Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Grassmannians and Gauss Maps in Piecewise-linear Topology

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To my Parents

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0. Introduction

This monograph brings together a number of results centered on an attempt to import into the study of PL manifolds some geometric ideas which take their inspiration from the origins of differential topology and differential geometry, ideas from which many important aspects of fiber-bundle theory have developed. The reader is presumed to be familiar with the central role that the theory of fiber bundles has played in the study of differentiable manifolds for the past four decades. The central theme here has been that a wide class of geometric problems can be reformulated as bundle-theoretic problems. Typical results flowing from this approach have been the Cairns-Hirsch Smoothing Theorem; The Hirsch Immersion Theorem, together with its generalization, the Gromov-Phillips Theorem, and much of the important work in foliation theory. The great advantage of a reduction to bundle theory as has been generally been thought, is that the geometric problem has become a homotopy - theoretic problem from whence, with a little luck, it can be made into an algebraic problem.

It is also presupposed that the reader is conversant with the generalizations of classical vector bundle theory, generalizations which appropriate much of the machinery developed for differentiable topology for use in the study of PL manifolds, topological manifolds, homology manifolds, Poincare-duality spaces and so forth. In particular, the notions of PL bundle, PL block-bundle, topological bundle, spherical fibration (together with their stable versions) are assumed to be familiar territory. So, too, the classifying spaces (and canonical bundles) associated with these notions:
$BO(k)$ for vector bundles, $B\widetilde{PL}(k)$ for k-dimensional PL-bundles, $BPL(k)$ for PL k-block-bundles, $BG(k)$ for (k-1) - spherical fibrations, and so forth.

I now wish to observe that these generalizations and the theorems that have exploited them have had a certain flavor, displaying, so to speak, an inclination to move into the homotopy theory as quickly as possible from the point of view of underlying constructions as well as that of ultimate results. A brief historical overview might make this clearer.

The notion of bundle and its applicability to topological questions goes back, of course, to Gauss, whose great work on curvature and its relation to the topology of surfaces exploits the Gauss map in its original and most literal sense. This of course is the map which, for any oriented surface immersed in 3-space, takes each point to the correctly-oriented unit normal vector to the surface at that point, the target space of the map being thought of as the standard unit 2-sphere.

In this century, the foundational work of Steenrod, Whitney, et. al. led to the formal definition of fiber bundles, with vector bundles along with principal Lie group bundles serving as the prime example. The discovery of the role of the Grassmann manifold as the "classifying space" for vector bundles preserved much of the original insight of Gauss' construction. As beginners in the subject soon learn, it helps one's intuition to picture vector bundles as tangent bundles to manifolds, particularly manifolds embedded or immersed in Euclidean space. In that case, one easily goes on to picture the classifying map, (frequently and quite appropriately called the "Gauss map") as that map which takes each point in the given n-manifold to the point in the appropriate Grassmannian corresponding to the unique n-dimensional linear subspace (of the given Euclidean space) parallel to the tangent space at the point.

In the intervening decades, generalizations of the notion of vector bundle have proliferated, and the notion of "universal classifying space" has become a familiar one for many contravariant homotopy functions beyond vector bundles and principal bundles. The chief tool here is E. Brown's Representability Theorem [Bro] and some of its generalizations, which guarantee that a homotopy functor is "representible" (i.e, has a classifying space within the category of CW complexes) under very unrestrictive conditions. In particular, Brown's Theorem is usually cited as the justification for asserting the existence of BPL, B Top, BG et.al.

Despite the beauty and usefulness of the Representability Theorem, however, I wish to assert that there is something problematical about its use in connection with intrinsically geometric problems. First of all, one sees that the classifing space B_F obtained for a given functor F is truly a "homotopy theoretic" object; it has no "natural" geometric structure and, indeed, is a geometric object only in the most shadowy and abstract sense. The same may be said of the map $X \to B_F$ classifying an element of F(X). This is no map at all strictly speaking, but rather a homotopy class of maps. In some sense, to the degree that we rely on the Representability Theorem, we "know" B_F or $[X, B_F]$ precisely as well as we know F or F(X). The roll of B_F as a space or an element of $[X, B_F]$ as a map is largely mataphorical. Note how far this is in spirit from the original Gauss construction, in which a specific geometric object (an embedded manifold) was seen to acquire an equally specific map into a concrete geometric object (the standard sphere), a map whose local properties, moreover, were of intense geometric interest. Gauss, after all, was not interested in the abstract classification of normal bundles of surface but rather in understanding the local geometry of curvature in its relation to global invariants.

