PLANE TRIGONOMETRY

RAYMOND W. BRINK

THE APPLETON-CENTURY MATHEMATICS SERIES

Plane Trigonometry

by

Raymond W. Brink

PROFESSOR EMERITUS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MINNESOTA

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PREFACE

In making this revision of his book on Plane Trigonometry, by one means and another the author has tried to arouse in the student reader a livelier interest in the analytical aspects of trigonometry.

To some extent this has been done by changing the order of topics. It is natural that a student's appreciation of a phase of the subject and his interest in it are greatest if he encounters it early in the course and becomes familiar with it through constant use. With the purpose of shifting the emphasis slightly away from the computational aspects and toward the analytical, the subject of logarithms and the logarithmic solution of right and oblique triangles has been postponed until nearly all the analytical topics have been considered. Moreover, many analytical portions of other sections have been moved forward and given increased emphasis.

Thus, in the first chapter there is now a more thoroughgoing consideration of sets and functions than was given in the two earlier editions. Many students feel embarrassed whenever they are supposed to recognize a problem requiring the use of radian measure; they remember the name, but cannot remember the face. In this revision, radian measure is introduced even before the discussion of directed angles or the definition of the trigonometric functions, and it is used whenever it is appropriate throughout the book. A somewhat novel derivation of formulas for the sum of two sines and the sum of two cosines, and hence of the addition formulas for sines and cosines, is given for limited ranges of the angles in connection with the very early treatment of vectors in Chapter 3. Even before that point, there is a discussion of inverse functions in general and of the inverse trigonometric functions in particular. The principal values of the latter are defined for negative values of the independent variable only after the chief properties of the trigonometric functions themselves have been considered.

In view of the origins of trigonometry and its wealth of interesting applications, it would be folly to neglect applied problems and the associated computation. Logarithms, too, should receive proper emphasis, as tools in computation, and because of their essential role in calculus and its applications. These aspects of the subject have not by any means been overlooked in the present revision. On the contrary, the collection of applied problems has been freshened and modernized. They are present in great variety and number, and an unusually large number of them deal with things that are of real interest to the student. Much of the illustrative material, as, for example, the discussion of the quality of musical sounds and the reference to harmonic analysis, has great appeal to many students.

In many colleges and universities a student will be taking courses in physics and in trigonometry at the same time. Very early in this book such a student will find the work with tables of natural functions, functions of acute angles, right triangles, vectors, vector quantities, projections and the like which he needs to have early in order to handle such material in his course in physics. This kind of experience gives him an invaluable feeling for the reality and utility of the subject matter, without causing any real delay in the progress of the analytical study.

Some of the proofs have been given a slightly more analytic flavor than they had in the earlier editions, as, for example, the proof of the law of cosines and the alternative proof of the addition formulas for sines and cosines. It is believed that the proofs set a good standard of generality and precision of statement. They are free from any attempt at extremes of formal, symbolic logic. In the author's opinion, such attempts should not be made in an elementary course in mathematics. With their forbidding special language and symbolism, for elementary students they are more difficult to understand than the mathematics itself, and are better postponed until the mathematics, which at first they tend rather to obscure than to clarify, has been mastered. In this, as in other respects, the present revision, with its increased emphasis on analytical methods, is modern in its spirit and material, but it is conservative in its details of presentation.

The subject of trigonometric identities and equations has offered serious difficulty to many students of average ability. The present edition tries to be especially helpful in this connection. There is preliminary practice with notation and mere algebraic manipulation of trigonometric expressions. Then there are a good many very simple problems, which progress gradually in difficulty in such a way as to bolster the student's courage and independence while increasing his technical skill in handling the standard formulas of trigonometry. Care is taken, too, to furnish as much motivation as possible for the work with identities and equations and also for the methods suggested.

This matter of motivation has dictated many details of treatment. Throughout the book there is an immediate application of principles to problems, either as illustrative examples solved in the text or as exercises for the student. So far as is practicable, the discussion of theoretical topics is broken into short passages immediately followed by examples and exercises. This arrangement not only reveals the utility of a method just learned, but enables the student to make steady daily progress in the theory while having ample time and material for practice.

