on an open subset U of B, with the line $\lambda(b)$ given by $[Y_1(b), Y_{n+3}(b)]$ for each $b \in U$. The Legendre map λ is called a Legendre submanifold if for a generic choice of Y_1 the forms ω_1^i, $3 \le i \le n+1$, are linearly independent, i.e.,

$$\wedge\, \omega_1^i \neq 0 \text{ on } U\ . \tag{3.8}$$

Here and later we pull back the structure forms to B^{n-1} and omit the symbols of such pull-backs for simplicity. Note that the Legendre condition is just

$$\omega_1^{n+2} = 0\ . \tag{3.9}$$

We now assume that our choice of Y_1 satisfies (3.8). By exterior differentiation of (3.9) and using (3.6), we get

$$\Sigma\, \omega_1^i \wedge \omega_{n+3}^i = 0\ . \tag{3.10}$$

Hence by Cartan's Lemma and (3.8), we have

$$\omega_{n+3}^i = \Sigma\, h_{ij}\omega_1^j\ , \text{ with } h_{ij} = h_{ji}\ . \tag{3.11}$$

The quadratic differential form

$$II(Y_1) = \sum_{i,j} h_{ij}\omega_1^i\omega_1^j\ ,$$

defined up to a non-zero factor and depending on the choice of Y_1, is called the second fundamental form.

This form can be related to the well-known Euclidean second fundamental form in the following way. Let e_{n+3} be any unit timelike vector in $\mathbb{R}_2^{n+3}$. For each $b \in U$, let $Y_1(b)$ be the point of intersection of the line $\lambda(b)$ with the hyperplane $e_{n+3}^{\perp}$. Y_1 represents the locus of point spheres in the Moebius space $Q^{n+1} \cap e_{n+3}^{\perp}$, and we call Y_1 the <u>Moebius projection of λ</u> determined by e_{n+3}. Let e_1 and e_2 be unit timelike, respectively spacelike, vectors orthogonal to e_{n+3} and to each other, chosen so that Y_1 is not the point at infinity $[e_1-e_2]$ for any $b \in U$. We can represent Y_1 by the vector

$$(3.13) \qquad Y_1 = \frac{1 + f\cdot f}{2} e_1 + \frac{1 - f\cdot f}{2} e_2 + f,$$

as in (2.2), where $f(b)$ lies in the space $\mathbb{R}^n$ of vectors orthogonal to e_1, e_2 and e_{n+3}. We will call the map $f: B \to \mathbb{R}^n$ the <u>Euclidean projection</u> of λ determined by the ordered triple e_1, e_2, e_{n+3}. The regularity condition (3.8) is equivalent to the condition that f be an immersion on U into $\mathbb{R}^n$. For each $b \in U$, let $Y_{n+3}(b)$ be the intersection of $\lambda(b)$ with the orthogonal complement of the lightlike vector $e_1 - e_2$. Y_{n+3} is distinct from Y_1 and thus $\langle Y_{n+3}, e_{n+3}\rangle \neq 0$. So we can represent Y_{n+3} by a vector of the form

$$(3.14) \qquad Y_{n+3} = h(e_1 - e_2) + \xi + e_{n+3},$$

where $\xi: U \to \mathbb{R}^n$ has unit length and h is a smooth function on U. Thus, according to (2.2), $Y_{n+3}(b)$ represents the plane in the pencil of oriented spheres in $\mathbb{R}^n$ corresponding to the line $\lambda(b)$ on Q^{n+1}. Note that the condition $\langle Y_1, Y_{n+3}\rangle = 0$ is equivalent to $h = f\cdot\xi$, while the Legendre condition $\langle dY_1, Y_{n+3}\rangle = 0$ is the same as the Euclidean condition

$$(3.15) \qquad \xi \cdot df = 0 .$$

Thus, ξ is a field of unit normals to the immersion f on U. Since f is an immersion, we can choose the Lie frame vectors $Y_3, \ldots, Y_{n+1}$ to satisfy

$$(3.16) \qquad Y_i = dY_1(X_i) = (f\cdot df(X_i))(e_1-e_2) + df(X_i), \quad 3 \le i \le n+1 ,$$

for tangent vector fields $X_3,\ldots,X_{n+1}$ on U. Then, we have

(3.17) $$\omega_1^i(X_j) = \langle dY_1(X_j), Y_i\rangle = \langle Y_j, Y_i\rangle = \delta_{ij} .$$

Now using (3.14) and (3.16), we compute

(3.18) $$\begin{aligned}\omega_{n+3}^i\ (X_j) &= \langle dY_{n+3}(X_j), Y_i\rangle = d\xi(X_j)\cdot df(X_i)\\ &= -\ df(AX_j)\cdot df(X_i) = -A_{ij} ,\end{aligned}$$

where $A = [A_{ij}]$ is the Euclidean shape operator (second fundamental form) of the immersion f. But by (3.11) and (3.17), we have

$$\omega_{n+3}^i(X_j) = \Sigma\ h_{ik}\omega_1^k(X_j) = h_{ij} .$$

Hence $h_{ij} = -A_{ij}$, and $[h_{ij}]$ is just the negative of the Euclidean shape operator A of f.

