Lecture Notes in Mathematics 1486

L. Arnold H. Crauel J.-P. Eckmann (Eds.)

Lyapunov Exponents

Proceedings, Oberwolfach 1990

§ 3. Мультипликативная эргодическая теорема. Формулировка и примеры

Теорема 1. *Если выполнено условие (5) § 2, то однопараметрическая группа измеримых изоморфизмов $\{\widetilde{T}^t\}$ правильна по Ляпунову при $t \to \pm\infty$.*

Из этой теоремы и теоремы § 1 вытекает.

Теорема 2. *Если выполнено условие (5) § 2, то для почти всех x по мере μ существуют точные характеристические показатели Ляпунова всех порядков, т. е.*

$$\chi_{\pm}(x, e^k; \mathfrak{n}) = \lim_{t \to \pm\infty} \frac{1}{|t|} \ln \lambda(e^k, \mathfrak{n}(t, x)),$$

где $\lambda(e^k, \mathfrak{n}(t, x))$ — коэффициент растяжения по k-мерному направлению e^k.

Springer-Verlag

L. Arnold H. Crauel J.-P. Eckmann (Eds.)

Lyapunov Exponents

Proceedings of a Conference held in
Oberwolfach, May 28 - June 2, 1990

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

Editors

Ludwig Arnold
Institut für Dynamische Systeme
Universität Bremen
Postfach 330 440
W-2800 Bremen 33, Germany

Hans Crauel
Fachbereich 9 Mathematik
Universität des Saarlandes
W-6600 Saarbrücken 11, Germany

Jean-Pierre Eckmann
Département de Physique Théorique
Université de Genève
CH-1211 Genève 4, Switzerland

Front cover:
V. I. Oseledets: A multiplicative ergodic theorem.
Trudy Moskov. Mat. Obsc. 19 (1968), 179-210

Mathematics Subject Classification (1980):
Primary: 58F
Secondary: 34F, 35R60, 58G32, 60F, 60G, 60H, 70L05, 70K, 73H, 73K, 93D, 93E

ISBN 3-540-54662-6 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-54662-6 Springer-Verlag New York Berlin Heidelberg

Typesetting: Camera ready by author
Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
46/3140-543210 - Printed on acid-free paper

Preface

These are the Proceedings of a conference on Lyapunov Exponents held at Oberwolfach May 28 – June 2, 1990. The volume contains an introductory survey and 26 original research papers, some of which have, in addition, survey character.

This conference was the second one on the subject of Lyapunov Exponents. The first one took place in Bremen in November 1984 and lead to the Proceedings volume Lecture Notes in Mathematics # 1186 (1986). Comparing those two volumes, one can realize pronounced shifts, particularly towards nonlinear and infinite-dimensional systems and engineering applications.

We would like to thank the 'Mathematisches Forschungsinstitut Oberwolfach' for letting us have the conference at this unique venue.

March 1991

Ludwig Arnold (Bremen)

Hans Crauel (Saarbrücken)

Jean-Pierre Eckmann (Genève)

Table of Contents

Chapter 3: Infinite-dimensional Random Dynamical Systems

Chapter 4: Deterministic Dynamical Systems

Chapter 5: Engineering Applications and Control Theory

Random Dynamical Systems

Ludwig Arnold
Institut für Dynamische Systeme
Universität Bremen
2800 Bremen 33

Hans Crauel
Fachbereich 9 MATHEMATIK
Universität des Saarlandes
6600 Saarbrücken 11

1 Introduction

The main purpose of this survey is to present and popularize the notion of a *random dynamical system* (RDS) and to give an impression of its scope. The notion of RDS covers the most important families of dynamical systems with randomness which are currently of interest. For instance, products of random maps — in particular products of random matrices — are RDS as well as (the solution flows of) stochastic and random ordinary and partial differential equations.

One of the basic results for RDS is the Multiplicative Ergodic Theorem (MET) of Oseledec [38]. Originally formulated for products of random matrices, it has been reformulated and reproved several times during the past twenty years. Basically, there are two classes of proofs. One makes use of Kingman's Subadditive Ergodic Theorem together with the polar decomposition of square matrices. The other one starts by proving the assertions of the MET for triangular systems, and then enlarges the probability space by the compact group of special orthogonal matrices, so that every matrix cocycle becomes homologous to a triangular one.

