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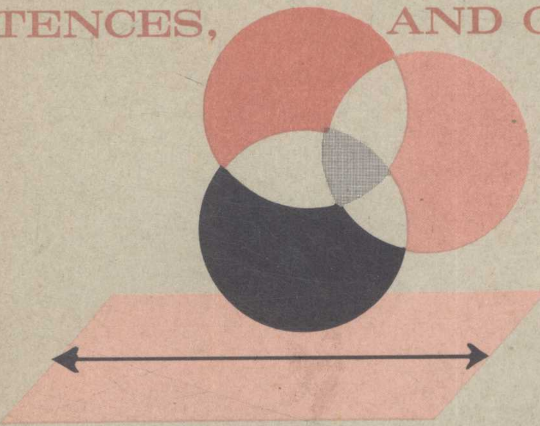

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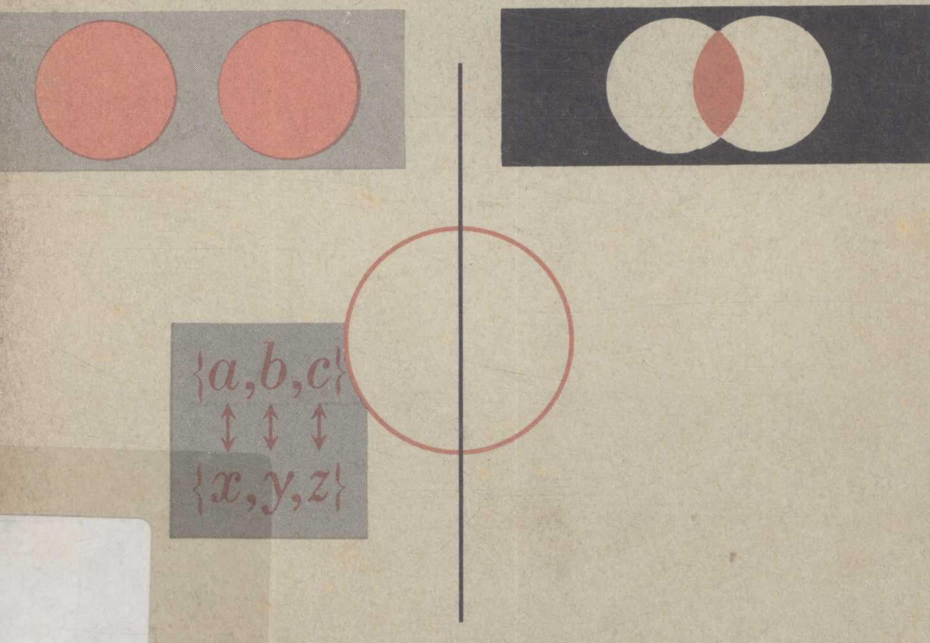

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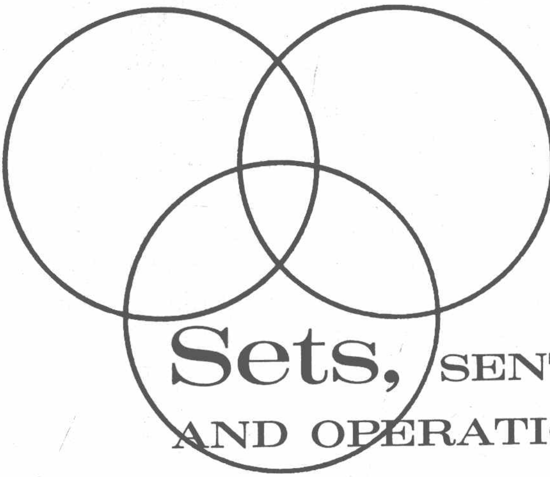