The present work is a first attempt at recovering something of this spirit for the study of combinatorial manifolds. Combinatorical manifolds, after all, are by definition, objects which support specific geometric structures, namely triangulations (more specifically, metric triangulations where each simplex has a metric consistent with its convex linear structure). There is a rough but useful analogy: triangulated manifolds are to combinatorial manifolds as Riemannian manifolds are to differentiable manifolds. That comparison suggests, among other implications, that the local properties of a triangulation ought to bear some relation to the global invariants of the manifold.

The problem, of course, is to give this insight some concrete point. The view taken in these notes is that the local geometry of a triangulated manifold gives rise to a map (and the emphasis here is on map rather than homotopy class of maps) into a universal example which, so to speak, is constructed from all possible prototypes of local geometrices. In view of tradition and of the naturality of the construction we call this map a Gauss map.

This usage is further justified by the fact that the Gauss map, as we define it, carries the appropriate bundle information. That is, the Gauss map is naturally covered by a bundle map (in the appropriate category) of the tangent bundle of the manifold to some canonical bundle over the universal space (which is thus naturally to be thought of as a kind of "Grassmannian"). Carrying this analogical mode of thinking yet further, we might consider a triangulated manifold embedded in Euclidean space so that the embedding is a convex-linear map on each simplex. The analogy here is to smooth a submanifold of Euclidean space. One ought to suspect that, just as there is a natural Grassmannian which receives the Gauss map of the embedded smooth manifold, there might be a natural space which receives the equally natural Gauss map of the embedded manifold. This suspicion is quite justified. Again, prototypes of local geometries (where now the embedding in Euclidean space is to be taken into account) can be assembled to form the appropriate PL Grassmannian which in turn supports an appropriate canonical bundle.

Once embarked upon this mode of thinking, we find ourselves naturally drawn into generalizations and extensions of the main idea of constructing Grassmannians and Gauss maps to handle different kinds of underlying geometric situations. To name but one example by way of suggesting the flavor of our approach, we might consider whether a combinatorical manifold M admits a "bundle of Grassmannians" so that given an immersion V M, there will be a Gauss map from V to that "bundle" covering the immersion.

Leaving aside for the moment an exact enumeration of those geometrical considerations which give rise to "Grassmannians" and "Gauss maps", we come to the further problem of justifying such constrictions beyond the limited appeal of abstract ingenuity.

First of all, we shall exploit the notion that a Gauss map (in contradistinction to a homotopy-theoretic classifying map into a homotopy-theoretic classifying space) is both concrete and locally determined. This can be used to convert global information into local information, at least in principle. The analogy to be borne in mind here is to the Chern-Weil theorem [Mi-St] on characteristic classes of Riemannian manifolds. Just as a universal differential form in the classical Grassmannian pulls back (given a classical Gauss map) to a de Rham co-cycle representing a characteristic class, a "universal co-cycle" in one of our "PL" Grassmannians performs a similar function. (In the subsequent chapter-by-chapter outline, we shall address this point more specifically.)

Beyond this, we are interested in the relation between "geometrical structure" on manifolds and Gauss maps. Geometrical structure, in our sense typically means immersion of the manifolds into a given ambient space, possibly with additional conditions as to the "local geometry" of the immersion. In the smooth case, such geometric questions usually are phrased in terms of infinitesimal data, so that a "geometry" for the manifold may be most usefully thought of as a cross section of some bundle of map germs satisfying, say, some further condition defined in terms of a jet bundle to which the original germ-bundle maps via differentials. The simplest example is an immersion, which is of course a smooth map whose 1-jet is of maximal rank every- where. The thematic result here is the theorem of Hirsch, Gromov and Phillips, [P] which assures us in a large number of cases that a section of the jet bundle with the appropriate properties is sufficient evidence for the existence of a section of the germ bundle itself, whose differential has the same properties as the original section. Again taking the simplest example, Hirsch's original result [Hi] tells us that a map between manifolds is homotopic to an immersion if it can be covered by a map of tangent bundles of maximal rank everywhere (with some additional assumptions necessary in codimension 0).