The book is especially adaptable to courses of various lengths and purposes. For students who study trigonometry chiefly as a tool for elementary work in physics or similar applications, even the first three chapters will provide the desired material. In more extensive courses in which the emphasis is still chiefly on computation, Chapters 8 and 9 can be studied immediately after Chapter 3. The first seven chapters treat most of the analytical trigonometry required by students who intend to continue their study of mathematics. There are thus several good stopping places, each one of which corresponds to a special purpose. The entire book, followed in the order in which the material is presented, offers an unusually rich and complete course in Plane Trigonometry.

R.W.B.

Minneapolis, Minnesota

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I

Distances, Sets, and Functions

1.1 DIRECTED DISTANCES

If a point moves in a straight line from a point A to a point C, it traces out the ***line segment*** $\overline{AC}$. The point A is the ***initial point,*** and C is the ***terminal point*** of the segment. We use $\overline{AC}$ to denote the segment itself, and we denote by AC, without the bar, the distance from A to C expressed in terms of some chosen unit of length, such as a foot or a centimeter. If B is a point on the line between A and C, in tracing $\overline{AC}$ the moving point traces first the segment $\overline{AB}$ and then the segment $\overline{BC}$. We can write

$$AB + BC = AC.$$

In order to make this rule hold even when B is not between A and C, we agree that distances measured in one direction along the line shall be counted positive, and distances measured in the opposite direction shall be counted negative. That is,

(1) $$\boldsymbol{AB = -BA,}$$

or

(2) $$\boldsymbol{AB + BA = 0.}$$

If a point starts at A and, after moving back and forth along the line in any way, ultimately arrives at C, the algebraic sum of all the distances traversed by the point is AC. This is true because every segment except $\overline{AC}$ is traversed the same number of times in each direction and the sum of the distances covered on these other segments is zero. In particular, if the tracing point moves from A to B and then

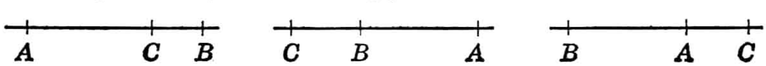

FIG. 1.1.1

to C,

(3) $$\boldsymbol{AB + BC = AC,}$$

for all arrangements of the points on a straight line.

It is to be noticed that, if A and C are distinct points on a line, $\overline{AC}$ denotes the line segment itself, of which A is the initial point and C is the terminal point. On the other hand, AC is a positive or negative *number* which measures the distance from A to C, in terms of the chosen unit of length and the chosen positive and negative directions.

Sometimes we shall not be concerned with directed distances, but only with the ***length*** of $\overline{AC}$ or the ***undirected distance*** between A and C. This length is simply the number of times that the unit of length can be applied to the segment $\overline{AC}$ or $\overline{CA}$, and is never negative. For reasons that will appear in § 1.3, we denote this length by $|AC|$. If AC is positive, $|AC| = AC$; if AC is negative, $|AC| = -AC$; and, in any case, $|AC| = |CA|$, though $AC = -CA$.

It is often convenient to represent an undirected distance, such as the length of a side of a triangle by a single letter, such as a, b, or c. In other cases a single letter, such as x or y, will be used to denote a directed distance.

While AC, used as a number, denotes the directed distance from A to C, we shall not hesitate to use AC as the name of an undirected line or segment through two points A and C, when it is convenient to do so. No confusion results from this notation, which will always be made clear by the context.

1.2 THE SCALE OF REAL NUMBERS

In this course, we shall deal primarily with real numbers. It will be assumed that every number symbol used represents a real number, such as 3, 0, $-\frac{5}{2}$, $-\sqrt{2}$, or π, rather than an imaginary number such as $\sqrt{-1}$, $2 + 3i$, or $5 - \sqrt{-3}$, unless there is a clear indication to the contrary.

