BARON VON VEGA'S

LOGARITHMIC TABLES

of NUMBERS and TRIGONOMETRICAL FUNCTIONS

TRANSLATED FROM THE
FORTIETH or DR. BREMIKER'S
THOROUGHLY REVISED AND ENLARGED EDITION

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PREFACE.

The existing logarithmic-trigonometrical Tables to seven figures may be divided into three classes. While the first class which contains the logarithms of the natural numbers is very nearly the same in all, they differ in the second, or trigonometrical part in this, that the first class contains the trigonometrical functions for the greatest part of the quadrant only for every full minute, the second for every tenth second, and the third for every single second. The oldest logarithmic table to seven figures belongs to the first class, viz. that of Sherwin, London, 1705. It contained the logarithms of the numbers from 1 to 101000 and besides the sines, tangents and secants, also their logarithms from minute to minute. This table had run through several editions by the end of the last century and served till quite recently as the basis of innumerable logarithmic tables, in some of which the natural sines and tangents, which are little wanted, were omitted, while others of them were enriched by more or less useful additions. Gardiner, who in the year 1741 prepared a third edition of Sherwin's table, constructed at the same time a table proceeding from 10 to 10 seconds, which was the first of its kind, and was published in London in the year 1742 in large 4to [1]. A second edition of this table also in large 4to with the addition of the logarithms of the sines and tangents for every single second up to 4^0 was published at Avignon in 1770 [2]. Upon these were founded Callet's "tables portatives" which were first published at Paris in 1783 and afterwards stereotyped. On account of their more convenient shape they soon displaced the former tables, and maintained the preference up to the most recent time. The first table of the third kind is the large one by Taylor pulished in London in 1792. Bagay's tables [3] founded upon it and published at Paris in 1829, appeared in a much smaller form in consequence of the smallness of the figures. To the tables of the first and second class belong also the logarithmic tables to seven figures calculated for the centesimal division of the quadrant, as that by Hobert and Ideler [4] proceeding from 100th to 100th of a grade (centesimal degree), and that published by Delambre [5] in the year IX of the republic, in which the interval is ten times as small, viz. 10 centesimal seconds.

1) Tables of Logarithms.
2) Tables de Logarithmes.
3) Nouvelles Tables astronomiques et hydrographiques. Paris, 1829.
4) Neue trigonometrische Tafeln. Berlin, 1799.
5) Tables trigonométriques décimales, calculées par Ch. Borda.

For tables to five figures the interval of one minute may be considered the proper one, because the differences will not be too large to admit of the proportional part for seconds being mentally calculated with readiness. In tables to seven figures this interval has always been recognized as highly inconvenient, on which account Gardiner directed his attention to the diminution of the interval. It is even necessary to diminish the interval to one second, if it be made a condition that the proportional part in this case too is to be mentally calculated. This very small interval, however, gives rise to several new inconveniences, which diminish the advantage of smaller differences to such a degree, that tables of this kind have never come much into use, at least with professional computers. Their large form, the excessive number of columns on every page, and the circumstance that even then there is not nearly room enough for all the figures, (the initial figures of the logarithms having to be placed at the top of the columns and the differences altogether omitted, so that they must be calculated by the computer,) make the advantage of the smaller interval very questionable.

The editions of Vega's *Logarithmic Tables* hitherto published had, in the trigonometrical part, the interval of 1 minute. In the present revised edition care has been taken not only to adopt the interval of 10 seconds for the whole quadrant in order to assign a higher rank to the tables, but also to secure all those advantages which by attention to size and arrangement might in any way contribute to facilitate the looking out of logarithms. Although the tables have in consequence been considerably enlarged, the price has not been raised, so that the advantages of the enlarged edition are offered to purchasers at a cost, which is unexampled for a work of this kind.

