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... Et si illa oblita fuerit, ego tamen non obliviscar tui,

Ecce in manibus mei descripsi te.

(Is. 49, 15-16)

Per Felice, Angelino ,Dulcineo e Michelino

In memoriam

Preface

The aim of these notes is to study the large-scale structure of the non-oscillation and disconjugacy domains of second order linear differential equations with two parameters and various extensions of the latter.

We were heavily influenced in this endeavor by a paper of Markus and Moore [Mo.2]. The applications to Hill's equation, Mathieu's equation, along with their discrete analogs, motivated many of the questions, some resolved, and some unresolved, in this work.

As we wished to consider linear systems of second order differential equations, Sturmian methods had to be avoided. Thus we chose to base the theory essentially on variational methods - For this reason many of the results herein will have analogs in higher dimensions (e.g., for Schrödinger operators) although we have not delved into this matter here.

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Angelo B. Mingarelli

S. Gotskalk Halvorsen

Ottawa, March 1985 and May 1988.

1. Introduction

The study of the solutions of equations of the form

$$y'' + (-\alpha + \beta B(x))y = 0, \qquad (1)$$

where α, β are real parameters and x varies over some real interval is of widespread importance in various branches of pure and applied mathematics. For example, the Mathieu equation, which arises naturally in connection with the problem of vibrations of an elliptic membrane, is of the form (1) with $B(x) = \cos 2x$. On the other hand, Lamé's equation, which occurs in the theory of the potential of an ellipsoid, is of the same type but with B(x) being a Jacobi elliptic function. Various eigenvalue problems may be cast in the form (1) - Fixing α and allowing β to be the parameter we obtain a weighted Sturm-Liouville equation with a possibly sign indefinite B(x). Such equations have received much attention lately and we refer the interested reader to the survey [Mi.3] for further information on this subject. Of interest is also the case when β is a function of α thus allowing for a nonlinear dependence on the parameter in question.

We are interested in studying the qualitative and spectral properties of equations of the form (1). In particular, we consider the set of all pairs (α, β) for which (1) has a solution which is positive in the interior of the interval under consideration and study topological properties of this set [defined below]. Central to our investigations is the paper by R.A. Moore [Mo1], who discusses the connection between the non-oscillation and periodicity of solutions of the Hill-type equation (1), in the case when B is continuous, periodic of period one and has mean value equal to zero.

Equation (1) is said to be **disconjugate on $(-\infty, \infty)$** if and only if every one of its nontrivial solutions has at most one zero in $(-\infty, \infty)$. It is said to be **non-oscillatory on $(-\infty, \infty)$** if and only if everyone of its nontrivial solutions has at most a finite number of zeros in $(-\infty, \infty)$. The collection of all $(\alpha, \beta) \in \mathbf{R}^2$ for which (1) is disconjugate (resp. non-oscillatory) on $(-\infty, \infty)$ will be dubbed the **disconjugacy domain** (resp. **non-oscillation domain**) of (1), and denoted by D (resp. N). Moore [Mo1] showed that, in fact, $D = N$ and that N is a closed, convex and unbounded subset of the $\alpha\beta$-plane which we call **parameter space** and label it $\mathbf{R}^2$ for simplicity. The results in [Mo1] were complemented by a paper of L. Markus and R.A. Moore [Mo2] in which B(x), appearing in (1), is now a (Bohr) almost-periodic function [Be.1] (or, for any sequence $\tau_n \in \mathbf{R}$, the sequence $B(x + \tau_n)$ has a subsequence which converges uniformly on $(-\infty, \infty)$).

The closedness of D (and/or N) and its convexity are what we will term the large-scale properties of D (or N) for reasons which will become clear below.

In this monograph we are concerned with the above-mentioned large-scale properties of D (and N) for the more general equation

$$y'' + (-\alpha A(x) + \beta B(x))y = 0 \quad (2)$$

on the closed half-line $I \equiv [0, \infty)$ (although many of the results herein are also valid on $(-\infty, \infty)$). In (2) we assume **minimal requirements** on A, B, in general, in the sense that, $A, B:[0, \infty) \rightarrow \mathbf{R}$, and $A, B \in L_1^{loc}(0, \infty)$, i.e., they are Lebesgue integrable on every compact subset of $[0, \infty)$. Thus we waive all types of "periodicity" assumptions on A, B, and examine the consequences on the large-scale properties of D and N.