Sets, SENTENCES, AND OPERATIONS

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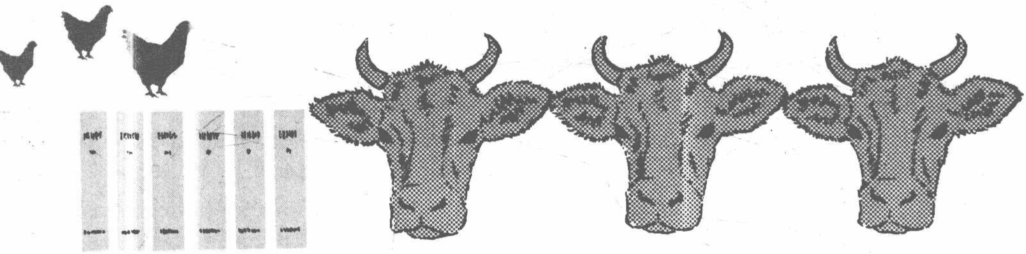
A Fresh New Look at Mathematics

Sets and New Mathematics

Mathematics is an old science. It probably started thousands of years ago when early man had to find a way of describing a herd of animals. In time, man learned to count with numbers. Over the centuries, other ideas in mathematics were developed to help man: the use of measures, an understanding of different kinds of shapes, and a knowledge of the logic of numbers. All of these ideas in mathematics, and many others, have been growing and changing, and new ideas are being added all the time to the storehouse of mathematics. One of the newest and most exciting ideas in mathematics is the study of sets.

What do we mean by a set? A *set* is simply a group or a collection of things. This idea of grouping things is not new to you. You already know the meaning of these expressions: "a set of books," "a collection of stamps," "a chain of stores," "a set of exercises." Objects are usually grouped because of some common feature. The set of books may all be encyclopedias; the stamps were perhaps issued by the same country; the stores in the chain are probably owned by the same company; the exercises may cover the same kind of material. We sometimes use other words to talk about collections or sets, such as *flock*, *herd*, *gang*, *team*, *company*, *group*, *club*, *class*, or *family*. You are even a member of groups or sets like these: your family, your class, or your scout troop.

The mathematician calls all of these collections by the term *set*. A set may be a collection of objects, numbers, persons, figures, or ideas. The objects or numbers making up a set belong to the set.



These items which belong to the set are called *members* or *elements* of a set.

The mathematical theory of sets began during the late years of the nineteenth century. A German mathematician, Georg Cantor, tackled a difficult mathematical problem concerned with the study of endless quantities. It involved such questions as: "How many whole numbers are there?" "How many points are there on a circle?" "How many instants of time elapse in one hour?" "Are there more numbers between 1 and 2 than there are points on a line?" Cantor solved his problem, and his work marked the beginning of the set idea in mathematics.

As sometimes happens with a new idea, Cantor's findings were not accepted at first. He was ridiculed for many years, but by 1920, his new way of thinking had gained some recognition from mathematicians. And now, the theory of sets has been applied to many fields of mathematics from algebra to probability.

The science of sets tells us how to combine sets and how to compare sets to determine relationships. Solving equations, drawing graphs, studying chance or probability, describing geometric figures become simpler by using the language and ideas of sets. Particularly exciting is the use of sets in solving problems, puzzles, and mysteries with electronic computers.

One of the first things you need to learn in the study of sets is to identify the members of a set. The exercise below will help you select members of sets.

EXERCISE SET 1

Discovering Sets

1. Write the members of each of these sets:
   a. the set of vowels.
   b. the set of odd numbers less than 10.
   c. the set of months beginning with the letter M.
   d. the set of United States senators from your state.

2. What are the members of the set of whole numbers less than 20 which are prime numbers? Prime numbers are whole numbers, except 1, evenly divisible only by themselves and 1.

3. Which of the following are members of the set of General Motors cars: Ford, Chevrolet, Rambler, Buick, Dodge, Pontiac?

4. a. How many members does the set of letters of our alphabet have?
   b. How many members does the set of vowels of our alphabet have?

5. a. Write the members of the set of numbers less than 99 which are divisible by 9, such as 9, 18, 27, and so on.
   b. Write the set of numbers less than 99 in which the sum of the digits is 9.
   c. Do the sets in *a* and *b* above have the same members?

