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in cooperation with

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Quasiconformal Mappings in the Plane:

Parametrical Methods

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FOREWORD

These lecture notes contain an exposition of analytic properties of quasiconformal mappings in the plane (Chapter I), a detailed and systematic study of the parametrical method with complete proofs (exploring these analytic properties and including results of the author), partly new and simplified (Chapter II), and a brief account of variational methods (Chapter III).

In contrast to the books by Lehto and Virtanen [1, 2] and by Ahlfors [5] the present author starts in Chapter I with defining the class of quasiconformal mappings as the closure of the Grötzsch class with respect to uniform convergence on compact subsets. Then an important part of this chapter is devoted to proving the fundamental theorem on existence and uniqueness of quasiconformal mappings with a preassigned complex dilatation, established in particular cases by Gauss [1], Lichtenstein [1] and Lavrentieff [2], and in the general case by Morrey [1]. The present author chooses the proof due to Bojarski [2] which is based a.o. on the fundamental results of Calderón and Zygmunt [2] connected with properties of the Hilbert transform and on reducing the problem in question to solving some linear integral equation.

Parametrical and variational methods belong to the most powerful research tools for extremal problems in the complex analysis.

The parametrical method for conformal mappings of the unit disc $\{z: |z| < 1\}$, initiated by Löwner [1], consists in fact of studying a partial differential equation for a function $w = f(z,t)$, $|z| < 1$, $0 \leq t \leq T$, whose solutions are homotopically contractible to the identity mapping within the class of conformal mappings of the unit disc and form in this class a dense subclass. This method was then extended to the case of doubly connected domains by Komatu [1] and Golusin [1]. With the help of this method it was possible to obtain a number of basic information on conformal mappings inaccessible for elementary methods. An analogous method for quasiconformal mappings of the unit disc onto itself was initiated by Shah Tao-shing [1] and then extended by other authors.

In Chapter II, concerned with the parametrical method, we concentrate on the results of Shah Tao-shing [1], Ahlfors and Bers [1], Ahlfors [5], Gehring and Reich [1], and the author. When considering more advanced and special topics as well as applications, a special attention is paid to the results of Reich and Strebel [1-3], Gehring and Reich [1], Kühnau [1-13], and Lehto [3-6]. These topics are mainly connected with Teichmüller mappings, quasiconformal mappings with preassigned boundary values, and conformal mappings with quasiconformal extensions.

Thanks to the books by Belinskiĭ [2], Kruškal [4, 5] and Schober [1], there is no necessity to describe the variational methods here in detail; they are only briefly reviewed. For conformal mappings they were initiated by Hadamard [1], Julia [1] and Courant [1], and then developed as a very effective research tool in the papers by Schiffer [1], Schaeffer and Spencer [1], Golusin [2], and other authors. Investigation of extremal properties of quasiconformal mappings, in particular a characterization of extremal mappings and their connection with quadratic differentials is a deep and unexpected discovery of Teichmüller [1], which initiated the development of more special variational methods for quasiconformal mappings (Belinskiĭ [1], Schiffer [2], and others).

In Chapter III, connected with variational methods, we concentrate on the results of Teichmüller [1, 2], Belinskiĭ [1], Schiffer [2], Renelt [1, 3], Schiffer and Schober [1, 2], Kruškal [1-8], Strebel [2-5], and Kühnau [1-15]. Section 26 gives a brief account of those aspects of the theory of extremal quasiconformal mappings and problems connected with the famous Teichmüller theorem (Teichmüller [1], Ahlfors [2]) which are necessary for full motivation of the Teichmüller mappings considered earlier in Sections 17 and 18. The concluding Section 27 indicates the importance of quasiconformal mappings, in particular of the analytic approach as well as of parametrical and variational methods, in electrical engineering.

In this place the author would like to thank Profs. L. V. Ahlfors, S. L. Kruškal, R. Kühnau, O. Lehto, A. Pfluger, and Dr. T. Iwaniec for reading various parts of the manuscript and critical remarks.

Łódź, Poland, November 1982 Julian Ławrynowicz

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NOTATION AND ABBREVIATIONS

Throughout these lecture notes, unless otherwise specified, we are concerned with points and sets of the closed plane $\mathbb{E}$, the one point compactification of the finite plane $\mathbb{C}$ with the usual metric. The real line with the usual metric is denoted by $\mathbb{R}$. The corresponding sets are denoted by $\mathbb{E}$, $\mathbb{C}$, and $\mathbb{R}$ as well.