It turns out that, in fact, D is always a closed and convex set which can even be a **bounded** set in $\mathbf{R}^2$. In general $D \subsetneq N$ and N is also convex though **not always** a closed set. (See §§ 2.1-2.2). In §2.3 we present general conditions on A, B which ensure that $D \subseteq N \subseteq H^+$ where $H^+ = \{(\alpha, \beta): \alpha > 0\} \cup \{0,0\}$ as is the case for Hill's equation [Mo1].

The lack of periodicity-type assumptions on A, B usually has the effect of splitting D and N, however we will see that $D = N$ may occur even in the "non-periodic" case (see § 2.1).

We then apply the foregoing results to the Sturm-Liouville equation (§ 2.5) and its extensions (§ 2.6) to potentials in which the eigen-parameter occurs quadratically.

In §2.7 we pose the general question - When is $D = N$? We show that, in particular, if A, B are Stepanov almost-periodic functions [Be1], then $D = N$ (§2.8). Further extensions of this result to the class of Weyl almost-periodic or more generally, Besicovitch almost-periodic functions appears doubtful, (see [Be1] for terminology).

In §3 we review the notions of disconjugacy for second order vector differential equations of the form (2) where, generally, A, B are $n \times n$ real matrix-functions whose entries are $L_1^{loc}(I)$. We introduce the new concepts of strong-and weak-disconjugacy and study the large-scale properties of D in these cases. As is to be expected, when A, B are real symmetric, and "disconjugate" has its usual meaning, many of the results of §2 allow extensions to the vector case.

In §4 we analyze the large-scale properties of D and N corresponding to Volterra-Stieltjes integral equations [At1], [Mi2] as they include in their structure, the theory of differential equations and, furthermore, the theory of difference equations (in this case, three-term recurrence relations).

The techniques which allow these extensions are basically variational in nature, unlike the ones in [Mo1, Mo2] which relied upon variable change and the nature of B(x) in (1). In general one cannot rely upon Sturmian arguments. Thus it is possible, although we shall not delve into this matter here, to extend many of the results herein to the setting of elliptic partial differential equations with two parameters.

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2. Scalar Linear Ordinary Differential Equations

In this chapter we introduce the sets D and N, dubbed the disconjugacy and non-oscillation domains respectively. In the former case the equation (1) is disconjugate (i.e., no non-trivial solution has more than one zero in the interior) while, in the latter case, the equation is non-oscillatory (i.e., every solution has a finite number of zeros; note that the interval may be a semi-axis). Of particular importance is the study of the location of these sets in parameter space, and the associated geometry. The prototype, which serves as a basis for the cited study, is a Mathieu equation with a B(x) term which is periodic with mean value equal to zero. In this case, D is contained in the right-half plane of parameter space- We seek to preserve this property by relaxing the assumptions on B(x). This leads to the following natural question: Which functions B have the property that the equation $y'' + \beta B(x)y = 0$ is oscillatory (on a semi-axis) for every real β different from zero? We apply the resulting theory to the singular Sturm-Liouville equation $y'' + (\lambda r(x) - q(x))y = 0$ as well as to the equation

$y'' + (\lambda^2 p(x) + \lambda r(x) - q(x))y = 0$, which often appears in the physical literature. We note that weakening the periodicity-type assumptions on B generally has the effect of splitting D and N. This, in turn, leads to the next natural question: When is $D = N$? Finally, we extend some of the results in [Mo2] to more general B's - namely those which are almost-periodic in the sense of Stepanov.

One of the key results in this chapter lies in the characterization of all those Bohr almost periodic functions B for which the equation $y'' + \beta B(x)y = 0$, has a solution which is positive on

$(-\infty,+\infty)$, for some real β different from zero.

In this section we will always assume, unless otherwise specified, that A,B are real-valued, α, β are real parameters and that A, B are locally Lebesgue integrable over I (generally taken to be either $[0,\infty)$ or $(-\infty,\infty)$). The notions of disconjugacy and non-oscillation in the case of the half-line I are similar to the one given in §1, (see e.g. [Ha3]). Hence we can define D and N in this case analogously for (2).