Let us emphasize that the MET is a *linear* result. It is possible to introduce Lyapunov exponent-like quantities for nonlinear systems directly à la (9) below, or, much more sophisticated, as by Kifer [27]. However, the wealth of structure provided by the MET is available for linear systems only. Speaking of an "MET for nonlinear systems" always means the MET for the *linearization* of a nonlinear system. What is new for nonlinear systems is the fact that the linearization lives on the tangent bundle of a manifold (instead of the flat bundle $\mathbb{R}^d \times \Omega$ as for products of random matrices). The MET yields nontrivial consequences for deterministic systems already. This case has been dealt with by Ruelle [39]. Ruelle's argument proceeds by trivialization of the nonflat tangent bundle. It is exactly the same argument that works for nonlinear random systems: infer the MET for the linearization of the system from the ordinary MET together with a trivialization argument. We reproduce the argument below.

Stochastic flows have entered the scene a couple of years ago. They are related to RDS, but they are not the same. We describe their relations, and point out their differences.

The final Section briefly reviews all contributions to the present volume.

2 Random Dynamical Systems and Multiplicative Ergodic Theory

2.1 RDS

Consider a set T (time), $T = \mathbb{R}$, $\mathbb{Z}$; $\mathbb{R}^+$, or $\mathbb{N}$, and a family $\{\vartheta_t : \Omega \to \Omega \mid t \in T\}$ of measure preserving transformations of a probability space $(\Omega, \mathcal{F}, P)$ such that $(t,\omega) \mapsto \vartheta_t\omega$ is measurable, $\{\vartheta_t \mid t \in T\}$ is ergodic, and $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$ for all $t, s \in T$ with $\vartheta_0 = \text{id}$. Thus $(\vartheta_t)_{t\in T}$ is a flow if $T = \mathbb{R}$ or $\mathbb{Z}$, and a semi-flow if $T = \mathbb{R}^+$ or $\mathbb{N}$. The set-up $((\Omega, \mathcal{F}, P), (\vartheta_t)_{t\in T})$ is a (measurable) dynamical system.

Definition A *random dynamical system* on a measurable space $(X, \mathcal{B})$ over $(\vartheta_t)_{t\in T}$ on $(\Omega, \mathcal{F}, P)$ is a measurable map

$$\varphi : T \times X \times \Omega \to X$$

such that $\varphi(0,\omega) = \text{id}$ (identity on X) and

$$\varphi(t+s,\omega) = \varphi(t, \vartheta_s\omega) \circ \varphi(s,\omega) \tag{1}$$

for all $t, s \in T$ and for all ω outside a P-nullset, where $\varphi(t,\omega) : X \to X$ is the map which arises when $t \in T$ and $\omega \in \Omega$ are fixed, and $\circ$ means composition. A family of maps $\varphi(t,\omega)$ satisfying (1) is called a *cocycle*, and (1) is the cocycle property.

We often omit mentioning $((\Omega, \mathcal{F}, P), (\vartheta_t)_{t\in T})$ in the following, speaking of a random dynamical system (abbreviated RDS) φ.
We do not assume the maps $\varphi(t,\omega)$ to be invertible a priori. By the cocycle property, $\varphi(t,\omega)$ is automatically invertible (for all $t \in T$ and for P-almost all ω) if $T = \mathbb{R}$ or $\mathbb{Z}$, and $\varphi(t,\omega)^{-1} = \varphi(-t, \vartheta_t\omega)$.

The following examples are quite distinct in many respects. However, they all are RDS.

1. The simplest case of a random dynamical system is a non-random — viz., deterministic — dynamical system. An RDS is deterministic if φ does not depend on ω, i. e., $\varphi(t,x,\omega) = \varphi(t,x)$. Then the cocycle property (1) reads $\varphi(t+s) = \varphi(t)\circ\varphi(s)$, hence $(\varphi(t))_{t\in T}$ consists of the iterates of a measurable map on X if $T = \mathbb{Z}^{(+)}$, and $(\varphi(t))_{t\in T}$ is a measurable (semi-) flow if $T = \mathbb{R}^{(+)}$, respectively.