<u>Remark 3.1</u>: The discussion above demonstrates how an immersion $f:B^{n-1} \to \mathbb{R}^n$ with field of unit normals ξ induces a Legendre submanifold $\lambda:B^{n-1} \to \Lambda$ defined by $\lambda(b) = [Y_1(b), Y_{n+3}(b)]$, for Y_1, Y_{n+3} as in (3.13), (3.14). Further, an immersed submanifold $f:M^k \to \mathbb{R}^n$ of codimension greater than one also gives rise to a Legendre submanifold $\lambda:B^{n-1} \to \Lambda$, where B^{n-1} is the bundle of unit normals to f in $\mathbb{R}^n$. As in the case of codimension one, $\lambda(b)$ is defined to be the line on Q^{n+1} corresponding to the oriented contact element determined by the unit vector b normal to f at the point $x = \pi(b)$, where π is the bundle projection from B^{n-1} to M^k.

As one would expect, the eigenvalues of the second fundamental form have geometric significance. Consider a curve $\gamma(t)$ on B. The set of points in Q^{n+1} lying on the lines $\lambda(\gamma(t))$ forms a ruled surface in Q^{n+1}. We look for the conditions that this ruled surface be developable, i.e., consist of tangent lines to a curve in Q^{n+1}. Let $rY_1 + Y_{n+3}$ be the point of contact. We have by (3.5) and (3.6)

(3.19) $$d(rY_1 + Y_{n+3}) \equiv \sum_i (r\omega_1^i + \omega_{n+3}^i)Y_i,\ \bmod\ Y_1, Y_{n+3} .$$

Thus, the lines $\lambda(\gamma(t))$ form a developable if and only if the tangent direction of $\gamma(t)$ is a common solution to the equations

$$\sum_j (r\delta_{ij} + h_{ij})\omega_1^j = 0, \quad 3 \le i \le n+1 . \tag{3.20}$$

In particular, r must be a root of the equation

$$\det(r\delta_{ij} + h_{ij}) = 0 . \tag{3.21}$$

By (3.11) the roots of (3.21) are all real. Denote them by $r_3, \ldots, r_{n+1}$. The points $r_i Y_1 + Y_{n+3}$, $3 \le i \le n+1$ are called the <u>focal points</u> or <u>curvature spheres</u> (Pinkall [P3]) on $\lambda(b)$. If Y_1 and Y_{n+3} correspond to an immersion $f: U \to \mathbb{R}^n$ as in (3.13) and (3.14), then these focal points on $\lambda(b)$ correspond by (2.2) to oriented spheres in $\mathbb{R}^n$ tangent to f at f(b) and centered at the Euclidean focal points of f. These spheres are called curvature spheres of f and the r_i are just the principal curvatures of f, i.e., eigenvalues of the shape operator A.

If r is a root of (3.21) of multiplicity m, then the equations (3.20) define an m-dimensional subspace T_r of $T_b B$, the tangent space to B at the point b. The space T_r is called a <u>principal space</u> of $T_b B$, the latter being decomposed into a direct sum of its principal spaces. Vectors in T_r are called <u>principal vectors</u> corresponding to the focal point $rY_1 + Y_{n+3}$. Of course, if Y_1 and Y_{n+3} correspond to an immersion $f: U \to \mathbb{R}^n$ as in (3.13) and (3.14), then these principal vectors are the same as the Euclidean principal vectors for f corresponding to the principal curvature r.

With a change of frame of the form

$$Y_i^* = \sum_i c_i^j Y_j , \quad 3 \le i \le n+1 , \tag{3.22}$$

where $[c_i^j]$ is an (n-1) x (n-1) orthogonal matrix, we can diagonalize $[h_{ij}]$ so that in the new frame, equation (3.11) has the form

$$\omega_{n+3}^i = - r_i \omega_1^i , \quad 3 \le i \le n+1 . \tag{3.23}$$

Note that none of the functions r_i is ever infinity on U because of the