In order to establish a correspondence between real numbers and points on a line, it is useful to form a ***scale of real numbers.*** To do this we draw an infinite straight line OX. On this we choose a point O as the ***origin*** or starting point, one direction on the line as the positive direction, and a unit of length in terms of which distances are measured. In Figure 1.2.1 the line is horizontal, the positive distances are measured to the right, and the unit of length is a quarter of an inch. We now attach the number 0 to the origin, the number 1 to the point which is one unit from the origin in the positive direction, the number -1 to the point which is one unit from the origin in the negative direction, and so on. *In general, each real number corresponds to the point the measure of whose directed distance from the origin is equal to the number.* Thus, in Figure 1.2.1, there is a correspondence between A and 5, B and -3, C and $\frac{5}{2}$,

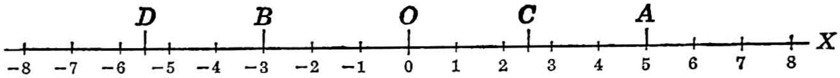

FIG. 1.2.1

and D and $-\sqrt{30}$. *We assume that there is just one real number that corresponds to each point of the number scale and that there is just one point that corresponds to each real number.*

The integers or whole numbers now correspond to the points marked 0, 1, -1, 2, -2, etc. A ***rational number*** is any number that can be expressed as the ratio of one integer to another. Rational numbers include the integers (for example, $-3 = (-3)/1$) and such fractional numbers as $2/3$, -0.6, 4.32. An irrational number is any real number that is not rational. It can be shown that such numbers as $\sqrt{2}$, $-\sqrt{3}$, and π (which is approximately but not exactly equal to 3.1416) are irrational. We have assumed that all of these numbers, rational or irrational, are represented by points on the number scale.

Suppose that P and Q are points on the number scale corresponding to the numbers x_1 and x_2, respectively; then

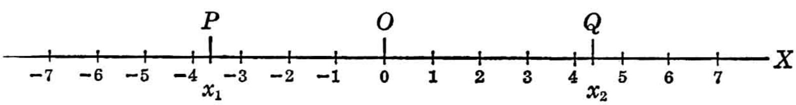

FIG. 1.2.2

(1) $$PQ = x_2 - x_1,$$

or, *the distance from one point to another point of the number scale is equal to the number corresponding to the terminal point minus the number corresponding to the initial point.*

Proof. By the construction of the number scale, and by equation (1) of § 1.1

$$OP = x_1, \qquad OQ = x_2, \qquad PO = -OP = -x_1.$$

By formula (3) of § 1.1, it follows that

$$PQ = PO + OQ = -x_1 + x_2 = x_2 - x_1.$$

Examples. As in Figure 1.2.1, let A correspond to 5, B to -3, C to $\frac{5}{2}$. Then

$$AB = (-3) - 5 = -8, \qquad BA = 5 - (-3) = 8,$$
$$AC = \tfrac{5}{2} - 5 = -\tfrac{5}{2}, \qquad CA = 5 - \tfrac{5}{2} = \tfrac{5}{2},$$
$$BC = \tfrac{5}{2} - (-3) = \tfrac{11}{2}, \qquad CB = (-3) - \tfrac{5}{2} = -\tfrac{11}{2},$$
$$|AB| = |BA| = 8, \qquad |BC| = |CB| = \tfrac{11}{2}.$$

Because of the correspondence between points and real numbers, we can use the letter a, for example, to mean either the number a or the point that corresponds to that number. Thus the point 3 is three units to the right of O and the point -3 is three units to the left of O. If point a is to the right of b, we say that $a > b$ or $b < a$, which we read "a is greater than b" or "b is less than a." The symbol $a \geqq b$ or $b \leqq a$ means that point a is either to the right of b or coincides with it. The symbol $a \geqq b$ is read "a is greater than or equal to b." If a is positive, $a > 0$. If $a > b$, then $a - b > 0$. For example, $5 > 3$, and $2 > -7$, since 5 lies to the right of 3 and 2 lies to the right of -7.

1.3 ABSOLUTE OR NUMERICAL VALUES

The ***absolute value*** or ***numerical value*** of a real number a is written $|a|$, and is equal to a if $a \geqq 0$ and to $-a$ if $a < 0$. Thus $|5| = 5$, $|-7| = 7$, $|0| = 0$.

It is important for the student to be thoroughly familiar with the language associated with numerical values, for we shall use it very often. To say that two numbers a and b are numerically equal or equal in absolute value means merely that $|a| = |b|$, and either $a = b$ or $a = -b$; that is, the points a and b are equidistant from 0 on the number scale, but may lie on the same side or on opposite sides of 0. To say that a is numerically greater than b or b is numerically less than a means that $|a| > |b|$. Thus -5 is numerically greater than 2, since $|-5| > |2|$.