Of course it is well known that the Hirsch Theorem admits a generalization into the PL category, with conditions being phrased in terms of PL tangent bundles. Yet if we wish to study immersions satisfying certain further restrictions, natural from the point of view of PL geometry, the general ideas of the Gromov-Phillips Theorem seem inadequate. There are no differentials, jet bundles etc. in the PL category.

However, we shall see that certain kinds of geometries on manifolds - certain kinds of immersions meeting local specifications - do correspond in natural ways to the Grassmannians we shall construct and, more particularly, to subspaces thereof. That is, an immersion whose local properties satisfy some restriction has a Gauss map whose image lies in an appropriate subspace of the Grassmannian. Thus, in the spirit of the Gromov-Phillips theorem but with much different constructions in hand, we may ask the converse question: Given an abstract map of a manifold to the indicated subspace of the Grassmannian, covered by a map from the tangent bundle to the cannonical bundle, can we then obtain an immersion with the appropriate geometry? We shall prove theorems of this kind usually with the proviso that the manifold in question be open.

We shall also address further questions in a related vein having to do with smoothing theory and with piecewise-differentiable, rather than piecewise linear maps. We shall also consider versions of these results in the context of actions by finite groups. The reader may find the following outline useful.

Chapter 1. Local formulas for characteristic classes.
The main topic in this section is an exposition of the author's joint work with C. Rourke [Le-R] proving the existence of local formulas for rational characteristic classes of PL manifolds. The methodology here is thematic. A semi-simplical complex $|Q_n|$ is constructed which is the natural target of a Gauss map from triangulated n-manifolds with a local ordering of vertices. $|Q_n|$ naturally supports a canonical n-block bundle which receives a natural n- block bundle map from the tangent block-bundle of such a manifold, which map covers the Gauss map. The existence of characteristic classes for the canonical block-bundle easily leads to the existence theorem. The chapter also contains a generalization to homology manifolds as well as a brief discussion of various attempts to find a concrete formula for the Pontrgagin classes and L-classes.

Chapter 2. Formal links and the PL Grassmannian $\mathcal{G}_{n,k}$.
In this chapter we construct the "PL Grassmannian" $\mathcal{G}_{n,k}$, together with its canonical PL n-bundle $\gamma_{n,k}$. This is the natural Grassmannian for simplex-wise linear immersions of triangulated n-manifolds into R^{n+k}. It is shown how a Gauss map arises naturally and automatically for such immersions.

Chapter 3. Some variations of the $\mathcal{G}_{n,k}$ construction.
This chapter briefly explores the construction of spaces akin to $\mathcal{G}_{n,k}$ and appropriate to geometric situations other than simplex-wise linear immersions of triangulated manifolds. In particular maps more general than immersions and complexes more general than combinatorical

manifolds correspond to certain spaces defined similarly to $\mathcal{G}_{n,k}$.

Chapter 4. The immersion theorem for subcomplexes of $\mathcal{G}_{n,k}$. In this section we define the notion of geometric subcomplex of $\mathcal{G}_{n,k}$. In spirit, this means a subcomplex which receives the Gauss map of manifolds immersed in such a way that additional geometric restrictions are observed. If $\mathcal{H}$ is such a subcomplex, we consider manifolds M^n whose tangent bundles map to the restriction to $\mathcal{H}$ of the canonical bundle $\gamma_{n,k}$. The main result, generalizations of which occupy much of the remaining text, is that if such a manifold be non-closed, then it will immerse in R^{n+k} so that the Gauss map has image in $\mathcal{H}$.

Chapter 5. Immersions equivariant with respect to orthogonal actions on R^{n+k}. Here we generalize the result of the last chapter to deal with triangulated manifolds on which a finite group acts simplicially and with orthogonal actions by that group on R^{n+k}. (The group then automatically acts on $\mathcal{G}_{n,k}$ as well.) The idea is to obtain equivariant immersions subject to additional geometric conditions corresponding to an invariant geometric subcomplex $\mathcal{H}$. The result holds for manifolds satisfying the so - called Bierstone condition.