The difference of two sets E and E' is denoted by $E \setminus E'$, the closure of E by $\mathrm{cl}\, E$, the interior of E by $\mathrm{int}\, E$, while the boundary, topological and oriented (positively) with respect to E, by $\mathrm{fr}\, E$ and ∂E, respectively. In the latter definition we assume E to be a domain bounded by disjoint Jordan curves (in particular, a Jordan domain), or its closure. Under Jordan curve we mean a homeomorphic image of a circle, under Jordan arc – a homeomorphic image of an interval, i.e. of a connected subset of $\mathbb{R}$ which does not reduce to a point. The open and closed segments with end points a, b are denoted by $(a;b)$ and $[a;b]$, respectively, while under (a,b) we mean the ordered pair of a, b and – more generally – (a_n) denotes the sequence with terms $a_1, a_2, \dots$ We also put

$$\Delta^r(s) = \{z\colon |z - s| \leq r\},\ \Delta^r = \Delta^r(0),\ \Delta = \Delta^1,\ \Delta_r = \Delta \setminus \mathrm{int}\, \Delta^r.$$

The Lebesgue plane measure of a measurable set $E \subset \mathbb{C}$ is denoted by $|E|$, the outer Lebesgue linear measure of a set $I \subset \mathbb{R}$ – by $[\![I]\!]^*$, while its Lebesgue linear measure, if I is measurable, by $[\![I]\!]$. Thereafter under a (plane, linear) <u>measurable</u> set we always understand a set which is Lebesgue (plane, linear) measurable. The diameter of a set E is denoted by $\mathrm{dia}\, E$.

If f is a function defined on E, and $E' \subset E$, then $f[E']$ denotes the image of E' under f (in case of a functional the convention concerning ( ) and [ ] is inverse). If, in particular, f is an elementary function: exp, arg, etc., and $z \in E$, we write fz instead of $f(z)$ in case where it does not lead to misunderstanding. An analogous convention is applied to linear operators. The zero and identity functions, defined on a set E, are denoted by 0_E and id_E, respectively. In the case where $E = \mathbb{E}$, the subscript is omitted. We say that f, defined on E, satisfies a property if this property is satisfied for all $z \in E$.

If f and g are functions defined on E and E', respectively, where $E' \cap f[E] \neq \emptyset$, then the composite function defined on the preimage of $E' \cap f[E]$ under f is denoted by $g \circ f$, and for any z of this preimage we write $g \circ f(z)$ instead of $(g \circ f)(z)$. Further, $\check{f}$ denotes the inverse of f, if it exists, while $f^{-1} = 1/f$. Correspondingly, $\check{f}[E']$ denotes the preimage of a set E' under f even in the case where f is not invertible. The notation $\check{f}$ for the inverse of f is used e.g. in Behnke and Sommer [1] and is much more convenient for our purposes than f^{-1} which e.g. in the expression f_w^{-1} (differentiation with respect to w) may lead to misunderstanding.

The support of f, i.e. the set $\mathrm{cl}\{z: f(z) \neq 0\}$, is denoted, following Morrey [2], by spt f; the abbreviation supp is more popular, but it is longer and similar to sup. A mapping (i.e. a function) from E into $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{E}$ is often denoted by $f: E \to \mathbb{R}$, $\mathbb{C}$, or $\mathbb{E}$, respectively. In the case where f is one-to-one, we always write $f: E \to E'$, $E' \subset \mathbb{R}$, $\mathbb{C}$, or $\mathbb{E}$, and this means that $f[E] = E'$. We also write $a_n \to a$ instead of $a_n \to a$ as $n \to +\infty$. If mappings $f: E \to \mathbb{E}$ and $f_n: E_n \to \mathbb{E}$, $n = 1, 2, \ldots$, are such that for every compact subset E_0 of $E \setminus \{\infty\} \setminus \check{f}[\{\infty\}]$ there is an index k such that $E_0 \subset E_n$ for $n > k$ and $f_n | E_0 \to f | E_0$ uniformly, we say, following Saks and Zygmund [1], that (f_n) _tends to_ f _almost uniformly_ and write $f_n \rightrightarrows f$. Finally, if $f: E \to \mathbb{C}$ is measurable, we put

$$\|f\|_p = \Big(\iint_E |f|^p dxdy \Big)^{1/p}, \quad 1 \le p < +\infty,$$

$$\|f\|_\infty = \begin{cases} 0, & |E| = 0, \\ \inf\limits_{E'} \sup\limits_{z \in E \setminus E'} |f(z)|, & |E| > 0, \end{cases}$$

where the infimum is taken over all sets E' with $|E'| = 0$.

The expressions _if and only if_, _almost every[where]_, and _with respect to_ are abbreviated by _iff_, _a.e._, and _w.r.t._, respectively, while _qc_ [_qcty_] means _quasiconformal_[_ity_] and ACL – absolutely continuous on lines.