Among the improvements in these tables we may mention:

1. The *systematic arrangement* of all the pages of each section, which has for its object that, when once the book is opened at the proper page, the eye with very little practice will involuntarily be directed to the place where the required logarithm is to be found. For nothing so much increases the difficulty of using tables as a want of uniformity in the position of the numbers. To attain this end main and sub-divisions have been introduced, which are easily distinguished and afford resting points to the eye. In the first part there are four such resting points on every page, viz. on the left page the lines corresponding to the numbers 10, 20, 30 and 40 are enclosed in double rules; on the right page, those corresponding to the numbers 60, 70, 80 and 90. The intermediate lines, opposite to the numbers 1 up to 9, are again separated by narrow spaces into groups of three each. In the second part each page contains 61 lines. Since five principal divisions are thus formed, the third, which is opposite to the number 30 either from the top or from the bottom, is marked by stronger lines, the subdivisions remaining the same as in the first part. Finally in the third part the lines corresponding to full minutes are inclosed between double rules, and every fifth minute is marked by thicker

rules. This arrangement which was first employed in a table of six figures [1]), has at the same time the advantage, that both with ascending and descending arguments, as in the 2nd and 3rd part, the rules have always the same position relatively both to the right and left margin of the page, an advantage which was unattainable with the single horizontal lines hitherto in use.

2. The *type*. On several occasions Mathematicians and Astronomers have remarked, that the strongly shaded and equally high figures that have become the fashion during the last twenty years, though they present on the whole a better appearance on the page, are yet far less legible than the older ones. Moreover the new type is easily injured in the finer strokes so that 1 and 4; 0, 6 and 9; 3, 5 and 8 become almost undistinguishable. Besides it often happens, that a figure thus strongly shaded, which dazzles the eye by its very blackness, does not leave sufficient blank space, the figures being both too close to each other and to the rules which mark the intervals. All these circumstances render the use of the tables more difficult and more fatiguing to the eye. Hence a type has been chosen which, being extremely slightly shaded, approaches the older form, projects partly above and below the parallel lines which inclose the main body and does not cover the white ground too much, while it is of a size which is neither too small to be easily recognized by moderately strong eyes, nor too large to get the requisite quantity into a moderate compass. The founding of this type is so characteristic that even after slight injuries which are inevitable after long use, there will be little fear of mistakes. Special attention has been paid to the proper distribution of the space which the figures and rules occupy; and the size of the page was not fixed upon till after repeated trials, and after all circumstances had been taken into account for facilitating the use of the tables.

3. In the trigonometrical part there is frequently a column next to that of the degrees, minutes and seconds, expressing the same arc in time; there are also in the first part one or two columns more which give the number, or the number increased tenfold, considered as a number of seconds expressed in degrees, minutes and seconds. This and similar other arrangements have here been omitted, as double and treble arguments, which are not out of place in nautical tables, but tend to obstruct the simple use of a table to seven figures without affording any equivalent advantage. On the other hand, in the first part at the foot of the page the twofold reduction of the argument into arc has been indicated, together with the logarithms of $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$ from 10 to 10 seconds, from 0 to 2° 46′ 40″. The latter are required for passing from log arc to log sin and log tan, and are of particular value in geodetic calculations, where long operations have sometimes to be performed with small arcs which

1) Logarithmorum VI decimalium nova tabula Berolinensis. Berlin, 1852.

are usually expressed in seconds, while four or five decimal places of the second have to be retained.

To facilitate the interpolation, small tables of proportional parts have been added in the first and third parts. The use of smaller type has enabled us to give them completely even in the first pages for every difference. These tables give accurately the tenths of the whole difference, so that the addition of the tenths, hundredths and thousandths gives the proportional part correct to the last figure, which is not the case with the usual arrangement. According to a very common arrangement the same small table of proportional parts has to be used for the whole extent of the table in its vicinity, without reference to the difference whether it be an unit more or less; but this has been rejected as inaccurate. In the third part want of space has prevented all the differences from being given; at the commencement, i. e. beginning at 5°, those only could be inserted which differed by 10 units, then those differing by 5, by 3 etc., while finally, from 24°, each difference is set down. The advantages which these tables afford in interpolation, will be best appreciated by those, who have felt their want in all other tables.

Special attention has been paid to the correctness of the seventh decimal place. Since the accurate value of a logarithm, unless it be a whole number, is always an interminable decimal fraction, the part which follows the first seven figures necessarily amounts to more or less than five units of the 8^{th} decimal place. In the former case the number in the seventh place must be increased by one unit, in the latter not, in order that the tabular value may always differ by less than half an unit of the last place from the true value of the logarithm. Therefore in order to set down the seventh place correctly, a knowledge of the following figures is required. A comparison with Vega's Thesaurus logarithmorum [1]) left only those logarithms of numbers doubtful, whose logarithms to ten figures ended with 500; these were therefore recalculated to fifteen places. The tenth figures proving accurate throughout, even in many other cases where the computation was repeated, it was found unnecessary to recalculate the logarithms ending in 499 and 501. The case was however far different in the trigonometrical section. Here the Thesaurus logarithmorum [1]), with the exception of a few cases as indicated by Vega himself and referring to that part of the table in which the interval is 10 seconds, can hardly be considered as anything but a reprint from Vlack [2]), since many comparisons of erroneous places have shown no difference between them. Now the uncertainty of the last figure amounts to 4 units, hence it became necessary to recalculate all the logarithms which in the Thesaurus end in 496, 497, 498, 499, 500, 501, 502, 503 and 504, since a correction amounting to more than 4 units might alter the last figure in logarithms to seven places [3]). These calculations have been made by means of