2.1 The disconjugacy Domain

In the sequel it may be helpful to view the parameter space $\mathbf{R}^2 = \{(\alpha,\beta): \alpha, \beta \in \mathbf{R}\}$ as a linear vector space over $\mathbf{R}$ with the usual operations of vector addition and scalar multiplication. The functions A, B are **linearly dependent** if there exists constants a, b $\in \mathbf{R}$, $a^2 + b^2 > 0$ such that $aA(x) + bB(x) = 0$ a.e. on I. A subspace S of $\mathbf{R}^2$ is said to be **proper** if $S \neq \mathbf{R}^2$ and $S \neq \{(0,0)\}$. It is clear that all the proper subspaces of $\mathbf{R}^2$ consist of full rays through the origin (0,0) in $\mathbf{R}^2$.

The symbol **AC[a, b]** (resp. $\mathbf{AC_{loc}(I)}$) will be used to denote the **class of all real-valued functions which are absolutely continuous (resp. absolutely continuous on every compact subinterval of I) on [a, b].**

For a given compact subinterval $[a, b] \subset I$ we define a vector space, with the usual operations.

$$A_1(a,b) = \{\eta \in AC\,[a, b]: \eta' \in L_2(a, b), \eta(a) = \eta(b) = 0\}$$

Now let $q: [a, b] \to \mathbf{R}$, $q \in L_1(a,b)$. Then for $\eta \in A_1(a,b)$ the functional

$$I(\eta, q; a, b) = \int_a^b \{(\eta'(t))^2 - q(t)(\eta(t))^2\}dt \tag{2.1.1}$$

is defined.

The first lemma is classical.

Lemma 2.1.1 **Let $q: I \to \mathbf{R}$, $q \in L_1^{loc}(I)$.. Then the equation**

$$y'' + q(x)y = 0, \quad x \in I \tag{2.1.2}$$

is disconjugate on I if and only if for every closed bounded subinterval [a, b] of I, the functional $I(\eta, q; a, b)$ is positive-definite on $A_1(a,b)$ i.e., $I(\eta, q; a, b) \geq 0$ for each $\eta \in A_1(a, b)$ and equality occurs if and only if $\eta = 0$.

Proof. A proof in the case when $q \in C(I)$ may be found in [Ha.4]. The general case follows an analogous line of thought. For a more general result see §4.

We will use lemma 2.1.1 in order to prove lemma 2.1.2 which is central in this subsection.

Lemma 2.1.2 **Let $q: I \to \mathbf{R}$, $q \in L_1^{loc}(I)$. Then the equation**

$$y'' + \lambda q(x)y = 0, \quad x \in I, \tag{2.1.3}$$

is disconjugate on I for each real value of λ, $-\infty < \lambda < +\infty$, if and only if $q(x) = 0$ a.e. on I.

Proof. The sufficiency is trivial since $y'' = 0$ is certainly disconjugate on I.

In order to prove the necessity of the condition, let (2.1.3) be disconjugate on I for every $\lambda \in \mathbf{R}$. Lemma 2.1.1 now implies that whenever $\eta \neq 0$, is in $A_1(a,b)$ where $[a, b] \subset I$,

$$I(\eta; \lambda q; a, b) > 0 \tag{2.1.4}$$

for every $\lambda \in \mathbf{R}$. This said, let [a, b] be a given subinterval of I and fix $\eta \not\equiv 0$ in $A_1(a, b)$. Since $\eta' \in L_2(a, b)$ it follows from (2.1.4) that

$$\lambda \int_a^b (\eta(t)^2) q(t)dt < \|\eta'\|_2^2 \tag{2.1.5}$$

where $\|\eta'\|_2$ is the usual norm on $L_2(a, b)$. We emphasize that (2.1.5) is valid for **every** $\lambda \in \mathbf{R}$. Since $|\lambda|$ can be chosen arbitrarily large it follows from (2.1.5) that

$$\int_a^b (\eta(t))^2 \, q(t)dt = 0 \tag{2.1.6}$$

Hence (2.1.6) holds for every $\eta \in A_1(a, b)$. Since $[a, b]$ is also arbitrary we find that (2.1.6) holds for every $\eta \in A_1(a, b)$ and for every $[a, b] \subset I$.