I. BASIC CONCEPTS AND THEOREMS IN THE ANALYTIC THEORY OF QUASICONFORMAL MAPPINGS

1. The class of regular quasiconformal mappings and its closure

We begin with the notion of a sense-preserving homeomorphism. It is well known that if $f\colon \mathrm{cl}\, D \to E$ is a homeomorphism and D a Jordan domain, then $E = \mathrm{cl}\, D'$, where D' is Jordan, and $f|\mathrm{fr}\, D\colon \mathrm{fr}\, D \to \mathrm{fr}\, D'$. Now, let $f_1\colon \mathrm{fr}\, \Delta \to \mathrm{fr}\, D$ and $f_2\colon \mathrm{fr}\, \Delta \to \mathrm{fr}\, D$ be homeomorphisms such that $\arg \check{f}_2 \circ f_1 | \mathrm{fr}\, \Delta \setminus \{-1\}$ is an increasing function of $\arg z$, $z \in \mathrm{fr}\, \Delta \setminus \{-1\}$. Then, as it is well known, also $g_1 = f \circ f_1$ and $g_2 = f \circ f_2$ are homeomorphisms such that $\arg \check{g}_2 \circ g_1 | \mathrm{fr}\, \Delta \setminus \{-1\}$ is an increasing function of $\arg z$, $z \in \mathrm{fr}\, \Delta \setminus \{-1\}$. Hence f induces a mapping between the orientation of $\mathrm{fr}\, D$ and the orientation of $\mathrm{fr}\, D'$. If they both are positive or negative w.r.t. the corresponding domains, $f\colon D \to D'$ is said to be <u>sense-preserving</u>. More general, if $f\colon E \to E'$ is a homeomorphism between two sets, f is said to be <u>sense-preserving</u> if $f|D$ is sense-preserving for every D such that $\mathrm{cl}\, D \subset E$. We note (cf. e.g. Newman [1], p. 198) that if $f\colon E \to E'$ is a homeomorphism and E is either a domain or the closure of a Jordan domain and there is a domain D such that $\mathrm{cl}\, D \subset E$ and $f|D$ is sense-preserving, then so is f. We also note that if f, f_1, and f_2 are sense-preserving, so are $\check{f}$ and $f_2 \circ f_1$ provided that $f_2 \circ f_1$ makes sense.

Suppose now that E is either an open set or the closure of a Jordan domain. A mapping $f\colon E \to \mathbb{E}$ is said to be <u>differentiable</u> at $z_0 \in \mathrm{int}\, E$ $(z_0, f(z_0) \neq \infty)$ if

$$(1.1)\quad f(z) = f(z_0) + f_z(z_0)(z - z_0) + f_{\bar z}(z_0)(\bar z - \bar z_0) + o(z - z_0),$$

where $f_z = \frac{1}{2}(f_x - i f_y)$, $f_{\bar z} = \frac{1}{2}(f_x + i f_y)$, $x = \mathrm{re}\, z$, $y = \mathrm{im}\, z$. It is clear that

$$(1.2)\quad \overline{f_z} = \bar f_{\bar z}, \quad \overline{f_{\bar z}} = \bar f_z.$$

A mapping f is said to be <u>differentiable</u> at ∞ if f^*, defined by

$f^*(z) = f(1/z)$, is differentiable at 0, and differentiable at z_0 such that $f(z_0) = \infty$ if f^{**}, defined by $f^{**}(z) = 1/f(z)$, is differentiable at z_0. A mapping $f\colon E \to \mathbb{E}$ is said to be differentiable if it is differentiable at every $z_0 \in \operatorname{int} E$. The directional derivatives $f_{|\alpha}$ are defined by $f_{|\alpha}(z) = e^{-i\alpha}[f_t(z + te^{i\alpha})]_{t=0}$, where α is real and $t \geq 0$. The Jacobian of f will be denoted by J or J_f. In the case where $z = \infty$ or $f(z_0) = \infty$ for some $z_0 \in E$, we do not define J_f but only $\operatorname{sgn} J_f(z_0) = \operatorname{sgn} J_{f^{**}}(z_0)$. It is easily verified that if $f\colon E \to \mathbb{C}$, where $E \subset \mathbb{C}$, is differentiable, then

$$(1.3)\quad f_{|\alpha} = e^{-i\alpha}(f_x \cos\alpha + f_y \sin\alpha) = f_z + e^{-2i\alpha} f_{\bar z},$$

$$(1.4)\quad \max|f_{|\alpha}| = |f_z| + |f_{\bar z}|, \quad \min|f_{|\alpha}| = ||f_z| - |f_{\bar z}||,$$

$$(1.5)\quad J = |f_z|^2 - |f_{\bar z}|^2 = \operatorname{sgn} J \max_\alpha |f_{|\alpha}| \min_\alpha |f_{|\alpha}|.$$

If f is differentiable at $z_0 \in \operatorname{int} E$ and $\operatorname{sgn} J_f \neq 0$, f is called regular at z and z is called a regular point of f. A mapping f is called regular if it is regular at every $z \in \operatorname{int} E$. A regular C^1-homeomorphism is called a diffeomorphism; in the case of sets E, E' containing ∞ the definition of C^1-functions should be extended similarly to that of differentiability. Suppose now that $f\colon E \to E'$ is a homeomorphism and E is either a domain or the closure of a Jordan domain. Then, by Newman's result quoted above, if $\operatorname{sgn} J(z) = 1$ at some regular z, f is sense-preserving and, conversely, if f is sense-preserving, $\operatorname{sgn} J(z) = 1$ at any regular $z \in \operatorname{int} E$.

