FUNDAMENTALS OF THE THEORY OF OPERATOR ALGEBRAS

Volume I

Elementary Theory

Richard V. Kadison

John R. Ringrose

FUNDAMENTALS OF THE THEORY OF OPERATOR ALGEBRAS

VOLUME I

Elementary Theory

Richard V. Kadison

Department of Mathematics
University of Pennsylvania
Philadelphia, Pennsylvania

John R. Ringrose

School of Mathematics
University of Newcastle
Newcastle upon Tyne, England

Academic Press
San Diego New York Boston
London Sydney Tokyo Toronto

ACADEMIC PRESS, INC.
A Division of Harcourt Brace & Company
525 B Street, Suite 1900
San Diego, California 92101-4495

United Kingdom Edition published by
ACADEMIC PRESS, LTD.
24–28 Oval Road, London NW1 7DX

Library of Congress Cataloging in Publication Data

Kadison, Richard V., date
Fundamentals of the theory of operator algebras.

(Pure and applied mathematics)
Includes index.
1. Operator algebras. I. Ringrose, John R.
II. Title. III. Series: Pure and applied mathematics
(Academic Press)
QA3.P8 [QA326] 512'.55 82-13768
ISBN 0-12-393304-8 (v. 1)

PRINTED IN THE UNITED STATES OF AMERICA

94 95 96 MV 5 4 3

FUNDAMENTALS OF THE THEORY OF OPERATOR ALGEBRAS

VOLUME I

Elementary Theory

This is a volume in
PURE AND APPLIED MATHEMATICS

A Series of Monographs and Textbooks

Editors: SAMUEL EILENBERG AND HYMAN BASS

A list of recent titles in this series appears at the end of this volume.

PREFACE

These volumes deal with a subject, introduced half a century ago, that has become increasingly important and popular in recent years. While they cover the fundamental aspects of this subject, they make no attempt to be encyclopaedic. Their primary goal is to *teach* the subject and lead the reader to the point where the vast recent research literature, both in the subject proper and in its many applications, becomes accessible.

Although we have put major emphasis on making the material presented clear and understandable, the subject is not easy; no account, however lucid, can make it so. If it is possible to browse in this subject and acquire a significant amount of information, we hope that these volumes present that opportunity—but they have been written primarily for the reader, either starting at the beginning or with enough preparation to enter at some intermediate stage, who works through the text systematically. The study of this material is best approached with equal measures of patience and persistence.

Our starting point in Chapter 1 is finite-dimensional linear algebra. We assume that the reader is familiar with the results of that subject and begin by proving the infinite-dimensional algebraic results that we need from time to time. These volumes deal almost exclusively with infinite-dimensional phenomena. Much of the intuition that the reader may have developed from contact with finite-dimensional algebra and geometry must be abandoned in this study. It will mislead as often as it guides. In its place, a new intuition about infinite-dimensional constructs must be cultivated. Results that are apparent in finite dimensions may be false, or may be difficult and important principles whose application yields great rewards, in the infinite-dimensional case.

Almost as much as the subject matter of these volumes is infinite dimensional, it is *non-commutative real analysis*. Despite this description, the reader will find a very large number of references to the "abelian" or "commutative" case—an important part of this first volume is an analysis of the abelian case. This case, parallel to function theory and measure theory, provides us with a major tool and an important guide to our intuition. A good part of what we know comes from extending to the non-commutative case results that are known in the commutative case. The "extension" process is usually difficult. The main techniques include elaborate interlacing of "abelian" segments. The reference to "real analysis" involves the fact that while we consider complex-valued functions and, non-commutatively, non-self-adjoint operators, the structures we study make simultaneously available to us the complex conjugates of those functions and, non-commutatively, the adjoints of those operators. In essence, we are studying the algebraic interrelations of systems of real functions and, non-commutatively, systems of self-adjoint operators. At its most primitive level, the non-commutativity makes itself visible in the fact that the product of a function and its conjugate is the same in either order while this is not in general true of the product of an operator and its adjoint.