Chapter 6. Immersions into triangulated manifolds. This chapter contains the thesis work of my student Regina Mladineo. As the title suggests, we study immersion theory where the target space is now a triangulated manifold rather than Euclidean space. We start by constructing, for a triangulated manifold, an analog to the Grassmannian bundle associated to the tangent bundle of a smooth manifold. If W^{n+k} is triangulated we construct $\mathcal{G}_{n,k}(W)$ which is the natural target of a Gauss map from M^n, where M^n is a triangulated manifold immersing in W^{n+k} in general position with respect to the triangulation. Here it is also assumed that inverse images of simplices of W are subcomplexes of M and that the map is simplex-wise convex-linear. In point of fact, $\mathcal{G}_{n,k}(W)$ is not a fiber bundle over W but rather a semisimplicial complex assembled from a collection of copies of $\mathcal{G}_{n-r,k}$ with one copy for each simplex of W of codimension r. Geometric subcomplexes are then defined and it is shown that a result analogous to that of Chapter 4 can be obtained. If W and M are further equipped with simplicial actions by a finite group then the analog to the result of Chapter 5 can be obtained as well.

Chapter 7. The Grassmannian for piecewise-smooth immersions. Here we broaden our considerations to study PL manifolds equipped not with a triangulation but rather with a stratification which is "linkwise simplicial" and where each stratum is provided with a smoothness structure so that inclusions of strata into higher strata are smooth. If we consider piecewise-smooth immersions of such manifolds M into Euclidean space R^{n+k}, it is natural to look for an appropriate notion of Grassmannian. This space, which we designate $\mathcal{G}^c_{n,k}$ turns out to be closely related to the $G_{n,k}$ of previous chapters. In fact, $\mathcal{G}^c_{n,k}$ is $\mathcal{G}_{n,k}$ retopologized as a the geometric realization of a simplicial space rather than a simplicial set. A theorem analogous to the main result of Chapter 4 is obtained.

Chapter 8. Some applications to smoothing theory. This chapter represents a detour from the main thrust of the foregoing Chapters 2-7 in that we are no longer concerned with immersion theory but with smoothing theory. We begin with the construction of a space A^{ord} which is, in some sense a simpler version of the $|Q_n|$ of chapter 1 and of $\mathcal{G}_{n,k}$ as well. A^{ord} is the natural target of Gauss map from a locally ordered triangulated manifold M^n, yet, N.B., it is not constructed with a view to supporting a canonical PL bundle. A^{ord} has, so to speak, one i-cell for each possible ordered triangulation of S^{i-1}. We then go on to construct another space A^{Br} which has one i-cell for each "Brouwer structure" on the cone on an ordered, triangulated S^{i-1}, where a Brouwer structure means a simplex-wise linear embedding in R^i. A^{Br} is then retopologized (as $\mathcal{G}_{n,k}$ was retopologized to produce $\mathcal{G}^c_{n,k}$) to yield yet another space A^{CBr}. A^{CBr} maps naturally into A^{ord}. Our theorem is that M^n is smoothable if and only if there is a homotopy lift in the diagram

$$\begin{array}{ccc} & & A^{CBr} \\ & & \downarrow \\ M^n & \to & A^{ord}. \end{array}$$

What is interesting about this result is that the property sought has no a priori connection with bundle theory.

Chapter 9. Equivariant piecewise differentiable immersions. We resume the main theme of these notes by considering piecewise-smooth manifolds supporting a compatible finite group actions and equivariant immersions into a Euclidean space on which the group acts orthogonally. We generalize the result of Chapter 7 just as Chapter 5 generalized that of Chapter 4.

Chapter 10. Piecewise differentaiable immersions into Riemannian manifolds.

We now consider piecewise-smooth immersions where the target is a smooth manifold equipped with a Riemannian metric. For such spaces W^{n+k} we construct an "associated Grassmannian bundle $\mathcal{G}^c_{n,k}(W)$ (now truly a bundle) whose fiber is the $\mathcal{G}^c_{n,k}$ of chapter 7. $\mathcal{G}^c_{n,k}(W)$ is the natural target of a Gauss map from M^n when M^n is piecewise-smoothly immersed. As Chapter 6 generalized the results of Chapters 4 and 5, this Chapter generalizes Chapters 7 and 9.