Suppose that f is a diffeomorphism and D a domain. The ratio

$$(1.6)\quad p(z) \equiv p_f(z) = \max_\alpha |f_{|\alpha}(z)| / \min_\alpha |f_{|\alpha}(z)|, \quad z \in D,$$

is called the dilatation of f at z. Clearly it is bounded on every compact subset of D and invariant under conformal mappings. The second conclusion enables us to extend the definition of p to the cases $z = \infty$ and $f(z) = \infty$ analogously as the definition of differentiability was extended. A sense-preserving diffeomorphism $f\colon D \to D'$, where D is a domain and $\sup p(z) \leq Q < +\infty$, is called a regular Q-quasiconformal (shortly: regular Q-qc) mapping. These mappings, but not their name, were first introduced by Grötzsch [1] (cf. also Lavrentieff [1]).

The very natural definition of Grötzsch has the disadvantage that the class of regular Q-qc mappings is not closed w.r.t. almost uniform convergence. We are thus led to the following definition due to Lehto and Virtanen ([1], p. 222, or [2], p. 211): a nonconstant mapping $f: D \to \mathbb{E}$, D being a domain, is said to be Q-quasiconformal, if there is a sequence of regular Q-qc mappings $f_n: D_n \to D'$ such that $f_n \rightrightarrows f$ and for a.e. z for which there exist finite partial derivatives $f_z(z)$, $f_{\bar{z}}(z)$ we have $f_{n\bar{z}}(z)/f_{nz}(z) \to f_{\bar{z}}(z)/f_z(z)$. It is clear that if, in particular, f is a homeomorphism it is sense-preserving. The fact that any qc mapping is a homeomorphism will be proved in Section 11.

Actually the above definition is not exactly the same as that given by Lehto and Virtanen since they suppose that $D_n = D$, but as noticed by Gehring [2] and follows from their book (pp. 79-82 or 74-78) it makes no difference. It is worth-while to mention that, as noticed by Gehring [2] and follows from the quoted book (pp. 217-222 or 207-211), the restriction concerning the partial derivatives is superfluous since if we find a sequence of regular Q-qc mappings $f_n \rightrightarrows f$, we can always find another sequence with the required property and even a sequence of real-analytic mappings with $(f_{n\bar{z}}/f_{nz})$ consisting of polynomials. Since the proofs of these results are rather long and at the same time the presentation in the book of Lehto and Virtanen [1, 2] is excellent, we prefer to leave these aspects aside, all the more that we never make any use of them. For the problem in question cf. also Strebel [6].

According to Gehring [2] this is the most natural definition of qcty. We complete it by the following: a homeomorphism $f: D \to D'$, where D is the closure of a domain bounded by disjoint Jordan curves (in particular, of a Jordan domain) is said to be Q-quasiconformal if $f|\operatorname{int} D$ is Q-qc. The same completion is accepted for conformal mappings.

It is clear that the class of 1-qc mappings is identical with the class of conformal mappings. Thus qc mappings are their natural generalization. In many results on conformal mappings only qcty is essential, and various extremal problems in qc mappings lead to conformal mappings. On the other hand qc mappings are less rigid than conformal mappings, so they are much more flexible as a tool (cf. Ahlfors [5], pp. 1-2).

2. Differentiability

We are going to give (in Section 11) an analytic characterization of qc mappings which involves in a more clear way differentiability properties. This characterization in its final form is due to Gehring and Lehto [1], but it was originated by Strebel [1] and Mori [1]. We shall often refer to Saks [2] (in the bibliography we also list Saks [1]).

We begin with the notion of ACL. A continuous function $f\colon D \to \mathbb{E}$ is said to be absolutely continuous on lines (shortly: ACL) if for any rectangle U, $\operatorname{cl} U \subset D \setminus \{\infty\} \setminus \check{f}[\{\infty\}]$, with sides parallel to the coordinate axes it is absolutely continuous on almost all line segments in U which are parallel to either side of U.

We claim that if f is Q-qc, it is ACL.

In fact, modelling an idea of Pfluger [2], take U and y so that $U \cap \{z\colon \operatorname{im} z = y\} \neq \emptyset$. The function σ, defined by $\sigma(y) = |f[U \cap \{z\colon \operatorname{im} z < y\}]|$, is an increasing function, whence there exists a finite derivative $\sigma'(y)$ for every y in question except perhaps for a set of linear measure zero. Consider now a sequence (f_n) of regular Q-qc mappings as in the definition of Q-qcty of f. We are going to show that there exists a subsequence (f_{n_j}) such that for a.e. y the corresponding $\sigma'_{n_j}(y) \to \sigma'(y)$ and next that any such y gives rise to f absolutely continuous for fixed y.