Now the test function $\phi_\varepsilon(t)$ defined by

$$\phi_\varepsilon(t) = \begin{cases} t-a & a \le t \le a+\varepsilon \\ \varepsilon & a+\varepsilon \le t \le b-\varepsilon \\ b-t & b-\varepsilon \le t \le b \end{cases}$$

is in $A_1(a, b)$, for each $\varepsilon > 0$. Inserting this in (2.1.6) and passing to the limit as $\varepsilon \to 0^+$ it is easily seen that (since $q \in L_1(a, b)$),

$$\int_a^b q(t)dt = 0 \tag{2.1.7}$$

Thus (2.1.7) holds for every compact subinterval $[a, b] \subset I$. Hence $q(t) = 0$ a.e. on I and this completes the proof.

Remark 2.1 Since lemma 2.1.1 is valid for disconjugacy on intervals other than a half-line, it follows that in the hypotheses of lemma 2.1.2 one may replace I by $(-\infty, \infty)$, (a, b) etc. (see e.g. [Ha.4]).

An interesting formulation of lemma 2.1.2 is its contrapositive.

Corollary 2.1.3 Let $q: I \to \mathbb{R}$, $q \in L_1^{loc}(I)$ and let $q(x) \neq 0$ on some set of positive Lebesgue measure contained in I. Then there exists at least one value of $\lambda \in \mathbf{R}$ such that (2.1.3) is not disconjugate on I.

Corollary 2.1.4 Let $r_i: I \to \mathbf{R}$, $r_i \in L_1^{loc}(I)$ for $i = 1, 2, ..., n$. If the single equation in n-parameters

$$y'' + (\lambda_1 r_1(x) + \lambda_2 r_2(x) + \cdots + \lambda_n r_n(x))y = 0 \tag{2.1.8}$$

is disconjugate on I for every point $(\lambda_1, \lambda_2, ..., \lambda_n)$ in $\mathbf{R}^n$, then $r_i(x) = 0$ a.e. on I for $i = 1, 2, ..., n$.

Remark 2.2 Note that, once again, the contrapositive of corollary 2.1.4 is of interest - Thus if none of the functions $r_i(x)$ vanishes identically (i.e., a.e.) then there exists at least one point $(\lambda_1,...,\lambda_n)$ in $\mathbf{R}^n$ for which (2.1.8) is not disconjugate on I.

Theorem 2.1.5 Let A, B as in the introduction to this section. Then the disconjugacy domain *D* of equation (2) is the whole space $\mathbf{R}^2$ if and only if A(x) = B(x) = 0 a.e. on I.

Proof. This is basically corollary 2.1.4 for an equation with two parameters, the proof of which is straightforward (since we may set $\lambda_j = 0$ for all j except one and apply lemma 2.1.2).

Corollary 2.1.6 If at least one of the functions A(x), B(x) is different from zero on a set of positive Lebesgue measure, then *D* is a proper subset of $\mathbf{R}^2$.

Proof. This is the contrapositive of theorem 2.1.5.

The question of the boundedness of non-boundedness of *D* is a difficult one. - The next result gives a necessary and sufficient condition for *D* to contain a full ray through the origin of $\mathbf{R}^2$, and thus a sufficient condition for the non-boundedness of *D* .

Theorem 2.1.7 *D* contains a proper subspace of the vector space $\mathbf{R}^2$ (other than the subspaces $\alpha = 0$, $\beta = 0$) if and only if the functions A, B are linearly dependent over I.

Proof. Let A, B be linearly dependent. Then there exists $c \neq 0$ in $\mathbf{R}$ such that A(x) = cB(x) a.e. on I. Equation (2) then becomes

$$y'' + (-\alpha c + \beta)B(x)y = 0.$$

Hence *D* contains the subspace $S = \{(\alpha,\beta) : \beta = c\alpha\}$.

Conversely, let *D* contain a proper subspace *S* of $\mathbf{R}^2$ other than the coordinate axes. Then $S = \{(\alpha,\beta):\beta = c\alpha\}$ for some $c \neq 0$. Hence, on this subspace, we must have

$$y'' + (-A(x) + cB(x))\alpha y = 0$$

disconjugate on I for every $\alpha \in \mathbf{R}$. Applying lemma 2.1.2 with q(x) = -A(x) + cB(x), we find that A(x) = cB(x) a.e. on I, so that A, B are linearly dependent.

Corollary 2.1.8 Whenever A, B are linearly independent functions on I, *D* cannot contain any full ray through the origin of parameter space.