1) Thesaurus logarithmorum completus. Lipsiae, 1794.

2) Trigonometria artificialis. Goudae 1633.

3) Other errors, too, occur which cannot be considered as errors of the press.

Briggs's table to 14 places [1]), the results of which were checked by the formation of differences, and it has been found, that in modern tables, including the former editions of these tables, many logarithms have been set down wrong in consequence of too much confidence being placed in the Thes. log. For the comparison of the sines and tangents for every second the Thes. log. gave the logarithms to ten figures only up to 2 degrees, hence other means had to be adopted for the interval from 2 to 5 degrees. Interpolation from the Trig. Britt. which first suggested itself to us was found upon repeated trials to be too laborious, as it required six series of differences to be taken into account. Interpolation between the logarithms to ten figures, given for every ten seconds, did appear less open to objection, since the uncertainty of the last figure would be increased in the interpolated numbers. Hence direct computation was preferred, the more so, as with proper arrangement it led most rapidly and safely to the desired result. First of all for every 400 seconds in the interval from 0 to 5 degrees the logarithms of $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$ were calculated; the series giving these functions arranged according to powers of x converge very rapidly, so that the first four terms suffice to give the logarithm accurate to 14 places, and for the interpolation four series of differences suffice. These logarithms having been tested by differences and reduced to the interval of 100 seconds by interpolation, the first ten places were further interpolated to an interval of one second; the logarithms of the numbers being added, all the sines and tangents to ten figures were obtained. Amongst these logarithms, those terminating in 499, 500 and 501 were recomputed to three more figures on account of an uncertainty of an unit and a half arising from the interpolation. This calculation was effected by means of the logarithms, previously computed to 14 places by interpolation, in which process the logarithms of natural numbers given to 14 figures in Briggs's table [2]), were used, and thus the data were obtained for assigning to every logarithm which the table was to contain its strict value, independently of other determinations. The seventh decimal place being fixed, the preceding six were tested by differences, and evidence of their correctness was thus established.

In order to guard as much as possible against errors of the press three proofs were read previously to stereotyping, and one subsequently. The first two were corrected by comparison with the manuscript, while the third was tested by differences in order to be secure against MS. errors. Finally the last figure has been compared with the tables of Gardiner [3]) and Callet [4]) and in part with those of Taylor [5]). Gardiner, who

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Thus the last three figures 517 of log tan 0° 2′ 7″ are given too great by 90 and the Sine and Cotangent present the same error.

1) Trigonometria britannica, edited by Gellibrand. Goudae 1633.
2) H. Briggii arithmetica logarithmica. London, 1624.
3) Table of logarithms. London, 1826.
4) Tables portatives de logarithmes. Paris, 1795. Tirage 1821.
5) Tables of logarithms. London, 1792.