1.4 A COORDINATE SYSTEM

To form a ***rectangular system of coordinates*** in a plane we first draw a horizontal line OX and a vertical line OY. Their point of intersection, O, is called the ***origin,*** OX is the ***x-axis*** of coordinates, and OY is the ***y-axis.*** We now lay off a scale of real numbers on each axis as in § 1.2, taking each origin at O, and the same unit of measure on both axes. Distances measured horizontally are counted positive if measured to the right and negative if measured to the left; vertical distances are positive if measured upward and negative if measured downward.

Now let P be any point of the plane, and let its directed distance from OY be x and its directed distance from OX be y. Then x is called the ***abscissa*** or x-coordinate of P, and y is the ***ordinate*** or y-coordinate of P. Point P is said to have the ***coordinates*** (x, y), written in that order, and P may be referred to as the point (x, y) or P: (x, y). The position of any point can be found if its coordinates are known, and, conversely, if a point is marked, its coordinates can be found by measurement. Just as a real number scale establishes a correspondence between the points of a line and the real numbers, so our coordinate system sets up a correspondence between the points of a plane and *ordered pairs* of

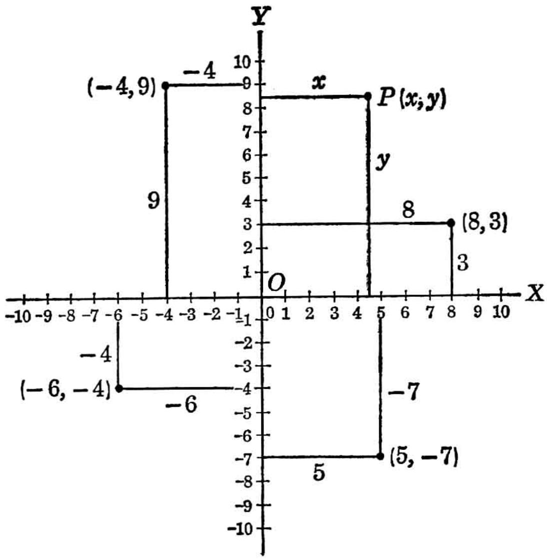

FIG. 1.4.1

real numbers. In Figure 1.4.1, points (8, 3), (−4, 9), (−6, −4), and (5, −7) are marked.

The undirected distance $|OP|$ between a point P and the origin is called the ***radius vector*** of P. Of course, if P is at the origin, its radius vector is 0. Otherwise, the radius vector of a point (x, y) is counted positive, regardless of the signs of x and y. (This notion is generalized in Chapter 11.) If r denotes the radius vector of P: (x, y), from the right triangle OMP in Figure 1.4.2 we have $r = |OP|$, and

$$r^2 = |OM|^2 + |MP|^2 = |x|^2 + |y|^2 = x^2 + y^2,$$

or

(1) $$\mathbf{r = \sqrt{x^2 + y^2}.}$$

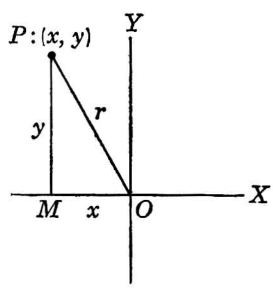

FIG. 1.4.2

More generally, if P_1: (x_1, y_1) and P_2: (x_2, y_2) are any two points, the distance between them is given by the formula

(2) $$\mathbf{d^2 = |P_1P_2|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.}$$

Proof. Draw $\overline{P_1M_1}$ and $\overline{P_2M_2}$ perpendicular to the x-axis. Draw $\overline{P_1N_1}$ and $\overline{P_2N_2}$ perpendicular to the y-axis. Point M_1 lies at x_1 and M_2 is at x_2 on the x-scale, so that by equation (1) of § 1.2, and by § 1.3,

(3) $$M_1M_2 = x_2 - x_1; \qquad \text{likewise} \qquad N_1N_2 = y_2 - y_1.$$