Proof. Note that the assumption of linear independence excludes the possibility that either one of A(x), B(x) vanishes a.e. on I. The result is now a direct consequence of theorem 2.1.7.

Remark 2.3 Note that if D contains two proper subspaces of $\mathbf{R}^2$ then $D = \mathbf{R}^2$. Thus whenever $D \subsetneq \mathbf{R}^2$, (which is generally the case) D contains at most one full ray through (0,0).

It is important to note that D need not always contain a full ray through (0, 0). In fact, in some cases, D may even be a **bounded** set (see the example below). However note that, for example, D is **always unbounded** whenever A, B are linearly dependent, or if A(x) = 1 a.e. on I and B(x) is arbitrary, as, in this case, $D \supset \{(\alpha, 0):\alpha \geq 0\}$.

Example 1 We present an example which shows that, under the stated general conditions on A, B, it may occur that D is a bounded set. In order to see this choose A, B as follows.: Let $\varepsilon > 0$, $\eta > 0$ be given numbers and let

$$A(x) = \begin{cases} -\varepsilon, & x \in [0, 4] \\ +\varepsilon, & x \in (4, 8] \\ 0, & x \in (8, \infty). \end{cases}$$

Define B by

$$B(x) = \begin{cases} +\eta, & x \in [0, 1] \cup (2, 3] \cup (4, 5] \cup (6, 7] \\ -\eta, & x \in (1, 2] \cup (3, 4] \cup (5, 6] \cup (7, 8] \\ 0, & x \in (8, \infty). \end{cases}$$

With A, B so defined it is easy to see that (2) is always non-oscillatory on I for any choice of (α, β) (since A, B both vanish identically for all x > 8). Thus $N = \mathbf{R}^2$ in this case.

Now it is not difficult, however tedious, to show that there exists $\alpha_o, \beta_o > 0$ such that $(\alpha, \beta) \notin D$ whenever $|\alpha| \geq \alpha_o$, $|\beta| \geq \beta_o$. That $D \neq \{(0, 0)\}$ can be seen by showing that the rhomb $|\alpha|\,\varepsilon + |\beta|\,\eta = \pi^2/256$ in parameter space, lies in D . Hence D is a nontrivial bounded set. We omit the details.

Lemma 2.1.9 Let q_i: I → R and in $L_1^{loc}(I)$ for i = 1, 2. Assume that each one of the equations $y'' + q_i(x)y = 0$, i = 1, 2, is disconjugate on I. Then the equation

$$y'' + ((1-\gamma)q_1(x) + \gamma q_2(x))y = 0 \qquad (2.1.9)$$

is disconjugate on I, for each $\gamma \in [0, 1]$.

Proof. Let $[a, b] \subset I$ be a compact subinterval. Then the quadratic functional $I(\eta, q_i; a, b)$, $i = 1, 2$, is positive on $A_1(a, b)$, (lemma 2.1.1). On account of the same lemma, it suffices to show that $I(\eta, (1-\gamma)q_1 + \gamma q_2; a, b)$ is positive on $A_1(a, b)$, (since $[a, b]$ is then arbitrary the result will follow). Now if $\gamma \in [0, 1]$ and $\eta \not\equiv 0$ is in $A_1(a, b)$,

$$(1-\gamma)I(\eta, q_1: a, b) + \gamma I(\eta, q_2; a, b) > 0 \qquad (2.1.10)$$

However the left side of (2.1.10) is equal to $I(\eta, (1-\gamma)q_1 + \gamma q_2;a, b)$ as a simple calculation will show. Hence the latter is positive definite on $A_1(a, b)$. This is valid for every $[a, b] \subset I$. Therefore the result follows.

Remark 2.4 The above lemma is similar in spirit to an early result of Adamov [Ad.1, §9] wherein the word "disconjugate" is replaced by "non-oscillatory" in the statement of our lemma. Adamov's result was rediscovered by Moore [Mo.1, lemma 2].

Lemma 2.1.10 **Let $q_n: I \to R$, $q_n \in L_1^{loc}(I)$ $n = 1, 2, \ldots$ be a sequence of functions such that**

$$y'' + q_n(x)y = 0 \qquad (2.1.11)$$

is disconjugate on I, for each $n = 1, 2, \ldots$

If $q_n(x) \to q(x)$ in the L_1-sense over each compact subinterval of I, then the limit equation

$$y'' + q(x)y = 0 \qquad (2.1.12)$$

is also disconjugate on I.