Obviously $\sigma_n - \sigma$ is of bounded variation and hence the integral of $|\sigma'_n - \sigma'|$ over any interval in the domain of $\sigma_n - \sigma$ does not exceed the total variation of $\sigma_n - \sigma$ in this interval which tends to 0 as $n \to \infty$ since $f_n \rightrightarrows f$. Therefore $|\sigma'_n - \sigma'| \to 0$ in measure (cf. e.g. Graves [1], p. 236), whence, by F. Riesz's theorem (e.g. ibid., p. 244), (σ'_n) contains a subsequence (σ'_{n_j}) such that $\sigma'_{n_j} \to \sigma'$ a.e. In what follows we may drop the index j.

Now let $(x_k; \tilde{x}_k)$, $k = 1, \dots, m$, belong to $\{x\colon x + iy \in U\}$ and be disjoint. Clearly, for each k and $\tilde{y} > y$ we have, by (1.1),

$$(2.1)\quad (\tilde{y} - y)|f_n(\tilde{x}_k + iy_{k,n}) - f_n(x_k + iy_{k,n})| \le \int_y^{\tilde{y}} \int_{x_k}^{\tilde{x}_k} (|f_{n\zeta}| + |f_{n\bar{\zeta}}|)\,d\xi\,d\eta,$$

where $\xi = \operatorname{re}\zeta$, $\eta = \operatorname{im}\zeta$, and $y_{k,n}$ is chosen so that the length of the image arc of $\{\xi + iy_{k,n}\colon x_k \le \xi \le \tilde{x}_k\}$ is a minimum. Owing to the Schwarz inequality and the fact that f_n are regular Q-qc mappings, the right-hand side of (2.1) squared is estimated from above by

$$\iint_{R_k} \frac{|f_{n\zeta}| + |f_{n\bar{\zeta}}|}{|f_{n\zeta}| - |f_{n\bar{\zeta}}|}\,d\xi\,d\eta \iint_{R_k} (|f_{n\zeta}|^2 - |f_{n\bar{\zeta}}|^2)\,d\xi\,d\eta \le Q(\tilde{x}_k - x_k)(\tilde{y} - y)|f_n[R_k]|,$$

where $R_k = \{\xi + i\eta\colon x_k \le \xi \le \tilde{x}_k,\ y \le \eta \le \tilde{y}\}$. Consequently, applying the Schwarz inequality for sums, we conclude that

$$\sum_{k=1}^{m} |f_n(\tilde{x}_k + iy_{k,n}) - f_n(x_k + iy_{k,n})| / [\sum_{k=1}^{m} (\tilde{x}_k - x_k)]^{\frac{1}{2}}$$

$$\leq [Q \frac{\sigma_n(\tilde{y}) - \sigma_n(y)}{\tilde{y} - y}]^{\frac{1}{2}}.$$

Hence, letting $\tilde{y} \to y$, we obtain

$$\sum_{k=1}^{m} |f_n(\tilde{x}_k + iy) - f_n(x_k + iy)| \leq [Q\sigma_n'(y)]^{\frac{1}{2}} [\sum_{k=1}^{m} (\tilde{x}_k - x_k)]^{\frac{1}{2}}$$

and an analogous inequality for f, what means that f is ACL, as desired, since in the above consideration the roles of x and y can be interchanged.

Now, since f is ACL, it is of bounded variation on a.e. line segment in U which is parallel to either side of U. We claim that f has finite partial derivatives f_x and f_y a.e. in D. Indeed, if U^* is the set of all points of U, where f_x exists, and χ_{U^*} is its characteristic function, then χ_{U^*} is measurable as U^* is a Borel set, so, by the theorem of Fubini, we have

$$|U^*| = \iint_U \chi_{U^*}\, dxdy = \int_{y:x+iy\in U} \| U^* \cap \{z:\ \operatorname{im} z = y\} \|\, dy.$$

But f_x exists a.e. on $U \cap \{z:\ \operatorname{im} z = y\}$ for a.e. $y \in \{y:\ x + iy \in U\}$, whence $|U^*| = |U|$. Consequently f_x exists a.e. in D and, clearly, the same is true for f_y, as desired. Hence also f_z and $f_{\bar{z}}$ exist a. e. in D. Furthermore, (1.4) and (1.6) with f_n substituted for f as well as the definition of Q-qcty yield

$$(2.2)\quad |f_z| + |f_{\bar{z}}| \leq Q(|f_z| - |f_{\bar{z}}|) \text{ a.e.}$$

Therefore we have proved

<u>LEMMA 1</u>. <u>If</u> $f: D \to \mathbb{E}$ <u>is</u> Q-<u>qc</u>, <u>it is ACL and possesses a.e. partial derivatives which satisfy</u> (2.2).

In Section 11 we shall prove that the converse is also true, provided f is a sense-preserving homeomorphism and D a domain.