In the sense that we consider an operator and its adjoint on the same footing, the subject matter we treat is referred to as the "self-adjoint theory." There is an emerging and important development of non-self-adjoint operator algebras that serves as a non-commutative analogue of complex function theory—algebras of holomorphic functions. This area is not treated in these volumes. Many important developments in the self-adjoint theory—both past and current—are not treated. The type I C^*-algebras and C^*-algebra K-theory are examples of important subjects not dealt with. The aim of teaching the basics and preparing the reader for individual work in research areas seems best served by a close adherence to the "classical" fundamentals of the subject. For this same reason, we have not included material on the important application of the subject to the mathematical foundation of theoretical quantum physics. With one exception, applications to the theory of representations of topological groups are omitted. Accounts of these vast research areas, within the scope of this treatise, would be necessarily superficial. We have preferred instead to devote space to clear and leisurely expositions of the fundamentals. For several important topics, two approaches are included.

Our emphasis on instruction rather than comprehensive coverage has led us to settle on a very brief bibliography. We cite just three textbooks (listed as [H], [K], and [R]) for background information on general topology and measure theory, and for this first volume, include only 25 items from the literature of our subject. Several extensive and excellent bibliographies are available (see, for example, [2,24,25]), and there would be little purpose in reproducing a modified version of one of the existing lists. We have included in our references items specifically referred to in the text and others that might provide profitable additional reading. As a consequence, we have made no attempt, either in the text or in the exercises, to credit sources on which we have drawn or to trace the historical background of the ideas and results that have gone into the development of the subject.

Each of the chapters of this first volume has a final section devoted to a substantial list of exercises, arranged roughly in the order of the appearance of topics in the chapter. They were designed to serve two purposes: to illustrate and extend the results and examples of the earlier sections of the chapter, and to help the reader to develop working technique and facility with the subject matter of the chapter. For the reader interested in acquiring an ability to work with the subject, a certain amount of exercise solving is indispensable. We *do not* recommend a rigid adherence to order—each exercise being solved in sequence and no new material attempted until all the exercises of the preceding chapter are solved. Somewhere between that approach and total disregard of the exercises a line must be drawn congenial to the individual reader's needs and circumstances. In general, we *do* recommend that the greater proportion of the reader's time be spent on a thorough understanding of the main text than on the exercises. In any event, *all* the exercises have been designed to be solved. Most exercises are separated into several parts with each of the parts manageable and some of them provided with hints. Some are routine, requiring nothing more than a clear understanding of a definition or result for their solutions. Other exercises (and groups of exercises) constitute small (guided) research projects.

On a first reading, as an introduction to the subject, certain sections may well be left unread and consulted on a few occasions as needed. Section 2.6, *Tensor products and the Hilbert–Schmidt class* (this "subsection" is the largest part of Section 2.6) will not be needed seriously until Chapter 11 (in Volume II). All the material on unbounded operators (and the material related to Stone's theorem) will not be needed until Chapter 9 (in Volume II). Thus Section 2.7, Section 3.2, *The Banach algebra* $L_1(\mathbb{R})$ *and Fourier analysis*, the last few pages of Chapter 4 (including Theorem 4.5.9), and Section 5.6, can be deferred to a later reading. Some readers, more or less familiar with the elements of functional analysis, may want to enter the text after Chapter 1 with occasional back references for notation or precise definitions and statements of results. The reader with a good general knowledge of basic functional analysis may consider beginning at Section 3.4 or perhaps with Chapter 4.

The various possible styles of reading this volume, related to the levels of preparation of the reader, suggest several styles and levels of courses for which it can be used. For all of these, a good working knowledge of point-set (general) topology, such as may be found in [K], is assumed. Somewhat less vital, but useful, is a knowledge of general measure theory, such as may be found in [H] and parts of [R]. Of course, full command of the fundamentals of real and complex analysis (we refer to [R] for these) is needed; and, as noted earlier, the elements of finite-dimensional linear algebra are used. The first three chapters form the basis of a course in elementary functional analysis with a slant toward operator algebras and its allied fields of group representations, harmonic analysis, and mathematical (quantum) physics. These chapters provide material for a brisk one-semester course at the first- or second-year graduate level or for a more leisurely one-year course at the advanced undergraduate or beginning graduate level. Chapters 3, 4, and 5 provide an introduction to the theory of operator algebras and have material that would serve as a one-semester graduate course at the second- or third-year level (especially if Section 5.6 is omitted). In any event, the book has been designed for individual study as well as for courses, so that the problem of a wide spread of preparation in a class can be dealt with by encouraging the better prepared students to proceed at their own paces. Seminar and reading-course possibilities are also available.

