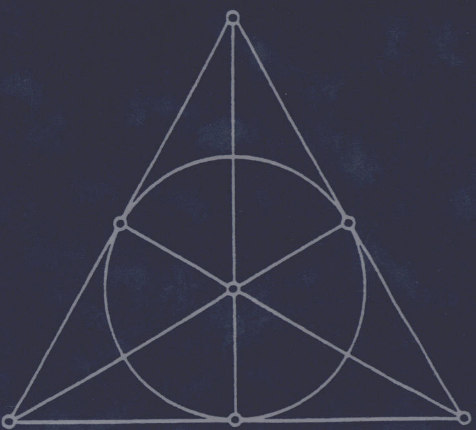
Wiley-Interscience Series in Discrete Mathematics and Optimization

COMBINATORICS

SECOND EDITION



RUSSELL MERRIS

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Combinatorics

SECOND EDITION

RUSSELL MERRIS
California State University, Hayward

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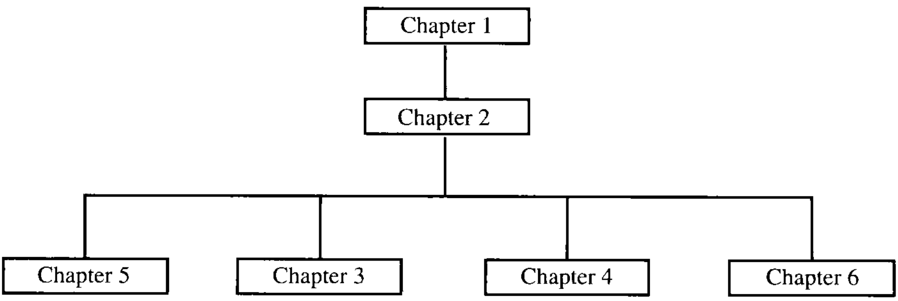
This book is dedicated to my wife, Karen Diehl Merris.

Preface

This book is intended to be used as the text for a course in combinatorics at the level of beginning upper division students. It has been shaped by two goals: to make some fairly deep mathematics accessible to students with a wide range of abilities, interests, and motivations and to create a pedagogical tool useful to the broad spectrum of instructors who bring a variety of perspectives and expectations to such a course.

The author's approach to the second goal has been to maximize flexibility. Following a basic foundation in Chapters 1 and 2, each instructor is free to pick and choose the most appropriate topics from the remaining four chapters. As summarized in the chart below, Chapters 3–6 are *completely independent of each other.* Flexibility is further enhanced by optional sections and appendices, by weaving some topics into the exercise sets of multiple sections, and by identifying various points of departure from each of the final four chapters. (The price of this flexibility is some redundancy, e.g., several definitions can be found in more than one place.)



Turning to the first goal, students using this book are expected to have been exposed to, even if they cannot recall them, such notions as equivalence relations, partial fractions, the Maclaurin series expansion for e^x, elementary row operations, determinants, and matrix inverses. A course designed around this book should have as specific prerequisites those portions of calculus and linear algebra commonly found among the lower division requirements for majors in the mathematical and computer sciences. Beyond these general prerequisites, the last two sections of Chapter 5 presume the reader to be familiar with the *definitions* of classical adjoint (adjugate) and characteristic roots (eigenvalues) of real matrices, and the first two sections of Chapter 6 make use of reduced row-echelon form, bases, dimension, rank, nullity, and orthogonality. (All of these topics are reviewed in Appendix A3.)

Strategies that promote student engagement are a lively writing style, timely and appropriate examples, interesting historical anecdotes, a variety of exercises (tempered and enlivened by suitable hints and answers), and judicious use of footnotes and appendices to touch on topics better suited to more advanced students. These are things about which there is general agreement, at least in principle.

There is less agreement about how to focus student energies on attainable objectives, in part because focusing on some things inevitably means neglecting others. If the course is approached as a *last chance* to expose students to this marvelous subject, it probably will be. If approached more invitingly, as a *first* course in combinatorics, it may be. To give some specific examples, highlighted in this book are binomial coefficients, Stirling numbers, Bell numbers, and partition numbers. These topics appear and reappear throughout the text. Beyond reinforcement in the service of retention, the tactic of overarching themes helps foster an image of combinatorics as a unified mathematical discipline. While other celebrated examples, e.g., Bernoulli numbers, Catalan numbers, and Fibonacci numbers, are generously represented, they appear almost entirely in the exercises. For the sake of argument, let us stipulate that these roles could just as well have been reversed. The issue is that beginning upper division students cannot be expected to absorb, much less appreciate, *all* of these special arrays and sequences in a single semester. On the other hand, the flexibility is there for willing admirers to rescue one or more of these justly famous combinatorial sequences from the relative obscurity of the exercises.