2. Let $\vartheta : \Omega \to \Omega$ be a measure preserving transformation, and let $\psi : X \times \Omega \to X$ be a measurable map. Put $\psi_n = \psi \circ \vartheta^{n-1}$. Then

$$\varphi(n,\omega) = \begin{cases} \psi_n(\omega) \circ \psi_{n-1}(\omega) \ldots \circ \psi_1(\omega) & \text{for } n > 0 \\ \text{id} & \text{for } n = 0 \\ \psi_{n+1}^{-1}(\omega) \circ \psi_{n+2}^{-1}(\omega) \circ \ldots \circ \psi_0^{-1}(\omega) & \text{for } n < 0, \end{cases}$$

defines an RDS (of course, defining $\varphi(n,\omega)$ for $n < 0$ needs ϑ and $\psi(\cdot,\omega)$ invertible P-a. s.). In particular, if $X = \mathbb{R}^d$ and $x \mapsto \psi(x,\omega)$ is linear, then φ is a product of random matrices.

3. Suppose $T = \mathbb{R}$, and M is a C^1 manifold. Denote by TM the total space of the tangent bundle of M, and let $Y : M \times \Omega \to TM$ be a measurable map such that for P-almost all ω the map $Y(\cdot,\omega)$ is a smooth vector field. Then the random differential equation

$$\dot{x}(t) = Y(x(t), \vartheta_t\omega), \qquad x(0) = x_0, \tag{2}$$

induces a map $\varphi(t,\omega) : M \to M$, such that $x(t,\omega) = \varphi(t,\omega)x$ solves (2) with $x(0) = x$ for $t \in (t^-(x,\omega), t^+(x,\omega))$, where $t^-(x,\cdot) < 0 < t^+(x,\cdot)$ (P-a. s.) describe the maximal intervals of definition of solutions. If $t^-(x,\cdot) = -\infty$ and $t^+(x,\cdot) = +\infty$ (for all $x \in M$ P-a. s.) then φ is an RDS. In addition, $x \mapsto \varphi(t,\omega)x$ is a diffeomorphism for all $t \in \mathbb{R}$ (P-a. s.) in this case. The maximal interval of definition is automatically all of $\mathbb{R}$ if M is compact. If $-\infty < t^-(x)$ or $t^+(x) < \infty$ for some x with positive probability we speak of a *local RDS* or *local random flow*.

4. Suppose M is a C^2 manifold, and Y_i, $0 \le i \le n$, are smooth vector fields on M. Then the stochastic differential equation

$$dx(t) = Y_0(x(t))\,dt + \sum_{i=1}^{n} Y_i(x(t)) \circ dW_i(t), \qquad x(0) = x_0, \tag{3}$$

induces a *(local) stochastic flow*. Usually (3) is understood for $t \ge 0$. We will describe below how to give (3) a meaning on the whole time axis. Once this is done, maximal intervals of solutions, containing $t = 0$ as an interior point, exist and have the same properties as for random flows described in the previous example.

We have introduced *local* random and stochastic flows because they play a role in stochastic bifurcation. For details see below.

As for deterministic systems, RDS may be classified according to their spatial properties.

If X is a topological space (with Borel σ-algebra), a random dynamical system is said to be *continuous* if $\varphi(t,\omega) : X \to X$ is continuous for all $t \in T$ and all $\omega \in \Omega$ outside a P-nullset.
If X is a C^r manifold, $r \ge 1$, an RDS φ on X is said to be *differentiable* or *smooth* if $\varphi(t,\omega) : X \to X$ is C^r differentiable for all $t \in T$ and all ω outside a P-nullset.
A random dynamical system on a topological vector space X is said to be linear if $\varphi(t,\omega) : X \to X$ is linear for all $t \in T$ and all ω outside a P-nullset.

If an RDS consists of non-invertible maps then T cannot contain negative times. An RDS φ consisting of invertible maps need not allow negative time, since ϑ_t need not be invertible. So we have to distinguish between two kinds of invertibility. An RDS is said to be *two sided* if $T = \mathbb{R}$ or $T = \mathbb{Z}$. It is said to be *invertible* if, for all $t \in T$ and P-almost all ω, $\varphi(t,\omega)$ is invertible in the corresponding class (measurable, continuous, smooth). Clearly 'two sided' is stronger than 'invertible'.