has likewise founded his logarithms to seven figures upon an abridgment of those to ten according to Vlack, has used great caution to insure the correctness of the 7th figure. He states in the preface to his tables, that he previously tested the last figure in Vlack by means of differences and corrected many errors, which he must have done with great care, for we found on comparison only two cases in which the last figure is not quite correct, viz. log 52943 and log cos 24° 55′ 30″, for which he gives 7238086 and 9.9575404, both of which are too large by an unit of the last place: the accurate values of these logarithms being 7238085 4683559 and 9.9575403 4999866. This care which also extends to that part of the table, which contains the logarithms for every second from 0 to 72 minutes, appears not to have met with due appreciation from later editors of logarithmic tables. Indeed it seems finally to have fallen completely into oblivion, as appears partly from the additions made in the later editions of Gardiner and partly from the circumstance, that after the publication of the Thesaurus logarithmorum in which, as has already been stated, only some few of Vlack's logarithms have been corrected, (most of them being reprinted unaltered), the editors of tables have taken its correctness for granted, either to avoid the trouble of a strict examination, or confiding in the accuracy implied by the offer of a ducat for the detection of every error, no similar reward having ever been previously offered. This affords the only explanation of the fact, that in more recent tables several logarithms have been brought back to the erroneous values which follow from Vlack's table, though they are correctly given by Gardiner. Thus 1) the log sin 2° 29′ 50″, whose accurate value is 8.6391970 5001389 has with Gardiner 1 in the seventh place which is correct; but since Vlack and Vega give the log to ten figures 6391970 499, we find in the recent tables and also in Taylor 0 in the seventh place. 2) log tan 5° 7′ 40″ is 8.9529682 4994086; Vlack and Vega give 9529682 501; Gardiner gives 2 in the seventh decimal place; the more recent tables, including Taylor's give 3. 3) log tan 7° 23′ 50″ is 9.1133684 5003401; and therefore 5 is the correct figure in the seventh place; but Vlack and Vega have 1133684 499, whence the recent tables including Taylor's give 4. 4) log tan 7° 59′ 0″ is 9.1468849 4989462, therefore 9 is the correct seventh figure; but Vega and Vlack have 1468849 501, and the modern tables have consequently 50 as the last two figures. 5) log sin 20° 9′ 0″ is 9.5371628 5003911, but in Vlack and Vega 5371628 499; Gardiner gives 9, the modern tables 8 in the 7th place.

The second edition of Gardiner published in the year 1770 at Avignon contains besides all the logarithms of the first edition, which were adopted without alteration [1]), the logarithms of the sines for every second from 72′ to 4°, and log tan for every second from 0 to 4°. The logarithms thus added have not the same degree of accuracy as those trans-

1) A few false prints, discovered during the 28 years that the tables had been used, were corrected, but many new errors were introduced.

ferred from the older table, though their correctness is still such as might well have served as a pattern for later tables. For a comparison shewed that out of 22032 logarithms only 22 were wrong by an unit. Both these editions are in large 4°, which renders them cumbrous. For this reason and in consequence of Gardiners's tables becoming scarce, Callet edited a new logarithmic-trigonometrical table, which was printed by Firmin Didot, and was the first, in which the printing from plates was successfully carried out. Besides a long introduction on the calculation and application of logarithms, tables of the natural sines and cosines, tangents and cotangents, and natural and common logarithms of numbers to 20 and 60 figures, this edition contained all the logarithms of the Avignon edition, with this addition that the log sin and log tan for every second which the latter contained up to 4° were carried one degree farther. But in this enlargement, the degree of accuracy introduced by Gardiner is utterly neglected. A comparison with this part of the tables which contains 6480 logarithms not occurring in Gardiner gave 1368 logarithms inaccurate by an unit in the last place. Finally the last figure of each of the 36000 logarithms of the second part was compared with Taylor's table, which is generally considered as a pattern of accuracy in the last figure, and every deviation thus found was subjected to a separate investigation. This examination gave 35 cases, in which the last figure in Taylor is wrong by an unit; of these 19 coincide with the erroneous statement in the Avignon edition.

A comparison of the logarithms of the first part with Babbage's most carefully edited table, in which elementary principles regarding the construction of tables were first established, furnished one case, that of log 52943 which, as in Gardiner, was given too great by an unit. The same error occurs also in the tables of Callet, Taylor, Delambre and others, evidently because all have copied Gardiner without farther examination.

The first two proofs of this table were corrected by MM. Lautensach, and Goldammer, and a third revision was instituted by myself, and checked by differences. After they were stereotyped the proofs from the plates were revised partly by Mr. Volkmann, and partly by Mr. Koch.

BREMIKER.

Berlin, August 1856.

INTRODUCTION.

The common Logarithms (Briggs's) are the only ones used to facilitate extensive numerical calculations. In this system, the logarithm of a number is the index of that power of 10 which is equal to the number. If for example a and b are the logarithms of the numbers A and B, we have

$$10^a = A \text{ and } 10^b = B$$

or

$$\log A = a \text{ and } \log B = b$$

and by the theory of powers and logarithms it follows, that

$$10^{a+b} = AB, \qquad 10^{a-b} = \frac{A}{B}, \qquad 10^{ac} = A^c, \qquad 10^{\frac{a}{c}} = \sqrt[c]{A},$$

or

$$\log AB = a + b, \qquad \log \frac{A}{B} = a - b, \qquad \log A^c = ca, \qquad \log \sqrt[c]{A} = \frac{a}{c},$$

from which we see that the logarithm of a product is the sum of the logarithms of the factors, the logarithm of a quotient the difference between the logarithms of the dividend and divisor, and that the logarithm of a power or root is found by multiplying or dividing the logarithm of the radix by the exponent.