While the overall framework of the first edition has been retained, everything else has been revised, corrected, smoothed, or polished. The focus of many sections has been clarified, e.g., by eliminating peripheral topics or moving them to the exercises. Material new to the second edition includes an optional section on algorithms, several new examples, and many new exercises, some designed to guide students to discover and prove nontrivial results for themselves. Finally, the section of hints and answers has been expanded by an order of magnitude.

The material in Chapter 3, Pólya's theory of enumeration, is typically found closer to the end of comparable books, perhaps reflecting the notion that it is the *last* thing that should be taught in a junior-level course. The author has aspired, not only to make this theory accessible to students taking a first upper division mathematics course, but to make it possible for the subject to be addressed right after Chapter 2. Its placement in the middle of the book is intended to signal that it *can* be fitted in there, not that it must be. If it seems desirable to cover some but not all of Chapter 3, there are many natural places to exit in favor of something else, e.g., after the application of Bell numbers to transitivity in Section 3.3, after enumerating the overall number of color patterns in Section 3.5, after *stating* Pólya's theorem in Section 3.6, or after proving the theorem at the end of Section 3.6.

Optional Sections 1.3 and 1.10 can be omitted with the understanding that exercises in subsequent sections involving probability or algorithms should be assigned with discretion. With the same caveat, Section 1.4 can be omitted by those not intending to go on to Sections 6.1, 6.2, or 6.4. The material in Section 6.3, touching on mutually orthogonal Latin squares and their connection to finite projective planes, can be covered independently of Sections 1.4, 6.1, and 6.2.

The book contains much more material then can be covered in a single semester. Among the possible syllabi for a one semester course are the following:

- Chapters 1, 2, and 4 and Sections 3.1–3.3
- Chapters 1 (omitting Sections 1.3, 1.4, & 1.10), 2, and 3, and Sections 5.1 & 5.2
- Chapters 1 (omitting Sections 1.3 & 1.10), 2, and 6 and Sections 4.1 – 4.4
- Chapters 1 (omitting Sections 1.4 & 1.10) and 2 and Sections 3.1 – 3.3, 4.1 – 4.3, & 6.3
- Chapters 1 (omitting Sections 1.3 & 1.4) and 2 and Sections 4.1 – 4.3, 5.1, & 5.3–5.7
- Chapters 1 (omitting Sections 1.3, 1.4, & 1.10) and 2 and Sections 4.1 – 4.3, 5.1, 5.3–5.5, & 6.3

Many people have contributed observations, suggestions, corrections, and constructive criticisms at various stages of this project. Among those deserving special mention are former students David Abad, Darryl Allen, Steve Baldzikowski, Dale Baxley, Stanley Cheuk, Marla Dresch, Dane Franchi, Philip Horowitz, Rhian Merris, Todd Mullanix, Cedide Olcay, Glenn Orr, Hitesh Patel, Margaret Slack, Rob Smedfjeld, and Masahiro Yamaguchi; sometime collaborators Bob Grone, Tom Roby, and Bill Watkins; correspondents Mark Hunacek and Gerhard Ringel; reviewers Rob Beezer, John Emert, Myron Hood, Herbert Kasube, André Kézdy, Charles Landraitis, John Lawlor, and Wiley editors Heather Bergman, Christine Punzo, and Steve Quigley. I am especially grateful for the tireless assistance of Cynthia Johnson and Ken Rebman.

Despite everyone's best intentions, no book seems complete without some errors. An up-to-date errata, accessible from the Internet, will be maintained at URL

http://www.sci.csuhayward.edu/~rmerris

Appropriate acknowledgment will be extended to the first person who communicates the specifics of a previously unlisted error to the author, preferably by e-mail addressed to

merris@csuhayward.edu

Hayward California RUSSELL MERRIS

Contents

Note: Asterisks indicate optional sections that can be omitted without loss of continuity.