The Sign Language of Sets

Suppose that we consider the set of odd numbers from 1 through 9. This set has the members 1, 3, 5, 7, 9. Mathematicians like to translate statements like this into symbols. The notation $\{1, 3, 5, 7, 9\}$ is another way of writing the statement "the set of odd numbers from 1 through 9." The symbols [ ] or $\{\ \}$ are used to group the members of a set. In this pamphlet, the second symbol, $\{\ \}$, will be used. Usually we use a capital letter to designate a set, like this: $A = \{1, 3, 5, 7, 9\}$. The notation $\{1, 3, 5, 7, 9\}$ is called a *tabulation* of the members of the set.

It is important that we be able to translate a statement into a set tabulation, but we must also be able to change a tabulation into a statement. For example, the tabulation $\{a, e, i, o, u\}$ can be translated as "the set of vowels of our alphabet." When we use a capital letter to designate the set, such as $X = \{a, e, i, o, u\}$, we read this as "X is the set whose members are a, e, i, o, and u" or "X is the set of vowels of our alphabet." The way we describe a set should tell us what items belong to the set and what items are not members of the set.

EXERCISE SET 2

Set Members

1. List or tabulate the members of the following sets:
   a. the names of the states of the United States that begin with the letter *M*.
   b. the set of symbols used in Roman numerals.
   c. the set of presidents of the United States whose last names begin with *J*.
   d. the set of the squares of all whole numbers between 0 and 10.
2. Describe the following sets in words:
   a. $K = \{5, 10, 15, 20, 25\}$
   b. $M = \{3, 6, 9, 12, 15, 18\}$
   c. $R = \{a, b, c, d, e\}$
   d. $Y = \{$ Tuesday, Thursday $\}$
3. Answer these questions about members of a set:
   a. Is a silver dollar a member of the set of United States coins?
   b. Is a whale a member of the set of all fish?
   c. Is meat a member of the set of foods containing protein?
   d. Is 64 a member of the set of squares of whole numbers?

Comparing Sets

In arithmetic, we often compare numbers to find out which one is the larger and how much larger. When we write equations or mathematical sentences like $2 + 3 = 5$ or $2x + 7 = 9$, we are interested in equivalent quantities. In a similar way, we solve problems with sets by comparing one set with another. In the study of chance or probability we need to know when sets are equal or how many members a set has. When solving problems by reasoning, we need to know if the members of one set are also the members of another set. In scientific research, the scientist needs to know how the members of one set of data are related to the members of another set.

One way to compare sets is to compare the *members* of one set with the *members* of another set. The sets $\{a, b, c\}$ and $\{x, y, z\}$ are different because they have different members. The sets $P = \{a, b, c\}$ and $Q = \{c, a, b\}$ are *equal* because they have the same members. We write this $P = Q$. It doesn't make any difference whether the members are in the same order or not. Any two sets with the same members are equal.

It is easy to recognize equal sets when the members are listed. But it is not always easy to decide about equality when the sets are described in words.

EXERCISE SET 3

Finding Equal Sets

Compare these sets to find out if they are equal:

1. {5, 10, 15, 20, 25} and {5, 15, 25, 10, 20}

2. $R = \{a, h, m, t\}$ and $T = \{m, a, t, h\}$

3. {1, 5, 7, 9} and the set of odd numbers less than 10.

4. the set of students in your class with an "A" average in mathematics and the set of boys in your class.

One-to-one Matching

Long ago shepherds kept track of their sheep by matching a pebble with each sheep. The number of pebbles represented the number of sheep in the flock. The matching of a pebble with a sheep gave a one-to-one relationship between the two. This was really a comparison between two sets: a set of sheep and a set of pebbles.