Any RDS induces a measurable *skew product (semi-) flow*

$$\begin{array}{rcl} \Theta_t : X \times \Omega & \to & X \times \Omega \\ (x,\omega) & \mapsto & (\varphi(t,\omega)x, \vartheta_t\omega), \end{array} \tag{4}$$

$t \in T$, where $\varphi(t,\omega)x = \varphi(t,x,\omega)$. The flow property $\Theta_{t+s} = \Theta_t \circ \Theta_s$ follows from the cocycle property of φ (see (1); we use the term flow for both continuous and discrete time T).

From the point of view of abstract ergodic theory, an RDS is nothing but an ordinary dynamical system $(\Theta_t)_{t\in T}$ with a factor $(\vartheta_t)_{t\in T}$ together with the extra bit of structure provided by the fact that the ergodic invariant measure P for the factor is given a priori. (This observation might serve as an abstract definition of RDS.)

A probability measure μ on $X \times \Omega$ (on the product σ-algebra $\mathcal{B} \otimes \mathcal{F}$) is said to be an *invariant measure for* φ if μ is invariant under Θ_t, $t \in T$, and if it has marginal P on Ω. Invariant measures always exist for continuous RDS on a compact X (which is in complete analogy with deterministic dynamical systems).

Denote by $Pr(X)$ the space of probability measures on X, endowed with the smallest σ-algebra making the maps $Pr(X) \to \mathbb{R}$, $\nu \mapsto \int_X h\,d\nu$, measurable with h varying over the bounded measurable functions on X.
Given a measure $\mu \in Pr(X \times \Omega)$ with marginal P on Ω, a measurable map $\mu_\cdot : \Omega \to Pr(X)$, $\omega \mapsto \mu_\omega$ will be called a *disintegration of* μ *(with respect to* P*)* if

$$\mu(B \times C) = \int_C \mu_\omega(B)\,dP(\omega)$$

for all $B \in \mathcal{B}$ and $C \in \mathcal{F}$.
Disintegrations exist and are unique (P-a. s.), e. g., if X is a Polish space. We will assume existence and uniqueness of a disintegration in the following.
A measure μ is invariant for the RDS φ if and only if

$$E(\varphi(t,\cdot)\mu_\cdot \mid \vartheta_t^{-1}\mathcal{F})(\omega) = \mu_{\vartheta_t\omega} \qquad P\text{-a. s. for every } t \in T. \tag{5}$$

If T is two sided then $\vartheta_t^{-1}\mathcal{F} = \mathcal{F}$, hence for T two sided (5) reads

$$\varphi(t,\omega)\mu_\omega = \mu_{\vartheta_t\omega} \qquad P\text{-a. s. for every } t \in T.$$

2.2 Lyapunov exponents and the Multiplicative Ergodic Theorem

For a differentiable manifold M denote by TM the total space of its tangent bundle. The linearization of a differentiable map $\psi : M \to M$ is denoted by $T\psi : TM \to TM$ with $T_x\psi : T_xM \to T_{\psi(x)}M$, $x \in M$, denoting the action of $T\psi$ on individual fibers.

Suppose φ is a smooth RDS on a d-dimensional Riemannian manifold M. The chain rule yields

$$T_x\varphi(t+s,\omega) = T_{\varphi(s,\omega)x}\varphi(t,\vartheta_s\omega) \circ T_x\varphi(s,\omega) \tag{6}$$

for all $t,s \in T$ and $x \in M$ with P-measure 1. Consequently, the linearization $T\varphi : T \times TM \times \Omega \to TM$ is a cocycle over the skew product flow $\Theta_t(x,\omega) = (\varphi(t,\omega)x, \vartheta_t\omega)$ on $M \times \Omega$ (cf (4)).