1

The Mathematics of Choice

> It seems that mathematical ideas are arranged somehow in strata, the ideas in each stratum being linked by a complex of relations both among themselves and with those above and below. The lower the stratum, the deeper (and in general the more difficult) the idea. Thus, the idea of an irrational is deeper than the idea of an integer.
>
> — G. H. Hardy (*A Mathematician's Apology*)

Roughly speaking, the first chapter of this book is the top stratum, the surface layer of combinatorics. Even so, it is far from superficial. While the first main result, the so-called fundamental counting principle, is nearly self-evident, it has enormous implications throughout combinatorial enumeration. In the version presented here, one is faced with a sequence of decisions, each of which involves some number of choices. It is from situations like this that the chapter derives its name.

To the uninitiated, mathematics may appear to be "just so many numbers and formulas." In fact, the numbers and formulas should be regarded as shorthand notes, summarizing *ideas*. Some ideas from the first section are summarized by an algebraic formula for multinomial coefficients. Special cases of these numbers are addressed from a combinatorial perspective in Section 1.2.

Section 1.3 is an optional discussion of probability theory which can be omitted if probabilistic exercises in subsequent sections are approached with caution. Section 1.4 is an optional excursion into the theory of binary codes which can be omitted by those not planning to visit Chapter 6. Sections 1.3 and 1.4 are partly motivational, illustrating that even the most basic combinatorial ideas have real-life applications.

In Section 1.5, ideas behind the formulas for sums of powers of positive integers motivate the study of relations among binomial coefficients. Choice is again the topic in Section 1.6, this time with or without replacement, where order does or doesn't matter.

To better organize and understand the multinomial theorem from Section 1.7, one is led to symmetric polynomials and, in Section 1.8, to partitions of n. Elementary symmetric functions and their association with power sums lie at the

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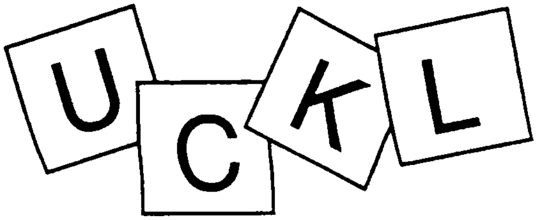
heart of Section 1.9. The final section of the chapter is an optional introduction to algorithms, the flavor of which can be sampled by venturing only as far as Algorithm 1.10.3. Those desiring not less but more attention to algorithms can find it in Appendix A2.

1.1. THE FUNDAMENTAL COUNTING PRINCIPLE



How many different four-letter words, including nonsense words, can be produced by rearranging the letters in LUCK? In the absence of a more inspired approach, there is always the brute-force strategy: Make a systematic list.

Once we become convinced that Fig. 1.1.1 accounts for every possible rearrangement and that no "word" is listed twice, the solution is obtained by counting the 24 words on the list.

While finding the brute-force strategy was effortless, implementing it required some work. Such an approach may be fine for an isolated problem, the *like* of which one does not expect to see again. But, just for the sake of argument, imagine yourself in the situation of having to solve a great many thinly disguised variations of this same problem. In that case, it would make sense to invest some effort in finding a strategy that requires less work to implement. Among the most powerful tools in this regard is the following commonsense principle.

1.1.1 Fundamental Counting Principle. Consider a (finite) sequence of decisions. Suppose the number of choices for each individual decision is independent of decisions made previously in the sequence. Then the number of ways to make the whole sequence of decisions is the product of these numbers of choices.

To state the principle symbolically, suppose c_i is the number of choices for decision i. If, for $1 \leq i < n$, c_{i+1} does not depend on which choices are made in

LUCK	LUKC	LCUK	LCKU	LKUC	LKCU
ULCK	ULKC	UCLK	UCKL	UKLC	UKCL
CLUK	CLKU	CULK	CUKL	CKLU	CKUL
KLUC	KLCU	KULC	KUCL	KCLU	KCUL

Figure 1.1.1. The rearrangements of LUCK.

decisions $1, \ldots, i$, then the number of different ways to make the sequence of decisions is $c_1 \times c_2 \times \cdots \times c_n$.