A comparison of any two sets may be made by matching members of one set with members of another set. The sets $A = \{a, b, c\}$ and $B = \{x, y, z\}$ are not equal. The members of A are not the same as the members of B, but each set has the same number of members. Every member of set A can be matched with a member of set B like this:

$$\begin{array}{c} \{a, \; b, \; c\} \\ \updownarrow \;\; \updownarrow \;\; \updownarrow \\ \{x, \; y, \; z\} \end{array}$$

This matching or pairing of members gives us what is called a *one-to-one correspondence.*

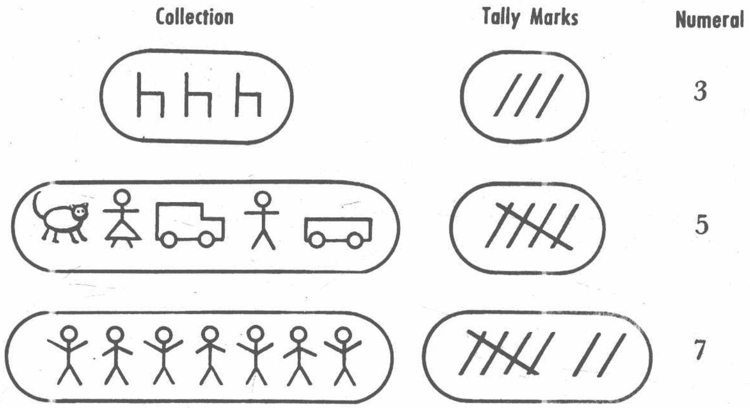


Figure 1

The collections and tally marks in Figure 1 show how a one-to-one correspondence between tally marks and sets of objects is related to our symbols for numbers.

A numeral is a symbol for a number and is often called "a name for a number." Sometimes we say a numeral describes a set. The numeral "5" describes all sets which have a one-to-one correspondence to five tally marks or five units. Frequently the word "number" is used to refer either to a number or to a name for it.

Whenever one set has a one-to-one correspondence to another set, the sets are said to be *equivalent*. For example, the sets $A = \{$ Tom, Bill, Mark $\}$ and $B = \{$ bicycle, skates, football $\}$ are equivalent. They are not equal because the members are different. But by matching or pairing members we get a one-to-one correspondence, like this:

$A = \{$ Tom, Bill, Mark $\}$
$B = \{$ bicycle, skates, football $\}$

We write this equivalence this way: $A \leftrightarrow B$. We read this "set A is equivalent to set B." Remember that equivalent and equal have different meanings.

Sometimes we also talk about a one-to-*two* correspondence. An example of one-to-two correspondence would be the set of people in a room and the set of hands of these people.

An easy way to find whether sets are equivalent or not is to count the members of each set. The number of members of set A is called the *number* of the set A and is written $n(A)$. The number of members in the set of vowels, $A = \{ a, e, i, o, u \}$, is 5. This is written $n(A) = 5$. The set $B = \{$ Tom, John, Bill, Brent, Gary $\}$ also has 5 members. Thus, $n(B) = 5$. Since $n(A) = 5$ and $n(B) = 5$, there is a one-to-one correspondence between the members of A and B. Hence $A \leftrightarrow B$, or A is equivalent to B. Is B equivalent to A? Again there is a one-to-one correspondence between the members of B and A, so $B \leftrightarrow A$. Now you see why the symbol of equivalence, $\leftrightarrow$, is an arrow pointing in both directions.

Set problems often refer to *natural numbers*. By natural numbers we mean the numbers that are positive whole numbers greater than zero, such as 1, 7, 95, 368.

EXERCISE SET 4

Making Set Comparisons

1. Which of these sets are equal?
  a. $\{ a, p, r, t \}$ and $\{ r, a, p, t \}$
  b. $\{ 1, 2, 3, 4 \}$ and $\{ 2, 4, 6, 8 \}$
  c. the set of prime numbers less than 10 and $\{ 1, 2, 3, 5, 7 \}$
  d. { California, Oregon, Alaska, Washington, Hawaii } and the set of states with borders on the Pacific Ocean.