Proof. Let $x_1, x_2 > 0$ be arbitrary but fixed and consider the interval $[x_1, x_2]$. It is known (see e.g. [Hl.1, Chapter III.1] that the solutions of (2.1.11) - (2.1.12) having the same initial values are "close" in the uniform norm over $[x_1, x_2]$ provided q_n and q are close in $L_1(x_1, x_2)$ which is the case, by assumption.

Now assume, on the contrary, that (2.1.12) is not disconjugate on I. Then it has a solution $y(x) \not\equiv 0$ with two zeros x_1, x_2, $x_1 < x_2$, say, in I. This said let $y_n(x)$ be solutions of (1) defined by $y_n(x_1) = 0$, $y'_n(x_1) = y'(x_1)$ $(\neq 0)$. Then for fixed $x > x_1$ and $\varepsilon > 0$, there exists N such that for each $n \geq N$, $\int_{x_1}^{x} |q_n - q| dx < \varepsilon$. Hence let $\eta > 0$ be fixed and set $x = x_2 + \eta$. For our ε, there is then a $\delta > 0$ such that

$$\sup_{x \in [x_1, x_2 + \eta]} |y_n(x) - y(x)| < \varepsilon$$

whenever

$$\int_{x_1}^{x_2+\eta} |q_n(x) - q(x)|\,dx < \delta.$$

Since y(x) must change sign at $x = x_2$, it follows that $y_n(x)$ must also change sign near $x = x_2$ if n is sufficiently large. Thus for such n, (2.1.11) is not disconjugate, which is a contradiction.

Theorem 2.1.11 In the usual topology of R^2, the disconjugacy domain D of (2) is a closed set.

Proof. Let (α_o, β_o) be a limit point of the sequence $(\alpha_n, \beta_n) \in D$, n = 1, 2, ... Then for each $\varepsilon > 0$ there exists an n such that $|\alpha_n - \alpha_o| < \varepsilon$, $|\beta_n - \beta_o| < \varepsilon$ and

$$y'' + (-\alpha_n A(x) + \beta_n B(x))y = 0 \qquad (2.1.13)$$

is not disconjugate. Now let y(x) be any nontrivial solution of (2) for $(\alpha, \beta) = (\alpha_o, \beta_o)$. Either y(x) never vanishes in which case $(\alpha_o, \beta_o) \in D$, (cf. [Ha.4]) or $y(x_o) = 0$ for some x_o. In the latter case let $y_n(x)$ be the solution of (2.1.13) which satisfies $y_n(x_o) = 0$, $y'_n(x_o) = y'(x_o)$. Then, by assumption, $y_n(x) \neq 0$ for $x \neq x_o$. However $\{y_n(x)\}$ uniformly approximates y(x) on each interval $[x_o, x_o + X]$ (by lemma 2.1.10), for $X > 0$ if ε sufficiently small. Hence y(x) can only change sign at $x = x_o$, and so $y(x) \neq 0$ for $x \neq x_o$ in I. Thus every solution $y(x) \not\equiv 0$ has at most one zero in I. Hence the result follows.

Theorem 2.1.12. When viewed as a subset of parameter space R^2, the disconjugacy domain of (2) is a convex set.

Proof. We must show that if $(\alpha_i, \beta_i) \in D$, i = 1, 2, then the line segment joining these two points is also in D, i.e., that the point $(1-\gamma)(\alpha_1, \beta_1) + \gamma(\alpha_2, \beta_2) \in D$ for each $\alpha \in [0, 1]$. This is equivalent to showing that

$$y'' + ((-(1-\gamma)\alpha_1 - \gamma\alpha_2) A(x) + ((1-\gamma)\beta_1 + \gamma\beta_2) B(x))y = 0$$

is disconjugate on I for each $\gamma \in [0, 1]$. Simplifying and rearranging terms in the potential of the last equation, we may rewrite it in the equivalent form

$$y'' + [(1-\gamma)(-\alpha_1 A(x) + \beta_1 B(x)) + \gamma(-\alpha_2 A(x) + \beta_2 B(x))]y = 0$$

for $\gamma \in [0, 1]$. Since $(\alpha_i, \beta_i) \in D$ for i = 1, 2, lemma 2.1.9 yields the conclusion.