A brief glossary of important definitions and constructions is provided in the appendix.

1. Local Formulae for Characteristic Classes

The point of view which looks at the characteristic classes of a manifold as global summaries of local data is a rather old one. Insofar as Stieffel-Whitney classes are concerned, this approach may be said to have been born with the subject. In particular, though Stieffel-Whitney classes were devised in connection with vector bundles and smooth manifolds, it became clear early on that the definition readily extended to combinatorial manifolds. [In fact, via the definition $w_i = (\cup\Phi)^{-1}Sq^i\Phi$, Φ the $Z/2Z$ Thom class of the bundle in question, it is easily seen that Poincare duality spaces have well-defined Stieffel-Whitney classes as well]. But the more interesting aspect of the definition of w_i on a combinatorial manifold, or, more correctly, a combinatorially triangulated manifold, is that the definition is local. We remind the reader how the formula works.

Let M^n be a combinatorially triangulated n-manifold. With respect to this fixed triangulation T, we have the first barycentric subdivision T'. The formula for w_i may be viewed as giving an i-co-cycle (co-efficients in $Z/2Z$) for the cell structure on M^n dual to T'. Alternatively, we may read the formula as giving a representative for the $n-i$ homology class w^*_{n-i} Poincare dual to w_i, by specifying an $(n-i)$ cycle in T' itself. The formula is extraordinarily simple:

Let $\gamma^*_{n-i} = \sum\tau_{n-i}$ where τ_{n-i} ranges over all the (n-i)-simplices of T' (in int M if $\partial\partial \neq 0$). Then

1.1 Theorem (Whitney [Whn]; see also [Ch_1], [H-T]). γ^*_{n-i} is a $Z/2Z$ cycle whose homology class is w^*_{n-i} $H_{n-i}(M, \partial M; Z/2Z)$.

Thus, one may read off directly, on the chain level, the Poincare duals of the standard Stieffel-Whitney classes. If one wishes to translate this into a corresponding statement about co-cycle representatives (in the dual cell structure) for the Stieffel Whitney cohomology classes themselves, it is useful to order the triangulation, at least so that each simplex is linearly ordered. The ordering ℓ canonically defines a subdivision map $\lambda: T' \to T$, and so we obtain a cycle $\lambda_* \gamma^*_{n-i} \in C_*(T, T \cap \partial M; Z/2Z)$. If we let γ^i be defined (with respect to the cell structure T^* Poincare dual to T) by $\gamma^i(\sigma^*)$ = (number of (n-i)-simplices in $\lambda^{-1}\sigma \subset T'$) (mod 2). Then

<u>1.2 Corollary</u>. γ^i is a cocycle in $C^i(T^*, Z/2Z)$ representing the Stieffel-Whitney class $w^i(M)$.

Note that the value of γ^i on a dual i-cell σ^* depends only on the structure of the ordered simplicial complex $st(\sigma)$. This sets the pattern for our generalization, at least on the level of existence theorems, to arbitrary characteristic co-homology classes of PL n-manifolds.

Let M^n be a PL manifold with a combinatorial triangulation T.

<u>1.3 Definition</u>. A <u>local ordering</u> for T is a partial ordering of the vertices of T such that each star $st(\sigma^k, T)$ (abbreviated $st(\sigma)$) (σ^k a k-simplex of T) is thereby linearly ordered.

Abstractly, an <u>n-star</u> of codimension i, $i \leq n$, shall mean a complex of the form $\Delta^{n-i} * \Sigma^{i-1}$, where Δ^{n-i} is the standard n-i simplex and Σ^{i-1} denotes a combinatorially triangulated (i-1)-sphere (= 0 if i = 0). An <u>ordered</u> codimension-i n-star is such an object with a linear ordering of its vertices, and an <u>oriented</u> star means one where Σ^{i-1} has been given an orientation ω. <u>Isomorphism</u> of ordered stars means a simplicial isomorphism preserving both the ordering and the factors Δ, Σ of the join.