When several (good) terms for a mathematical construct are in common use, we have made no effort to choose one and then to use that one term consistently. On the contrary, we have used such terms interchangeably after introducing them simultaneously. This seems the best preparation for further reading in the research literature. Some examples of such terms are weaker, coarser (for topologies on a space), unitary transformation, and Hilbert space isomorphism (for structure-preserving mappings between Hilbert spaces). In cases where there is conflicting use of a term in the research literature (for example, "purely infinite" in connection with von Neumann algebras), we have avoided all use of the term and employed accepted terminology for each of the constructs involved. Since the symbol * is used to denote the adjoint operations on operators and on sets of operators, we have preferred to use a different symbol in the context of Banach dual spaces. We denote the dual space of a Banach space $\mathfrak{X}$ by $\mathfrak{X}^\sharp$. However, we felt compelled by usage to retain the terminology "weak *" for the topology induced by elements of $\mathfrak{X}$ (as linear functionals on $\mathfrak{X}^\sharp$).

Results in the body of the text are italicized, titled Theorem, Proposition, Lemma, and Corollary (in decreasing order of "importance"—though, as usual, the "heart of the matter" may be dealt with in a lemma and its most usable aspect may appear in a corollary). In addition, there are Remarks and Examples that extend and illuminate the material of a section, and of course there are the (formal) Definitions. None of these items is italicized, though a crucial phrase or word frequently is. Each of these segments of the text is preceded by a number, the first digit of which indicates the chapter, the second the section, and the last one- or two-digit number the position of the item in the section. Thus, "Proposition 5.5.18" refers to the eighteenth numbered item in the fifth section of the fifth chapter. A back or forward reference to such an item will include the title ("Theorem," "Remark," etc.), though the number alone would serve to locate it. Occasionally a displayed equation, formula, inequality, etc., is assigned a number in parentheses at the left of the display—for example, the "convolution formula" of Fourier transform theory appears as the display numbered (4) in the proof of Theorem 3.2.26. In its own section, it is referred to as (4) and elsewhere as 3.2(4).

The lack of illustrative examples in much of Chapter 1 results from our wish to bring the reader more rapidly to the subject of operator algebras rather than to dwell on the basics of general functional analysis. As compensation for their lack, the exercises supply much of the illustrative material for this chapter. Although the tensor product development in Section 2.6 may appear somewhat formal and forbidding at first, it turns out that the trouble and care taken at that point simplify subsequent application. The same can be said (perhaps more strongly) about Section 5.6. The material on unbounded operators (their spectral theory and function calculus) is so vital when needed and so susceptible to incorrect and incomplete application that it seemed well worth a careful and thorough treatment. We have chosen a powerful approach that permits such a treatment, much in the spirit of the theory of operator algebras.

Another (general) aspect of the organization of material in a text is the way the material of the text proper relates to the exercises. As a matter of specific policy, we have not relegated to the exercises whole arguments or parts of arguments. Reference is occasionally made to an exercise as an illustration of some point—for example, the fact that the statement resulting from the omission of some hypothesis from a theorem is false.

During the course of the preparation of these volumes, we have enjoyed, jointly and separately, the hospitality and facilities of several universities, aside from our home institutions. Notable among these are the Mathematics Institutes of the Universities of Aarhus and Copenhagen and the Theoretical Physics Institute of Marseille–Luminy. The subject matter of these volumes and its style of development is inextricably interwoven with the individual research of the authors. As a consequence, the support of that research by the National Science Foundation (U.S.A.) and the Science Research Council (U.K.) has had an oblique but vital influence on the formation of these volumes. It is the authors' pleasure to express their gratitude for this support and for the hospitality of the host institutions noted.