Let's apply this principle to the word problem we just solved. Imagine yourself in the midst of making the brute-force list. Writing down one of the words involves a sequence of four decisions. Decision 1 is which of the four letters to write first, so $c_1 = 4$. (It is no accident that Fig. 1.1.1 consists of four rows!) For each way of making decision 1, there are $c_2 = 3$ choices for decision 2, namely which letter to write second. Notice that the specific letters comprising these three choices depend on how decision 1 was made, but their *number* does not. That is what is meant by the number of choices for decision 2 being independent of how the previous decision is made. Of course, $c_3 = 2$, but what about c_4? Facing no alternative, is it correct to say there is "no choice" for the last decision? If that were literally true, then c_4 would be zero. In fact, $c_4 = 1$. So, by the fundamental counting principle, the number of ways to make the sequence of decisions, i.e., the number of words on the final list, is

$$c_1 \times c_2 \times c_3 \times c_4 = 4 \times 3 \times 2 \times 1.$$

The product $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ is commonly written n! and read *n-factorial.** The number of four-letter words that can be made up by rearranging the letters in the word LUCK is $4! = 24$.

What if the word had been LUCKY? The number of five-letter words that can be produced by rearranging the letters of the word LUCKY is $5! = 120$. A systematic list might consist of five rows each containing $4! = 24$ words.

Suppose the word had been LOOT? How many four-letter words, including nonsense words, can be constructed by rearranging the letters in LOOT? Why not apply the fundamental counting principle? Once again, imagine yourself in the midst of making a brute-force list. Writing down one of the words involves a sequence of four decisions. Decision 1 is which of the three letters L, O, or T to write first. This time, $c_1 = 3$. But, what about c_2? In this case, the number of choices for decision 2 depends on how decision 1 was made! If, e.g., *L* were chosen to be the first letter, then there would be two choices for the second letter, namely O or T. If, however, O were chosen first, then there would be three choices for the second decision, L, (the second) O, or T. Do we take $c_2 = 2$ or $c_2 = 3$? The answer is that *the fundamental counting principle does not apply to this problem* (at least not directly). The fundamental counting principle applies *only* when the *number* of choices for decision $i + 1$ is *independent* of how the previous i decisions are made.

To enumerate all possible rearrangements of the letters in LOOT, begin by distinguishing the two O's. maybe write the word as LOoT. Applying the fundamental counting principle, we find that there are $4! = 24$ different-*looking* four-letter words that can be made up from L, O, o, and T.

*The exclamation mark is used, not for emphasis, but because it is a convenient symbol common to most keyboards.

LOoT	LOTo	LoOT	LoTO	LTOo	LToO
OLoT	OLTo	OoLT	OoTL	OTLo	OToL
oLOT	oLTO	oOLT	oOTL	oTLO	oTOL
TLOo	TLoO	TOLo	TOoL	ToLO	ToOL

Figure 1.1.2. Rearrangements of LOoT.

Among the words in Fig. 1.1.2 are pairs like OLoT and oLOT, which look different only because the two O's have been distinguished. In fact, every word in the list occurs twice, once with "big O" coming before "little o", and once the other way around. Evidently, the number of different words (with indistinguishable O's) that can be produced from the letters in LOOT is not 4! but $4!/2 = 12$.

What about TOOT? First write it as TOot. Deduce that in any list of all possible rearrangements of the letters T, O, o, and t, there would be $4! = 24$ different-looking words. Dividing by 2 makes up for the fact that two of the letters are O's. Dividing by 2 again makes up for the two T's. The result, $24/(2 \times 2) = 6$, is the number of different words that can be made up by rearranging the letters in TOOT. Here they are

TTOO TOTO TOOT OTTO OTOT OOTT

All right, what if the word had been LULL? How many words can be produced by rearranging the letters in LULL? Is it too early to guess a pattern? Could the number we're looking for be $4!/3 = 8$? No. It is easy to see that the correct answer must be 4. Once the position of the letter U is known, the word is completely determined. Every other position is filled with an L. A complete list is ULLL, LULL, LLUL, LLLU.