The application of these four formulæ, to which all calculations with logarithms are limited, rests upon the condition that, given a number we can find with facility its logarithm, and conversely, given a logarithm we can readily find the corresponding number. It is the object of logarithmic tables to solve these two problems.

From the above definition of logarithms which is perfectly sufficient for practical purposes, it follows further that

$$\log 1 = 0, \quad \log 10 = 1, \quad \log 100 = 2, \quad \log 1000 = 3, \text{ etc.}$$
$$\log 0.1 = -1, \quad \log 0.01 = -2, \quad \log 0.001 = -3, \text{ etc.}$$

and these are the only numbers, whose logarithms are integers. The logarithm of any other number consists of an integer (termed the index or characteristic) followed by a decimal fraction (mantissa), which has the property of being incapable of being fully written out, since it goes on *ad infinitum*. Fortunately it is only in rare cases, that more than seven decimal places of this fraction are required: in general indeed four or five suffice, and it is only for the most accurate computations that six or seven are used.

TABLE I.

To find the logarithm of a given number, or the number corresponding to a given logarithm, a table is used which contains the logarithms of all whole numbers from 10000 to 100000. Such a table is that extending from p. 6 to p. 185. It is, indeed, preceded by a few pages containing the logarithms of 1 to 1000; but this is done more for the sake of convenience than because it is necessary, since all the same logarithms also occur in the rest of the table.

Given a number, to find its logarithm.

If the given number consists of five figures, the first four are to be sought in the first column headed N (number), and the last in the horizontal line at the top. Then the last four figures of the logarithm will be found where the horizontal line, corresponding to the first four figures of the number meets the vertical line corresponding to the fifth figure. The first three figures of the logarithm stand in the column headed 0, and are common to all the figures in the same horizontal line. Where these are not given on the horizontal line, the immediately preceding three figures have to be taken, except when the first of the last four figures is marked with a line above it, in which case the first three figures of the logarithm are those which occupy that position in the next horizontal line.

Example 1. Required the logarithm of 24818.

At p. 35 looking down the first vertical column we stop at 2481, and then follow the horizontal line to the right as far as the column headed 8. Here we find the number 7668. These figures are annexed to the number 394, which remain the same for several lines, and are placed three lines higher up in the column headed 0; so that we get 394 7668. The index which is not set down in the table is 4 throughout, hence $\log 24818 = 4.3947668$.

Example 2. Required the logarithm of 24833.

Here the last four figures are 0292, the first of which is marked with a line ($\bar{0}$), hence they are to be combined with 395; therefore the logarithm of 24833 is 4.3950292.

When the given number besides an integer part of five figures contains a decimal fraction, the difference of the logarithms of the two whole numbers of five figures between which the given number lies has to be multiplied with the decimal fraction, and the product to be added to the logarithm of the whole number. Thus let the number be 24833.73; this being contained between 24833 and 24834 whose logarithms differ by $0467 - 0292 = 175$, we multiply 175 by 0.73 which gives 128. Adding this to log 24833, which was 4.3950292, we get 4.3950420 for the log of 24833.73. To facilitate this multiplication every page contains small additional tables in a column headed P. P. (proportional parts) which for each of the differences occurring on that page give the several tenths, whence the hundredths, thousandths, etc. are easily deduced. In the last example we make use of the table headed 175, from which we get

for 0.7	122.5
for 0.03	5.25
together	127.75

of which the decimal fraction is rejected, but the last figure is increased by an unit, since the omitted fraction amounts to more than 0.5.

By means of the logarithms of the numbers, having an integer part of five figures, it is easy to find those of all other numbers written with the same significant figures; for their logarithms will only differ in the index. Thus if the logarithm of 24.83373 is required, we first find the log of 24833.73 and subtract from it log 1000. Since the latter $= 3$, we get $\log 24.83373 = 1.3950420$. In the same way $\log 248337300 = \log 24833.73 + \log 10000 = \log 24833.73 + 4 = 8.3950420$. If the given number is a proper fraction, as for instance 0.06103, we find at p. 108 $\log 61030 = 4.7855434$, and since $0.06103 = \frac{61030}{1000000}$ therefore log $0.06103 = \log 61030 - 6 = 4.7855434 - 6$, which would be negative, viz. -1.2144566; but it is more convenient for calculation to write $8.7855434 - 10$, so that the fraction in the logarithm may be positive. The general rule then, is: form a number of five figures in the integer part from the given number either by annexing cyphers or by separating the first five figures; find the fractional part of the logarithm of this number, and give an index less by one unit than the number of figures in the integer part of the given number. The index of a proper fraction is found by subtracting the number of cyphers between the decimal point and the first significant figure from 9.