Suppose μ is an invariant measure for φ such that

$$(x,\omega) \mapsto \sup_{0<t\leq t_0} \log^+(\|T_x\varphi(t,\omega)\|) \in L^1(\mu), \tag{7}$$

where $\log^+ = \max\{\log, 0\}$, and $\|\cdot\|$ denotes the norm induced by the Riemannian metric. Denote by $\lambda_1^\mu(x,\omega) \geq \lambda_2^\mu(x,\omega) \geq \ldots \geq \lambda_d^\mu(x,\omega)$ the *Lyapunov exponents of* φ *associated with* μ, where the $\Theta_\cdot$-invariant maps $(x,\omega) \mapsto \lambda_i^\mu(x,\omega)$ are defined via

$$\sum_{i=1}^k \lambda_i^\mu(x,\omega) = \lim_{t\to\infty} \frac{1}{t} \log \|\textstyle\bigwedge^k T_x\varphi(t,\omega)\|, \tag{8}$$

$1 \leq k \leq d$. Here $\bigwedge^k$ denotes the k-fold exterior product of $T_x\varphi$. Existence of the limits in (8) follows from Kingman's subadditive ergodic theorem (Kingman [29]). Though (7) guarantees $\lambda_1 < \infty$, the last λ_i's may equal $-\infty$. If μ is ergodic, the Lyapunov exponents do not depend on (x,ω). If M is compact, the Lyapunov exponents do not depend on the choice of the Riemannian metric.

Sometimes it is more convenient to count only the distinct Lyapunov exponents, denoted here by $\Lambda_1 > \Lambda_2 > \ldots > \Lambda_r$, where r is the number of distinct exponents, $1 \leq r \leq d$ (we assume μ ergodic to ease notation). Denote by $d_i = \max\{p - q + 1 \mid \lambda_p = \lambda_q = \Lambda_i\}$ the multiplicity of Λ_i.

There is another classical way to introduce Lyapunov exponents (see for instance Arnold and Wihstutz [8]). Put

$$\lambda(v,\omega) = \limsup_{t\to\infty} \frac{1}{t} \log \|T_x\varphi(t,\omega)v\|. \tag{9}$$

The map $\lambda(\cdot,\omega) : TM \to \mathbb{R} \cup \{-\infty\}$ satisfies $\lambda(cv) = \lambda(v)$ for all $c \neq 0$, $v \in TM$, and $\lambda(c_1v_1 + c_2v_2) \leq \max\{\lambda(v_1), \lambda(v_2)\}$ for all $c_1, c_2 \in \mathbb{R}$ and $v_1, v_2 \in T_xM$, $x \in M$ (sometimes called a *characteristic exponent*); we dropped ω, which is fixed here. These two properties imply that λ takes only finitely many values $\tilde\Lambda_1 > \tilde\Lambda_2 > \ldots > \tilde\Lambda_{\tilde r}$ as v varies over T_xM, $v \neq 0$. The Lyapunov exponents in this approach are the $\tilde\Lambda_i$. By definition of $v \mapsto \lambda(v,\omega)$, the sets

$$V_\delta(x,\omega) = \{v \in T_xM \mid \lambda(v,\omega) \leq \delta\}$$

are linear subspaces of T_xM for $\delta \in \mathbb{R}$ arbitrary. Put $V_i = V_{\tilde\Lambda_i}$ and $\tilde d_i = \dim V_i - \dim V_{i+1}$.

The two definitions of Lyapunov exponents presented above are in general not equivalent. However, they are equivalent if (and only if) (x,ω) is a *forward regular point* for $T_x\varphi(t,\omega)$ (see Arnold and Wihstutz [8]) pp. 2–3). In terms of the present paper, (x,ω) is forward regular if $\sum_1^r d_i\Lambda_i = \sum_1^{\tilde r} \tilde d_i\tilde\Lambda_i$. It is clear from (6) that the bundle of linear subspaces

$$V_i(x,\omega) = \{v \in T_xM \mid \lambda(v,\omega) \leq \tilde\Lambda_i(x,\omega)\},$$

is invariant under $T\varphi$ in the sense that $T_x\varphi(t,\omega)V_i(x,\omega) \subset V_i(\Theta_t(x,\omega))$. We refer to the family

$$T_xM = V_1(x,\omega) \supset V_2(x,\omega) \supset \ldots \supset V_r(x,\omega) \supset \{0\}$$

as the *Oseledec flag* associated with φ.
In the following we will be concerned with regular systems only, so that we need not distinguish between Λ and $\tilde{\Lambda}$.

We now recall the Multiplicative Ergodic Theorem (MET) of Oseledec for nonlinear RDS and sketch Ruelle's trivialization argument which reduces the nonflat bundle case to the flat bundle one.