To find out why 4!/3 is wrong, let's proceed as we did before. Begin by distinguishing the three L's, say L_1, L_2, and L_3. There are 4! different-looking words that can be made up by rearranging the four letters L_1, L_2, L_3, and U. If we were to make a list of these 24 words and then erase all the subscripts, how many times would, say, LLLU appear? The answer to this question can be obtained from the fundamental counting principle! There are three decisions: decision 1 has three choices, namely which of the three L's to write first. There are two choices for decision 2 (which of the two remaining L's to write second) and one choice for the third decision, which L to put last. Once the subscripts are erased, LLLU would appear 3! times on the list. We should divide $4! = 24$, not by 3, but by $3! = 6$. Indeed, $4!/3! = 4$ is the correct answer.

Whoops! if the answer corresponding to LULL is 4!/3!, why didn't we get 4!/2! for the answer to LOOT? In fact, we did: $2! = 2$.

Are you ready for MISSISSIPPI? It's the same problem! If the letters were all different, the answer would be 11!. Dividing 11! by 4! makes up for the fact that there are four I's. Dividing the quotient by another 4! compensates for the four S's.

Dividing that quotient by 2! makes up for the two P's. In fact, no harm is done if that quotient is divided by $1! = 1$ in honor of the single M. The result is

$$\frac{11!}{4!\,4!\,2!\,1!} = 34,650.$$

(Confirm the arithmetic.) The 11 letters in MISSISSIPPI can be (re)arranged in 34,650 different ways.*

There is a special notation that summarizes the solution to what we might call the "MISSISSIPPI problem."

1.1.2 Definition. The *multinomial coefficient*

$$\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1!r_2!\cdots r_k!},$$

where $r_1 + r_2 + \cdots + r_k = n$.

So, "multinomial coefficient" is a *name* for the answer to the question, how many n-letter "words" can be assembled using r_1 copies of one letter, r_2 copies of a second (different) letter, r_3 copies of a third letter, ..., and r_k copies of a kth letter?

1.1.3 Example. After cancellation,

$$\binom{9}{4,3,1,1} = \frac{9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{4\times 3\times 2\times 1\times 3\times 2\times 1\times 1\times 1}$$
$$= 9\times 8\times 7\times 5 = 2520.$$

Therefore, 2520 different words can be manufactured by rearranging the nine letters in the word SASSAFRAS. □

In real-life applications, the words need not be assembled from the English alphabet. Consider, e.g., POSTNET† barcodes commonly attached to U.S. mail by the Postal Service. In this scheme, various numerical delivery codes‡ are represented by "words" whose letters, or *bits*, come from the alphabet {ı, |}. Corresponding, e.g., to a ZIP + 4 code is a 52-bit barcode that begins and ends with |. The 50-bit middle part is partitioned into ten 5-bit zones. The first nine of these zones are for the digits that comprise the ZIP + 4 code. The last zone accommodates a *parity*

*This number is roughly equal to the number of members of the Mathematical Association of America (MAA), the largest professional organization for mathematicians in the United States.

†Postal Numeric Encoding Technique.

‡The original five-digit Zoning Improvement Plan (ZIP) code was introduced in 1964; ZIP+4 codes followed about 25 years later. The 11-digit Delivery Point Barcode (DPBC) is a more recent variation.

0 = 1 = 2 = 3 = 4 =

5 = 6 = 7 = 8 = 9 =

Figure 1.1.3. POSTNET barcodes.

check digit, chosen so that the sum of all ten digits is a multiple of 10. Finally, each digit is represented by one of the 5-bit barcodes in Fig. 1.1.3. Consider, e.g., the ZIP +4 code 20090-0973, for the Mathematical Association of America. Because the sum of these digits is 30, the parity check digit is 0. The corresponding 52-bit word can be found in Fig. 1.1.4.

20090-0973

Figure 1.1.4

We conclude this section with another application of the fundamental counting principle.

1.1.4 Example. Suppose you wanted to determine the number of positive integers that exactly divide $n = 12$. That isn't much of a problem; there are six of them, namely, 1, 2, 3, 4, 6, and 12. What about the analogous problem for $n = 360$ or for $n = 360,000$? Solving even the first of these by brute-force list making would be a lot of work. Having already found another strategy whose implementation requires a lot less work, let's take advantage of it.