Given a logarithm, required the corresponding number

A logarithm being given, and it being required to find the corresponding number, we first look out the first three figures of the decimal fraction in the logarithm in the column headed 0, and then in the columns headed 0, 1, 2 . . . those four figures, which are next less than the remaining four figures of the given logarithm. We now take out from the vertical column headed N the number consisting of the four figures, which are on the same horizontal line that contains the last four figures, and as fifth figure, the one at the head of the column. For example, if the logarithm 2.5833980 be given, we find at p. 62 the logarithm 583 3915, to which corresponds the number 38317. The difference between it and the next greater logarithm is 113, whilst the difference between it and the given logarithm is 65. Now the figures following the first five places are found by dividing 65 by 113, that is to say, the next figure is obtained by taking this fraction 10 times, the two next by taking it 100 times, the three next by taking it 1000 times. It would be inexpedient to take more, because even the third figure is incertain. But this division may be facilitated by the table of proportional parts headed 113. We take from it the next figure 5 corresponding to 56.5 which is next inferior to 65; subtracting 56.5 from 65, there remains 8.5. Taking this tenfold, which gives 85, we find opposite to the next inferior number 79.1 the number 7 as the next figure in the number, and again increasing the difference 85 — 79.1 = 5.9 tenfold, which gives 59, and looking for the next inferior number in the table (which is 56.6) we find the figure 5, hence the figures following the first five are 575, and the number is written with the figures 38317575; and since the given logarithm has the index 2, the decimal point must be put after the third figure; therefore the required number is 383.17575. The index, however, might have any other value, the number would still consist of the same figures, only the position of the decimal point would change; thus if the logarithm were 9.5833980, the corresponding number would contain an integer part of 10 figures. But since the tables furnish only the first eight figures of the number, as found above, two cyphers must be annexed, giving the number 3831757500. If the logarithm were 7.5833980 — 10, the corresponding number would be 0.0038317575. Hence the computation of the proportional part which with a little practice and by making use of the small table of proportional parts may be made mentally, (so that no other figures need be written down than those actually sought), would stand thus:

	Given logarithm		2.5833980
	pag. 62. number 383.17 . .		2.5833915
		Difference	65
by the table for the difference 113	5	. . . .	56.5
			85
again	7	. . . .	79.1
. . . .	5	. . . .	59
required number =	383.17575		

In logarithmic calculations cases occur, in which the logarithm of a logarithm has to be taken. If for example a number is to be raised to a power, whose exponent is a number of many figures, or a similar root has to be extracted, we should have to multiply or divide the logarithm of the given number by another large number; but this is again best done by the aid of logarithms. As a first example let it be required to raise the number 23.90087 to the power 1.1087023; here we take the log of 23.90087 which is = 1.3784137; and this must be multiplied by 1.1087023; hence we add the logarithms of both the numbers

$$\log 1.3784137 = 0.1393796$$
$$\log 1.1087023 = \underline{0.0448149}$$
$$\log 1.5282504 = 0.1841945$$

and look out the number 1.5282504 corresponding to their sum. This is the logarithm of the required power; hence this power is the number, corresponding to this logarithm, viz. 33.748186. Again, to take a second example, required the 7.001705th root of 0.791; here we should have to divide the log of 0.791 which is 9.8981765 — 10 by the index of the root. Now in this case the dividend is a difference, and it is simpler to restore the negative logarithm and to divide it than to divide the two terms of the difference separately, and to get the quotient by the subtraction of the results. Now the negative logarithm is = — 0.1018235; hence

$$\begin{aligned} \log 0.1018235 &= 9.0078480 - 10 \\ \log 7.001705 &= 0.8452038 \\ \log 0.0145427 &= 8.1626442 - 10 \end{aligned}$$

Corresponding to the difference 8.16226442 — 10 we find the number 0.0145427, which is the logarithm of the root; but it is negative, since it has arisen out of the division of a negative logarithm by a positive number; hence the tabular-logarithm is 10 — 0.0145427 — 10 = 9.9854573 — 10 and the corresponding number is — 0.9670687 which is the required root.