2. Which of these pairs of sets have a one-to-one correspondence?
  a. $\{ a, b, c, d, e \}$ and $\{ 3, 7, 4, 8, 5 \}$
  b. the set of odd numbers below 20 and the set of prime numbers below 20.
  c. { Jim, Dave, Mike } and { Mary, Cathy, Karen, Molly }
  d. the set of even natural numbers less than 20 and the set of odd natural numbers less than 20.

3. What is the number of members of each of these sets?
  a. $A =$ { shoe, stocking, boot, rubbers, slipper }
  b. the set of states of the United States.
  c. the set of United States senators.
  d. the set of natural numbers that divide 24 without a remainder.

4. Which of these sets are equivalent?
  a. $\{ 1, 2, 3, 4, 5, 6 \}$ and $\{ 7, 14, 21, 28, 35, 42 \}$
  b. { 5, a, ⊗, △, ~~~, ≀ } and $\{ \mathcal{J}, R, P, B, C, M \}$

c. the set of states of the United States, and the set of United States senators.

d. $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\}$ and $\{\frac{3}{4}, \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, \frac{3}{10}\}$

5. Find the number of members of these sets:

a. n { the set of letters in our alphabet }

b. n { the set of even natural numbers less than 50 }

c. n { the set of prime numbers between 30 and 40 }

d. n { the set of number symbols in the decimal number system }

Finite, Infinite, and Empty Sets

For some sets, the number of members is so large that it would be very tiresome to list them all; for example, the set of people in your town, or the set of natural numbers less than 1000. We use dots to show that we have left some numbers out:

$$\{1, 2, 3, 4, 5, \ldots 999, 1000\}.$$

Sometimes there is no end to the number of members in a set; for example, the set of natural numbers. We say this set has an *infinite* number of members. We write this set this way: $N = \{1, 2, 3, 4, 5, \ldots\}$. We cannot count the number of members of this set. Another way of saying this is that the set of natural numbers is an infinite set. When a set is not infinite, we can count the members in the set with a number and we call it a *finite* set. The number of members in an infinite set is always greater than any number we can count. Some examples of infinite sets are: the set of points on a line, the set of minutes in the future, the number of prime numbers, the set of even numbers.

Let's compare the members of two infinite sets. Let $N = \{1, 2, 3, 4, 5, \ldots\}$ be the set of all natural numbers and $E = \{2, 4, 6, 8, 10 \ldots\}$ be the set of even natural numbers. What happens when we match or pair members of these sets?

$$\begin{array}{ccccccc} N = \{ & 1, & 2, & 3, & 4, & 5, & \ldots\} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ E = \{ & 2, & 4, & 6, & 8, & 10, & \ldots\} \end{array}$$

We see that every natural number can be matched with an even number. This seems to indicate that there are as many even numbers as there are natural numbers! This is expected, since an operation (multiplication by 2) changes the members of one set to the other.

Is it possible for the collection of even numbers, which is contained in the set of natural numbers, to have as many members as the set of natural numbers? For many years, this problem troubled the greatest mathematicians of the world. But it was solved when Georg Cantor, the German mathematician mentioned earlier in this booklet, proposed his ideas on infinite collections that became the basis for set theory. He showed that the distinguishing feature of an infinite set is that it can be placed in a one-to-one correspondence with a part of itself. Although the conclusion that there are as many even numbers as there are natural numbers is difficult to accept, set ideas show it to be logical.

In contrast to infinite sets, there are also sets with no members. For example, what is the set of even numbers that are divisors of 13? This set is called an *empty set* or *null set.*

The empty set is similar to zero in our number system. It is usually shown by the symbol ϕ, which is a zero with a line through it. The empty set can also be shown like this: $\{\ \}$. Other examples of the empty set are: the set of living persons more than 200 years old, the set of rectangles with five sides, the set of even prime numbers between 20 and 30.