3. Distributional derivatives

Let $f: D \to \mathbb{C}$ be a locally integrable function in a domain $D \subset \mathbb{C}$. Functions $f_z', f_{\bar{z}}': D \to \mathbb{C}$ locally L^p (i.e. functions which are L^p on arbitrary compact subsets of D) are said to be distributional L^p-derivatives of f, if

$$(3.1)\quad \iint_D f'_z h\,dxdy = -\iint_D f\,h_z\,dxdy,\quad \iint_D f'_{\bar z} h\,dxdy = -\iint_D f\,h_{\bar z}\,dxdy$$

for all C^1-functions $h: D \to ¢$ with compact support. When considering the classes L^p we always assume that $p \geq 1$. This definition can immediately be generalized to the case $D, f[D] \subset \mathbb{E}$, assuming the above conditions for D replaced with $D \setminus \{\infty\} \setminus \check{f}[\{\infty\}]$. It is clear that the idea of defining the distributional L^p-derivatives with the help of (3.1) originates from Green's formula which, in its classical version, yields

$$(3.2)\quad \iint_{D^*} h^*_z\,dxdy = \tfrac{1}{2}i \int_{\partial D^*} h^* d\bar z,\quad \iint_{D^*} h^*_{\bar z}\,dxdy = -\tfrac{1}{2}i \int_{\partial D^*} h^* dz,$$

where D^* is a domain in ¢ with rectifiable ∂D^* consisting of disjoint Jordan curves and $h^*: \mathrm{cl}\, D^* \to ¢$ is a C^1-function.

We are going to show that any qc mapping f possesses distributional L^p-derivatives f'_z, $f'_{\bar z}$, and $f'_z = f_z$, $f'_{\bar z} = f_{\bar z}$ a.e. We begin with a lemma due to Gehring and Lehto [1].

By an open mapping we mean such a mapping $f: D \to \mathbb{E}$ that if $U \subset D$ and U is open, f[U] is open as well. Under a Borel function we understand a function $f: E \to \mathbb{R}$, ¢, resp. $\mathbb{E}$ such that the preimage $\check{f}[U]$ of each open set $U \subset \mathbb{R}$, ¢, resp. $\mathbb{E}$ is Borel. A point z_0 is said to be a point of linear density for a set E in the x (resp. y) direction if for the collection of closed line segments I containing z_0 and parallel to the x (resp. y) axis we have $|\!|E \cap I|\!|^*/|\!|I|\!| \to 1$ as $|\!|I|\!| \to 0$.

LEMMA 2. *If $f: D \to ¢$, D being a domain in ¢, is continuous and open, and possesses finite partial derivatives a.e. in D, it is differentiable a.e. in D.*

Proof. It is sufficient to prove the assertion for a compact $U \subset D$. We claim that for any $\eta > 0$ there is a closed set $U \subset D$ such that $|U \setminus E| < \eta$ and $f_x|E$, $f_y|E$ are continuous. In fact, let D^* denote the set where the both partial derivatives exist and are finite. For every $z \in D^*$ we define

$$(3.3)\quad f(z,t) = \left| \frac{f(z+t) - f(z)}{t} - f_x(z) \right| + \left| \frac{f(z+it) - f(z)}{t} - f_y(z) \right|,$$

where $t \in \mathbb{R} \setminus \{0\}$ and $z + t,\ z + it \in D$. For sufficiently small $|t|$, the functions $f(\ ,t)$ and, consequently, the supremum g_n of them taken, at a fixed z, over all rational $t \in (0; 1/n)$, n being sufficiently large, are defined a.e. in U and Borel. Since $g_n \to 0$ a.e. in D, we can apply Egoroff's theorem (cf. e.g. Saks [2], p. 18), and this yields the existence of a closed set $E \subset U$ such that $|U \setminus E| < \eta$ and the functions of z and t, defined on E at a given t by $(1/t)[f(z+t) - f(z)]$ and $(1/t)[f(z+it) - f(z)]$, both tend uniformly to $f_x|E$ and $f_y|E$, respectively. Hence the latter two functions are continuous, as desired. This means that for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\max(|z^* - z_0|, |t|) < \delta$ implies

(3.4) $$\max\{|f(z,t)|,\ |f_x(z^*) - f_x(z_0)|,\ |f_y(z^*) - f_y(z_0)|\} < \varepsilon;\quad z_0, z^*, z \in E.$$