Consider $360 = 2^3 \times 3^2 \times 5$, for example. If $360 = dq$ for positive integers d and q, then, by the uniqueness part of the *fundamental theorem of arithmetic*, the prime factors of d, together with the prime factors of q, are precisely the prime factors of 360, multiplicities included. It follows that the prime factorization of d must be of the form $d = 2^a \times 3^b \times 5^c$, where $0 \le a \le 3, 0 \le b \le 2$, and $0 \le c \le 1$. Evidently, there are four choices for a (namely 0, 1, 2, or 3), three choices for b, and two choices for c. So, the number of possibile d's is $4 \times 3 \times 2 = 24$. □

1.1. EXERCISES

1 The Hawaiian alphabet consists of 12 letters, the vowels *a*, *e*, *i*, *o*, *u* and the consonants *h*, *k*, *l*, *m*, *n*, *p*, *w*.

(a) Show that 20,736 different 4-letter "words" could be constructed using the 12-letter Hawaiian alphabet.

(b) Show that 456,976 different 4-letter "words" could be produced using the 26-letter English alphabet.*

(c) How many four-letter "words" can be assembled using the Hawaiian alphabet if the second and last letters are vowels and the other 2 are consonants?

(d) How many four-letter "words" can be produced from the Hawaiian alphabet if the second and last letters are vowels but there are no restrictions on the other 2 letters?

2 Show that

(a) $3! \times 5! = 6!$.

(b) $6! \times 7! = 10!$.

(c) $(n + 1) \times (n!) = (n + 1)!$.

(d) $n^2 = n![1/(n - 1)! + 1/(n - 2)!]$.

(e) $n^3 = n![1/(n - 1)! + 3/(n - 2)! + 1/(n - 3)!]$.

3 One brand of electric garage door opener permits the owner to select his or her own electronic "combination" by setting six different switches either in the "up" or the "down" position. How many different combinations are possible?

4 One generation back you have two ancestors, your (biological) parents. Two generations back you have four ancestors, your grandparents. Estimating 2^{10} as 10^3, approximately how many ancestors do you have

(a) 20 generations back?

(b) 40 generations back?

(c) In round numbers, what do you estimate is the total population of the planet?

(d) What's wrong?

5 Make a list of all the "words" that can be made up by rearranging the letters in

(a) TO. (b) TOO. (c) TWO.

6 Evaluate multinomial coefficient

(a) $\binom{6}{4,1,1}$. (b) $\binom{6}{3,3}$. (c) $\binom{6}{2,2,2}$.

*Based on these calculations, might it be reasonable to expect Hawaiian words, on average, to be longer than their English counterparts? Certainly such a conclusion would be warranted if both languages had the same vocabulary and both were equally "efficient" in avoiding long words when short ones are available. How efficient is English? Given that the total number of words defined in a typical "unabridged dictionary" is at most 350,000, one could, at least in principle, construct a new language with the same vocabulary as English but in which every word has four letters—and there would be 100,000 words to spare!

(d) $\binom{6}{3,2,1}$. (e) $\binom{6}{1,3,2}$. (f) $\binom{6}{1,1,1,1,1,1}$.

7 How many different "words" can be constructed by rearranging the letters in

(a) ALLELE? **(b)** BANANA? **(c)** PAPAYA?

(d) BUBBLE? **(e)** ALABAMA? **(f)** TENNESSEE?

(g) HALEAKALA? **(h)** KAMEHAMEHA? **(i)** MATHEMATICS?

8 Prove that

(a) $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$.

(b) $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n + 1)! - 1$.

(c) $(2n)!/2^n$ is an integer.

9 Show that the barcodes in Fig. 1.1.3 comprise *all possible* five-letter words consisting of two |'s and three ı's.

10 Explain how the following barcodes fail the POSTNET standard:

(a)

(b)

(c)

11 "Read" the ZIP+4 Code

(a)

(b)

12 Given that the first nine zones correspond to the ZIP+4 delivery code 94542-2520, determine the parity check digit and the two "hidden digits" in the 62-bit DPBC

(*Hint:* Do you need to be told that the parity check digit is last?)

13 Write out the 52-bit POSTNET barcode for 20742-2461, the ZIP+4 code at the University of Maryland used by the Association for Women in Mathematics.

14 Write out all 24 divisors of 360. (See Example 1.1.4.)

15 Compute the number of positive integer divisors of

(a) 2^{10}. **(b)** 10^{10}. **(c)** 12^{10}. **(d)** 31^{10}.

(e) 360,000. **(f)** 10!.