CONTENTS OF VOLUME II

Advanced Theory

Chapter 6. **Comparison Theory of Projections**

Chapter 7. **Normal States and Unitary Equivalence of von Neumann Algebras**

Chapter 8. **The Trace**

Chapter 9. **Algebra and Commutant**

Chapter 10. **Special Representations of C^*-Algebras**

Chapter 11. **Tensor Products**

Chapter 12. **Approximation by Matrix Algebras**

Chapter 13. **Crossed Products**

Chapter 14. **Direct Integrals and Decompositions**

Bibliography

CONTENTS

Chapter 3. **Banach Algebras**

Chapter 4. **Elementary C^*-Algebra Theory**

Chapter 5. **Elementary von Neumann Algebra Theory**

CHAPTER 1

LINEAR SPACES

This chapter contains an account of those basic aspects of linear functional analysis that are needed, later in the book, in the study of operator algebras. The main topics – continuous linear operators, continuous linear functionals, weak topologies, convexity – are studied first in the context of linear topological spaces, then in the more restricted setting of normed spaces and Banach spaces. In preparation for this, some related material is treated in the purely algebraic situation (that is, without topological considerations).

1.1. Algebraic results

In this section we shall consider linear spaces (that is, vector spaces) over a field $\mathbb{K}$, and it will be assumed throughout that $\mathbb{K}$ is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. We sometimes distinguish between these two cases by referring to *real* vector spaces or *complex* vector spaces. Our main concern is with linear functionals, convex sets, and the separation of convex sets by hyperplanes.

Suppose that $\mathscr{V}$ is a linear space with scalar field $\mathbb{K}$. If X and Y are non-empty subsets of $\mathscr{V}$, and $a \in \mathbb{K}$, we define further subsets aX, $X \pm Y$ by

$$aX = \{ax : x \in X\}, \qquad X + Y = \{x + y : x \in X, y \in Y\},$$

and

$$X - Y = X + (-1)Y.$$

When X consists of a single element x, we write $x \pm Y$ in place of $X \pm Y$. To avoid ambiguity in the use of the symbol $-$, the set theoretic difference $\{x \in \mathbb{A} : x \notin \mathbb{B}\}$ of two sets $\mathbb{A}$ and $\mathbb{B}$ will be denoted by $\mathbb{A} \setminus \mathbb{B}$. A vector of the form $a_1x_1 + \cdots + a_nx_n$, where $x_1, \ldots, x_n \in X$ and $a_1, \ldots, a_n \in \mathbb{K}$, is called a (*finite*) *linear combination* of elements of X. The zero vector is always of this form (in a trivial way), with $\{x_1, \ldots, x_n\}$ an arbitrary finite subset of X, and $a_j = 0$ for each j. If it can be expressed as a *non-trivial* linear combination of elements of X (that is, with $x_1, \ldots, x_n$ distinct, and at least one a_j non-zero), then X is said to be *linearly dependent*; otherwise X is *linearly independent*. The set of all finite linear combinations of elements of X is a linear subspace of $\mathscr{V}$, the smallest containing X; we refer to it as the *linear subspace generated by* X.

If $\mathscr{V}_0$ is a linear subspace of $\mathscr{V}$, we denote by $\mathscr{V}/\mathscr{V}_0$ the set of all cosets $x + \mathscr{V}_0$ $(x \in \mathscr{V})$ in the additive group $\mathscr{V}$. Of course, $\mathscr{V}/\mathscr{V}_0$ is a group, with addition defined by $(x + \mathscr{V}_0) + (y + \mathscr{V}_0) = (x + y) + \mathscr{V}_0$. If $a \in \mathbb{K}$, and $x_1 + \mathscr{V}_0 = x_2 + \mathscr{V}_0$, we have $ax_1 - ax_2 = a(x_1 - x_2) \in \mathscr{V}_0$, so $ax_1 + \mathscr{V}_0 = ax_2 + \mathscr{V}_0$. From this it follows easily that $\mathscr{V}/\mathscr{V}_0$ becomes a linear space over $\mathbb{K}$, the *quotient* of $\mathscr{V}$ by $\mathscr{V}_0$, when multiplication by scalars is defined (unambiguously) by $a(x + \mathscr{V}_0) = ax + \mathscr{V}_0$. If $\mathscr{V}/\mathscr{V}_0$ has finite dimension n, we say that $\mathscr{V}_0$ has finite *codimension* n in $\mathscr{V}$.