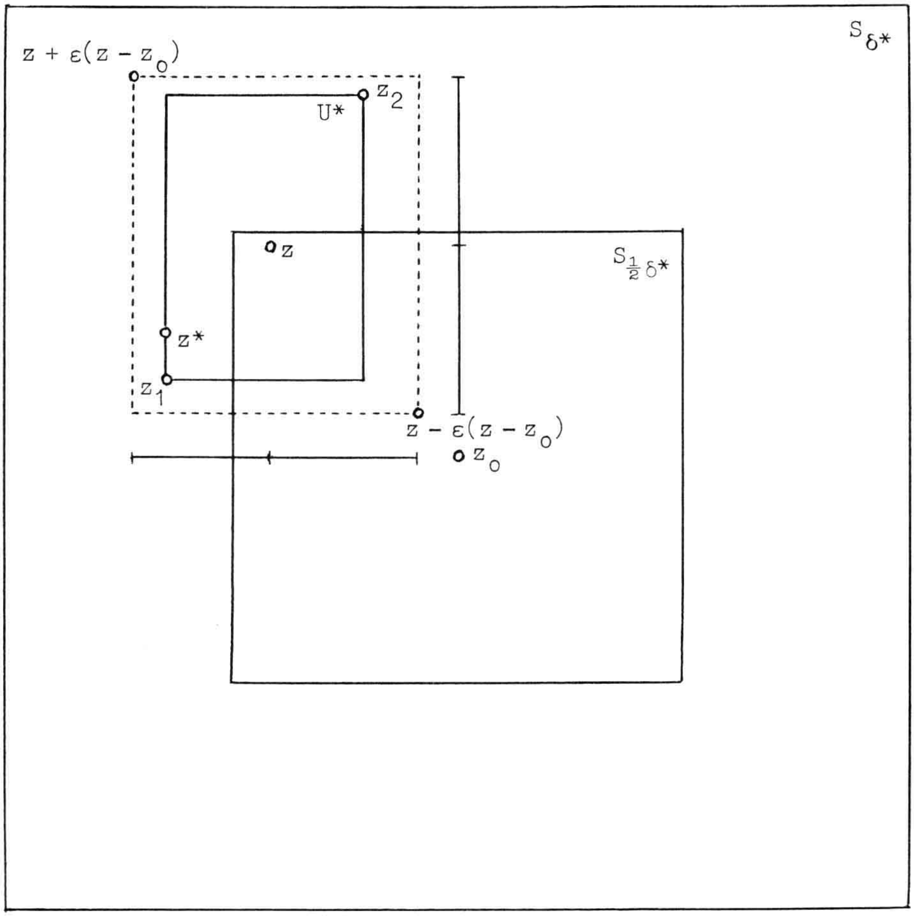

Fig. 1

Suppose now that z_o is a point of linear density for E in both x and y-directions. We are going to prove that f is differentiable at z_o. By the definition of z_o there is a square $S_{\delta^*} = \{x + iy: \max(|x - x_o|, |y - y_o|) < \delta^* < \delta\}$ such that for any closed line segment I, $I \subset S_{\delta^*}$, containing z_o and parallel to either coordinate axis, we have

$$(3.5)\quad [\![I \cap E]\!] \geq [\![I]\!]/(1 + \varepsilon).$$

We suppose next, in addition, that $z \in S_{\frac{1}{2}\delta^*} \setminus \{0\}$ and $\varepsilon < 1$. By (3.5), each of the line segments $(x + iy_o - \varepsilon(x - x_o);\ x + iy_o)$, $(x + iy_o;\ x + iy_o + \varepsilon(x - x_o))$, $(x_o + iy - \varepsilon i(y - y_o);\ x_o + iy)$, $(x_o + iy;\ x_o + iy + \varepsilon i(y - y_o))$ contains at least one point of E; let us denote those we have chosen by $x_1 + iy_o$, $x_2 + iy_o$, $x_o + iy_1$, $x_o + iy_2$, respectively. Let $U^* = \{x + iy: x_1 \leq x \leq x_2,\ y_1 \leq y \leq y_2\}$. Since f is open, it satisfies the maximum principle. Hence there is a $z^* \in \operatorname{fr} U^*$ such that $|f(s) - L(z)| \leq |f(z^*) - L(z)|$ for $s \in \operatorname{int} U^*$, where $L(z) = f(z_o) + f_z(z_o)(z - z_o) + f_{\bar z}(z_o)(z - z_o)$. In particular, we have

$$(3.6)\quad |f(z) - L(z)| \leq |f(z^*) - L(z)| \leq |f(z^*) - L(z^*)| + |L(z^*) - L(z)|$$
$$\leq |f(z^*) - L(z^*)| + 2\varepsilon\{|f_z(z_o)| + |f_{\bar z}(z_o)|\}|z - z_o|.$$

By the definition of U^*, either $x^* + iy_o \in E$ or $x_o + iy^* \in E$. Suppose e.g. that $x^* + iy_o \in E$. By (3.3) and (3.4), we get

$$\begin{aligned}
|f(z^*) - L(z^*)| &\leq |f(x^* + iy^*) - f(x^* + iy_o) - f_y(x^* + iy_o)(y^* - y_o)| \\
&\quad + |f(x^* + iy_o) - f(x_o + iy_o) - f_x(x_o + iy_o)(x^* - x_o)| \\
&\quad + |f_y(x^* + iy_o) - f_y(x_o + iy_o)|\,|y^* - y_o| \\
&\leq |f(x^* + iy_o, y^* - y_o)|\,|y^* - y_o| + |f(x_o + iy_o, x^* - x_o)|\,|x^* - x_o| \\
&\quad + |f_y(x^* + iy_o) - f_y(x_o + iy_o)|\,|y^* - y_o| \\
&\leq (1 + 2\varepsilon)\{|f(x^* + iy_o, y^* - y_o)| + |f(x_o + iy_o, x^* - x_o)| \\
&\quad + |f_y(x^* + iy_o) - f_y(x_o + iy_o)|\}\,|z - z_o|.
\end{aligned}$$