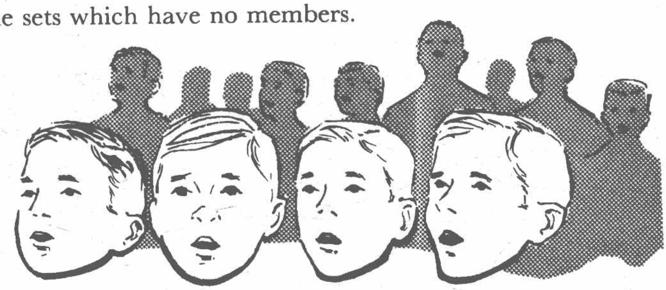
EXERCISE SET 5

Infinite Sets and Empty Sets

1. Which of the following sets are infinite?
   a. the grains of sand on all the beaches of the world.
   b. the population of the world.
   c. the atoms in our world.
   d. the natural numbers evenly divisible by 13.
2. Which of the following sets have a one-to-one correspondence?
   a. $\{1, 2, 3, 4, \ldots\}$ and $\{5, 10, 15, 20, \ldots\}$
   b. the set of natural numbers and the set of squares of natural numbers.
   c. the set of words in the English language and $\{1, 2, 3, 4, \ldots\}$
   d. the set of multiples of 5 and the set of multiples of 1,000,000.
3. Which of the following sets are empty sets?
   a. the set of foreign-born presidents.
   b. the set of men 15 feet tall.

c. the set of even prime numbers.
d. the set of squares of odd numbers that are even.

4. Name some sets which have no members.



Sets within Sets

Jill was talking about last Friday's party for the ninth grade. She said, "Jane and Ruth had beautiful blue dresses. James and Chuck came late. The quartet of 4D's — Dill, Dan, Dick, and Dave — won the talent contest."

Jill was talking about different groups who were members of the set of ninth graders. These groups, such as Jane and Ruth, are called *subsets* of the set of ninth graders. Jane and Ruth are members of the set of ninth graders and also members of the subset of ninth-grade girls with beautiful blue dresses. Likewise, James and Chuck are members of a subset, the subset of tardy boys. Similarly, the 4D's are a subset of the ninth graders participating in the talent contest.

In the set $K = \{a, b, c, d, e\}$, we may wish to consider the vowels a and e. The letters a and e may be called the set of vowels of the set K. We say that a and e are members of a subset of K. A subset is part of a set. Each member of the subset is also a member of the whole set.

A Homecoming Committee is made up of this set: {John, Sue, Carla, Ralph, David}. The set of boys on the committee, {John, Ralph, David}, is a subset of the Homecoming Committee set. Other subsets of this committee could be {Sue, Carla} and {John, Sue, David}.

Sometimes we need to know how many subsets can be formed for a given set. This is like asking how many committees we could have made up of these people: {John, Ralph, David}. Here they are:

A = {John, Ralph, David}	E = {John}
B = {John, Ralph}	F = {Ralph}
C = {John, David}	G = {David}
D = {Ralph, David}	H = { }

You may think it strange to think of a committee of one or a committee of none. In sets, this is possible as long as we don't say how many members each committee should have. Notice also that we think of the original set as a subset of itself. The members of every subset are members of the original committee. So when we ask how many subsets can be formed, we include the empty set and the original set.

Sometimes the set of subsets which can be formed from a set is called the *power* set of the original set. This idea has many applications in the mathematics of chance or probability, where the number of subsets tells us the number of ways different events may occur.