TABLE II.

Table II, (p. 188 to p. 287) serves to find log sin and log tan of an arc not greater than 5°, or the log cos and log cot of an arc lying between 85° and 90°. The left hand page of the table contains the logarithms of the sines of arcs between 0 and 5 degrees. Degrees and minutes are given at the top, the second on the left hand margin. These logarithms are at the same time the log cos of the complementary angles, the degrees and minutes of which are indicated at the bottom of the page, the seconds on the right hand margin. Thus according to p. 210 log sin 1° 11′ 46″ = 8.3196173, which is also log cos 88° 48 14″. The right hand page is constructed exactly similarly with regard to log tan of arcs between 0 and 5° and log cot of arcs between 85° and 90°

Given an arc, required its log sin or log tan.

For every full second the log sin and log tan are given in the table. If the arc contains additionally a fractional part of a second, the proportional part has to be calculated in the same manner as for the logarithms of numbers. For this purpose, the logarithm corresponding to the integer part of the seconds in the given arc is subtracted from the log corresponding to the next full second, or *vice versa*, the difference is multiplied by the fractional part of the second in the given arc, and the product added to, or substracted from the former logarithm, according as the logarithms go on increasing or decreasing; the former being here the case for descending arguments and the latter for ascending arguments. Thus let it be required to find log sin 2° 19′ 49″.71. At page

234 we find log sin 2° 19′ 49″ = 8.6091653, differing from the next greater log of sin 2° 19′ 50″ by 518, which multiplied by 0.71 gives 367.78 instead of which we take 368. Adding this proportional part to the preceding logarithm we get 8.6092021 as the required log sin 2° 19′ 49″.71.

As a second instance let us find log cot 86° 53′ 11″.374. According to p. 251 this log lies between 8.7355695 and 8.7355307 the difference of which is 388; multiplying this by 0.374 we get 145 which has to be subtracted from the former of these logarithms since they decrease for ascending arguments; hence the required log is 8.7355550.

Given the log sin or log tan of an arc, required the arc.

The process of finding the arc corresponding to a given log sin or log tan by means of table II is exactly the same which is adopted in finding the number corresponding to a given logarithm in table I. We look out the logarithm either next above or next below the given one, according as the logarithms increase or decrease with the increase of the arc; form the difference between it and the given logarithm, and divide it by the whole difference between the two consecutive logarithms of the table between which the given log lies. The quotient is the fractional part of a second which has to be added to the whole number of seconds contained in the arc corresponding to the logarithm, from which we had started; for example let 8.5139150 be the given log cos. The next greater log of the table is according to page 224, 8.5139642, differing from the given one by 492, but from the next smaller one in the tables by 645; hence the arc is that corresponding to the preceding logarithm, which is 88° 7′ 43″ increased by $\frac{492}{645}$ or 0″.763, i. e. 88° 7′ 43″.763.

When a calculation refers to arcs under 30 minutes, and it is required that the seventh decimal place of the logarithms shall be perfectly accurate, a condition which occurs in higher geodesy, we have to take into account five decimal places in the seconds in passing from the arc to log sin and log tan, or *vice versa*. In this case the use of table II becomes inconvenient both in consequence of the long multiplications and divisions in calculating the proportional parts, and because it would be necessary to take into account second differences. Here the application of table I is preferable. To find from it the log sin or log tan of an arc, convert it into seconds, take the log of the number of seconds, and add to it the corresponding value of S or T, given at the bottom of the page, each of which begins with 4.685. Thus if it be required to find log sin and log tan of 22′ 59″.7083, we have (see p. 13)

$$\begin{aligned} \log 1377.7083 &= 3.1391573 \\ S\ 22'\ 58'' &= 4.6855716 \\ T\ 22'\ 58'' &= \underline{4.6855813} \\ \log \sin &= \underline{7.8247289} \\ \log \tan &= 7.8247386 \end{aligned}$$

where in interpolating we have only to take account of whole seconds. If the arc were 137".77083, we should as before take log arc from p. 13, but the numbers S and T with the argument 2′ 18″; from p. 2; thus we should have

$$\begin{aligned} \log 137''.77083 &= 2.1391573 \\ S\ 2'\ 18'' &= 4.6855748 \\ T\ 2'\ 18'' &= \underline{4.6855749} \\ \log\sin &= 6.8247321 \\ \log\tan &= 6.8247322 \end{aligned}$$