Suppose that $\mathscr{V}$ and $\mathscr{W}$ are linear spaces over $\mathbb{K}$. By a *linear operator* (or *linear transformation*) from $\mathscr{V}$ into $\mathscr{W}$, we mean a mapping $T\colon \mathscr{V} \to \mathscr{W}$ such that

$$T(ax + by) = aTx + bTy$$

whenever $x, y \in \mathscr{V}$ and $a, b \in \mathbb{K}$ (the notation $T\colon \mathscr{V} \to \mathscr{W}$ indicates that T is defined on $\mathscr{V}$ and takes values in $\mathscr{W}$; it can be read "T, from $\mathscr{V}$ into $\mathscr{W}$"). If $\mathscr{V}_0$ is a linear subspace of $\mathscr{V}$, the equation $Qx = x + \mathscr{V}_0$ defines a linear operator Q from $\mathscr{V}$ onto $\mathscr{V}/\mathscr{V}_0$, the *quotient mapping*. When $T\colon \mathscr{V} \to \mathscr{W}$ is a linear operator, the *null space* of T is the linear subspace $\{x \in \mathscr{V} : Tx = 0\}$ of $\mathscr{V}$, and the image (or range) $T(\mathscr{V}) = \{Tx : x \in \mathscr{V}\}$ is a linear subspace of $\mathscr{W}$. If $T(\mathscr{V}_0) = \{0\}$, the condition $x + \mathscr{V}_0 = y + \mathscr{V}_0$ entails $x - y \in \mathscr{V}_0$, and hence $Tx - Ty = 0$; moreover, if $\mathscr{V}_0$ is the null space of T, $Tx = 0$ entails $x \in \mathscr{V}_0$. From this, the equation $T_0(x + \mathscr{V}_0) = Tx$ defines (unambiguously) a linear operator T_0 from $\mathscr{V}/\mathscr{V}_0$ onto $T(\mathscr{V})$ $(\subseteq \mathscr{W})$, when $T(\mathscr{V}_0) = \{0\}$; and T_0 is one-to-one if $\mathscr{V}_0$ is the null space of T. Note that $T = T_0Q$, a fact sometimes described by saying that *T factors through $\mathscr{V}/\mathscr{V}_0$* when $T(\mathscr{V}_0) = \{0\}$. Given any linear operators $S, T\colon \mathscr{V} \to \mathscr{W}$ and scalars a, b, the equation $(aS + bT)x = aSx + bTx$ $(x \in \mathscr{V})$ defines another such operator $aS + bT$, and in this way, the set of all linear operators from $\mathscr{V}$ into $\mathscr{W}$ becomes a linear space over $\mathbb{K}$.

By a *linear functional* on $\mathscr{V}$ we mean a linear operator $\rho\colon \mathscr{V} \to \mathbb{K}$ (of course, $\mathbb{K}$ is a one-dimensional linear space over $\mathbb{K}$). The set of all linear functionals on $\mathscr{V}$ is itself a linear space over $\mathbb{K}$, the *algebraic dual space of* $\mathscr{V}$. When ρ is a *non-zero* linear functional on $\mathscr{V}$ (that is, ρ does not vanish identically on $\mathscr{V}$) the image $\rho(\mathscr{V})$ is $\mathbb{K}$.

1.1.1. Proposition. *If ρ is a linear functional on a linear space $\mathscr{V}$, then every linear functional on $\mathscr{V}$ that vanishes on the null space $\mathscr{V}_0$ of ρ is a scalar multiple of ρ. If $\rho \neq 0$, $\mathscr{V}_0$ has codimension 1 in $\mathscr{V}$. Conversely each linear subspace of codimension 1 in $\mathscr{V}$ is the null space of a non-zero linear functional. If $\rho_1, \ldots, \rho_n$ are linear functionals on $\mathscr{V}$, then every linear functional on $\mathscr{V}$ that vanishes on the intersection of the null spaces of $\rho_1, \ldots, \rho_n$ is a linear combination of $\rho_1, \ldots, \rho_n$.*