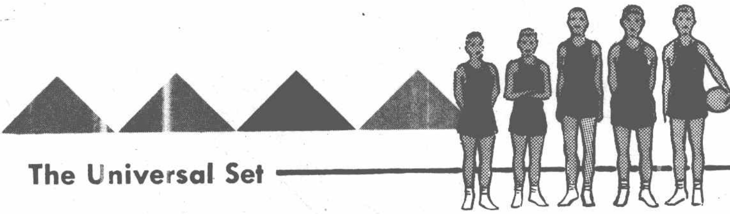
EXERCISE SET 6

Working with Subsets

1. If $A = \{1, 3, 5, 7, 9\}$, answer the following questions:
   a. What is the subset of prime numbers in set A?
   b. What is the subset of numbers evenly divisible by 3 in set A?
   c. What is the subset of numbers evenly divisible by 1 in set A?
   d. What is the subset of numbers evenly divisible by 2 in set A?
2. Make up one subset for each of the following sets:
   a. $\{w, x, y, z\}$
   b. { Ford, Chevrolet, Plymouth, Rambler }
   c. the set of holidays in a year.
   d. the set of four-sided geometric figures.
3. List all the possible subsets for each of the following sets:
   a. $\{a, b, c\}$
   b. $\{7, 11\}$
   c. { Peg, Sue, Kay }
   d. $\{p, g, r, s\}$
4. Copy and complete this table, showing the number of subsets for each set:

Set	Number of Members of the Original Set	Number of Subsets
$\{a\}$	1	
$\{a, b\}$	2	
$\{a, b, c\}$	3	
$\{a, b, c, d\}$	4	

5. a. How many subsets do you think there will be if the original set in Example 4 has 5 elements?
   b. Can you write a formula for the number of subsets, P, if the number of members of the original set is n?



The Universal Set

Whenever we talk about sets, we have in mind a collection of things which have some common characteristic. For example, we may be talking about even numbers, basketball players who are 7 feet tall, or triangles. In these sets, the common characteristic of each is evident: the numbers are divisible by 2; the people are all 7 feet tall; and the triangles are geometric figures with three sides. However, the members of each of these sets can be considered subsets of a much larger set, called the *universe* or *universal set*. For example, the even numbers are a subset of the universal set of all numbers; the tall basketball players are a subset of the universal set of all basketball players; and the triangles are a subset of the universal set of all geometric figures.

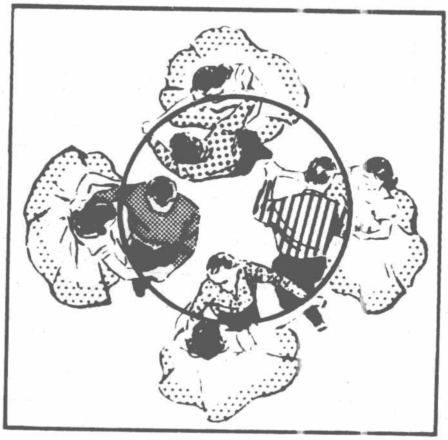
Sometimes it is not clear what the universal set is unless we describe it specifically. Suppose the members of a set of cars are all Chevrolets. The universal set may be all General Motors cars, all American cars, all low-priced cars, or all cars in the world.

The universal set is designated by the capital letter U. If, for a set of numbers, no other description is given for the universal set, it is usually interpreted as "the set of all real numbers" or $U = \{$ all real numbers $\}$. This set includes all positive and negative whole numbers, fractions, and irrationals (numbers that cannot be expressed as the ratio of two whole numbers, such as $\sqrt{2}$).

EXERCISE SET 7

Universal Set Problems

1. What is a universal set for these sets?
   a. the set of students at Central High School with "A" grades.
   b. the set of Ford cars.
   c. the set of cashmere sweaters.
   d. the set of whole numbers between 10 and 20.
2. Name a subset of each of the following universal sets:
   a. the set of students in your school.
   b. the set of whole numbers greater than 3 and less than 30.
   c. the set of four-sided figures.
   d. the set of books in your library.



Problems Solved by Sets and Pictures

Set Diagrams

"One picture is worth a thousand words" is a very old saying. When solving mathematical problems, we often have to find relationships between sets of values, objects, or events, and a good way to see how sets are related is to represent them with drawings.

Mathematicians usually use a rectangle to represent a universal set. All the members of the universal set are considered as points on or inside the rectangle. For example, U = the set of all people in Crow Valley, is diagramed in Figure 2.