If conversely, it be required to find the arc corresponding to a given log sin or log tan, we first take the arc to the nearest second below from table II, and look out in table I the corresponding values of S and T; we subtract these from the given logarithm and take out the number corresponding to the difference, which gives the required arc in seconds. Thus let 7.1690522 be a given log tan. According to p. 189 the corresponding arc lies between 5′ 4″ and 5′ 5″; at p. 3 the value of T is 4.6855752, hence

$$\begin{array}{lr} \log\tan\ \ldots\ldots & 7.1690522 \\ T\ 5'\ 4''\ \ldots\ldots & \underline{4.6855752} \\ \text{Difference} & 2.4834770 = \log 304.4227' \end{array}$$

hence arc = 304″.4227 or 5′ 4″.4227.

In this method it would be necessary to refer to two or even three different places of the tables; to avoid this inconvenience a table has been added at p. 575 which for arcs up to 35′ gives with much more readiness all transitions between S, arc and log sin as well as those between T, arc and log tan. For extensive geodetic calculations this table may be taken out of the book and mounted on cardboard, to be used as an auxiliary table, besides which we require only the table of logarithms of the numbers of the seconds in the arc. In regard to the construction of this table, it is to be observed, that the value of S = 4.6855749 holds up to 1′ 39″ and log sin 6.681 From 1′ 40″ or log sin = 6.682 . . . the succeeding value of S = 4.6855748 is to be used. All other values of S answer accurately to the arcs and log sin on the same line with them, so that in passing from the arc or log sin to the number S that value of S has to be taken, which lies nearest to the given arc or log sin. The same rule holds for T. Hence, except for a few values of S and T towards the end of the table which increase by 10 units at a time, all interpolation is avoided, as with a glance at the table we can at once take out the required S or T.

TABLE III.

This table contains for every ten seconds of the quadrant the logarithms of the sines, cosines, tangents, and cotangents. From 0 to 45°

the degrees are placed at the top of the page; the minutes and seconds, denoted with ′ and ″ on the left hand margin, which is to be combined with the heading at the top of the page. From 45° to 90° the degrees are put at the bottom of the page, the minutes and seconds at the right hand margin, and the name of the trigonometrical function at the foot of the column. Arguments that are opposite one another are complementary (i. e. make together 90°) and the sines and tangents (i. e. their logarithms) of an arc above 45° are also placed opposite to the sines and tangents of its complement, because the former are respectively the cosines and cotangents of the latter, and *vice versa*. For the sake of symmetry the sine is placed next to the argument. next to it the tangent, which is followed by the cotangent, and this by the cosine; and since for complementary angles the same logarithms hold, but in an inverse order, there is again next to the argument the sine, followed in order by the tangent, cotangent, and cosine, as may be seen by the names at the top and bottom of the columns. The differences of any two consecutive logarithms are placed for the sine and cosine immediately to the right on the intermediate line, and are headed d (differentia); for the tangents and cotangents they are also placed between these columns on the intermediate line and are headed d. c. (differentia communis) since these differences hold for both. The sines and cosines of all arcs in the quadrant are proper fractions, and so are the tangents for arcs below 45°; hence their logarithms would be negative. But the index and the fraction have been made positive by the addition of 10, so that — 10 has to be supplied. For tangents of arcs between 45° and 90° this supplement must be omitted.

Given an acute angle, required its sine (log sin).

As the table contains the sine only for every tenth second, we have to interpolate for the units and its decimal parts. This is done, as in the case of the logarithms of numbers, by multiplying the decimal fraction formed by the tenth part of the units with the appended fraction by the difference, and adding the product to the log sin of the next inferior multiple of 10″; for example let it be required to find log sin 18° 51′ 27″.21. Here the sine of 18° 51′ 20″ is 9.5094491, which differs from the next in the table by 616; multiplying this by 0.721 we get 444, which added to the preceding logarithm gives 9.5094935, the number required. The formation of this product, when no separate auxiliary table of proportional parts[1]) is employed, is facilitated by the small tables of differences on the margin. Since for want of space it has not been possible to put down all these differences we in every case take the nearest; in the present instance we take that for 614. It gives

1) Bremiker, Tafel der Proportionaltheile (Table of proportional parts). Berlin, Ferd. Dümmler, 1843.