Hence, by (3.4), we obtain

$$(3.7)\quad |f(z^*) - L(z^*)| < 3(1 + 2\varepsilon)\varepsilon|z - z_o|.$$

An analogous calculation leads to (3.7) if $x_o + iy^* \in E$. Relations (3.6) and (3.7) yield (1.1), so f is differentiable at z_o, as desired.

3. Distributional derivatives

Finally we note that almost all points of a measurable set in $\mathbb{C}$ are their points of linear density in both x and y-directions (cf. Saks [2], p. 298). Hence our assertion follows.

Suppose now that $f: D \to D' \subset \mathbb{C}$, where D is a domain in $\mathbb{C}$, is an ACL sense-preserving homeomorphism and (2.2) holds. Let z_o be a point of differentiability and U a closed square containing z_o and with sides t. We have $|L[U]| = t^2 J(z_o)$, $\operatorname{dia} L[U] \le 2t\{|f_x(z_o)| + |f_y(z_o)|\}$, and $|f(z) - L(z)| \le |z - z_o| \varepsilon(t)$, $z \in U$, where $\varepsilon(t)$ is the maximum of $(1/|z - z_o|)|f(z) - L(z)|$ over all $|z - z_o| \in (0; 2t)$. Therefore

$$|f[u]| \le t^2 J(z_o) + 16\{|f_x(z_o)| + |f_y(z_o)|\} t^2 \varepsilon(t) + 4\pi t^2 \varepsilon^2(t),$$

$$|f[U]| \ge t^2 J(z_o) - 16\{|f_x(z_o)| + |f_y(z_o)|\} t^2 \varepsilon(t) + 16 t^2 \varepsilon^2(t),$$

whence $(1/t)|f[U]| \to J(z_o)$ as $t \to 0+$. On the other hand, a theorem of Lebesgue (cf. e.g. Saks [2], pp. 115 and 119) states that a non-negative totally additive bounded set function $\tau: D \to \mathbb{R}$ possesses a.e. a finite measurable derivative τ' and

$$(3.8) \qquad \tau(E) \ge \iint_E \tau' dxdy$$

for any Borel set $E \subset D$ with equality iff τ is locally absolutely continuous (concerning D in this theorem only $D \subset C$ is assumed). With $\tau(E) = |f[E]|$ we have $\tau'(z) = J(z)$ a.e. in D so that, by (3.8), J is locally L^1. Since, by (1.5) and (2.2), $(|f_z|^2 + |f_{\bar z}|)^2 \le QJ$ a.e., thus f_z and $f_{\bar z}$ are locally

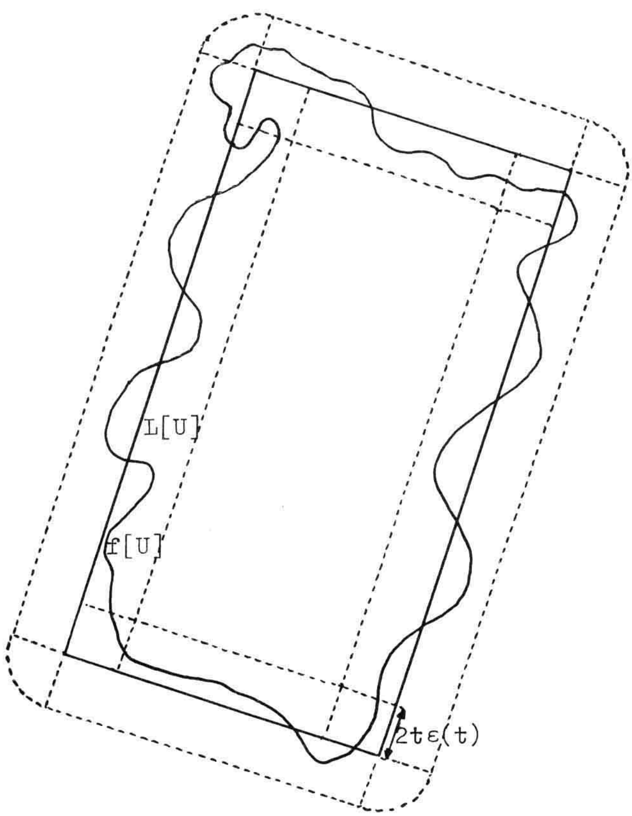

Fig. 2