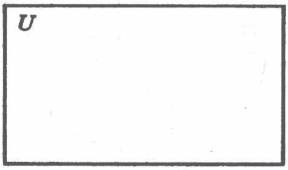


Figure 2

A subset of the universe is usually represented by a circle. Figure 3 shows how to represent A, the set of all men in Crow Valley.

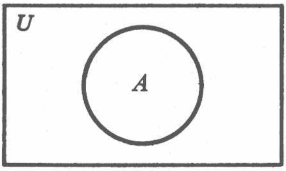


Figure 3

A subset of set A is represented by another circle. Let B = the set of men of Crow Valley who are older than 50 years of age. Figure 4 pictures the relationship between the sets.

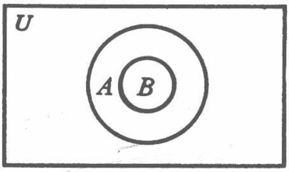


Figure 4

These set diagrams are called *Venn diagrams.*

If you stop to think about the ways any two circles can be related, you will know how any two sets can be related. They can be related in these ways:

1. Circles can be separate, like those in Figure 5.

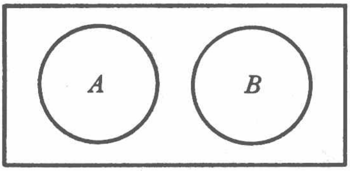


Figure 5

Then we say that A misses B, or A does not intersect B. The sets represented are completely separate and independent, with no members in common. Set A is said to be *disjoint* from set B.

An example of disjoint sets is:

X = the set of boys in our class
Y = the set of girls in our class.

Another example is:

A = the set of all odd numbers
B = the set of all even numbers.

2. Circles can meet, like those in Figure 6.

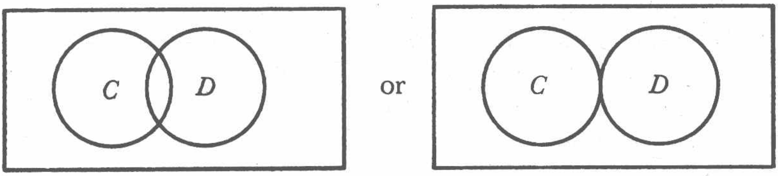


Figure 6

Then we say set C meets or *intersects* set D. This happens when some of the members of set C are also members of set D.

An example of intersecting sets is:

C = the set of boys in the tenth grade

D = the set of boys in the high school science club.

If C and D intersect, we know that at least one tenth-grade boy is also a member of the science club. We assume that the club has members from other classes.

3. One circle can completely coincide with another, as in Figure 7.

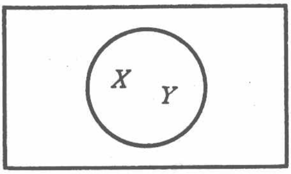


Figure 7

When circles coincide, the sets represented have exactly the same members. We know that sets with exactly the same members are said to be identical or equal. We write this relationship as $X = Y$.

If X = the set of even natural numbers less than 20 and Y = the set of natural numbers less than 20 which are divisible by 2, then

$$X = \{2, 4, 6, 8, 10, 12, 14, 16, 18\},$$
$$\text{and } Y = \{2, 4, 6, 8, 10, 12, 14, 16, 18\},$$
$$\text{or } X = Y.$$

4. One circle can be inside the other, as in Figure 8.

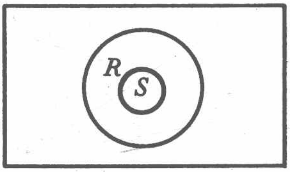


Figure 8

This means that all members of set S are also members of set R. In other words, S is a subset of R. This would be the case if R = the set of letters of the alphabet and S = the set of vowels. When S has fewer members than R, we say S is a *proper* subset of R. This relationship is written with symbols this way: $S \subset R$. The expression $S \subset R$ is read "S is included in R" or "S is a subset of R."