Theorem
(i) (Multiplicative Ergodic Theorem without invertibility)
Suppose φ is a smooth RDS on a d-dimensional Riemannian manifold M and let μ be an invariant measure for φ such that (7) is satisfied. Then the linearization $T_x\varphi(t,\omega)$ of φ is forward regular at μ-almost all points $(x,\omega) \in M \times \Omega$.
(ii) (Multiplicative Ergodic Theorem with invertibility)
Suppose φ is a smooth two sided RDS on a d-dimensional Riemannian manifold M and let μ be an invariant measure for φ such that

$$(x,\omega) \mapsto \sup_{0<t\leq t_0} \{\log^+(\|T_x\varphi(t,\omega)\|) + \log^+(\|(T_x\varphi(t,\omega))^{-1}\|)\} \in L^1(\mu). \tag{10}$$

Then the linearization $T_x\varphi(t,\omega)$ of φ is bi-regular[1] *at μ-almost all points $(x,\omega) \in M \times \Omega$.*

Note that in the invertible case regularity implies that for μ-almost all (x,ω) the spaces

$$E_i(x,\omega) = \{v \in T_xM \mid \lambda^+(v,\omega) = \lambda^-(v,\omega) = \Lambda_i(x,\omega)\}$$

form a splitting of T_xM (with $\lambda^+(v,\omega) = \lambda(v,\omega)$ as in (9), and $\lambda^-(v,\omega)$ defined as in (9) with $t \to -\infty$). $TM = \bigoplus E_i$ is referred to as the *Oseledec splitting*.

PROOF OF THE MET Denote the tangent bundle by (TM, π, M). Choose a countable covering of M by bundle charts (M_i, ψ_i) trivializing TM locally in an isometrical manner. That means, M_i is an open subset of M, and $\psi_i : \pi^{-1}(M_i) \to M_i \times \mathbb{R}^d$, where $\pi : TM \to M$ denotes the canonical bundle projection, such that ψ_i restricted to $\pi^{-1}\{x\}$ is linear for all $x \in M_i$. In addition, ψ_i may be chosen to be an isomorphism with respect to the scalar product on $\pi^{-1}\{x\}$ induced by the Riemannian structure on M and the standard scalar product on $\mathbb{R}^d$ for all $x \in M_i$, see Klingenberg [31] Theorem 1.8.20. (It is not really essential to choose isometric bundle charts, it is simply more convenient.)
Next put $B_0 = M_0$ and $B_n = M_n \setminus \bigcup_{j<n} B_j$ to obtain a countable covering $\{B_n \mid n \in \mathbb{N}\}$ of M by disjoint Borel sets. Putting

$$\begin{aligned} \Sigma : TM &\to M \times \mathbb{R}^d \\ u &\mapsto \psi_i(x) \qquad \text{for } x \in B_i \end{aligned}$$

yields a bimeasurable bundle map from TM to the flat bundle $M \times \mathbb{R}^d$ such that $\Sigma_x : T_xM \to \{x\} \times \mathbb{R}^d$ is an isomorphism for all $x \in M$. Finally, put

$$\Psi(t;x,\omega) = \Sigma_{\varphi(t,\omega)x} \circ T_x\varphi(t,\omega) \circ \Sigma_x^{-1}.$$

[1] 'regular' in the terminology of Arnold and Wihstutz [8] p.4

Then Ψ is a linear RDS on $\mathbb{R}^d$ over the (enlarged) probability space $(M \times \Omega, \mu)$. Since Σ_x is an isomorphism for all $x \in M$,

$$\|\bigwedge^k \Psi(t; x, \omega)\| = \|\bigwedge^k T_x\varphi(t, \omega)\|$$

for all k, $1 \leq k \leq d$, and

$$\|\Psi(t; x, \omega)y\| = \|T_x\varphi(t, \omega)(\Sigma_x^{-1}y)\|$$

for all $y \in \mathbb{R}^d$, $t \in T$, $x \in M$, and P-almost all $\omega \in \Omega$. Thus regularity of Ψ implies regularity of $T\varphi$. But Ψ satisfies the integrability conditions of the 'ordinary' MET by (7) and (10), respectively, hence Ψ is forward or bi-regular, respectively, for μ-almost all (x, ω). □

Note that the Theorem does not require compactness of the manifold M. For a given smooth RDS with an invariant measure it thus only remains to check the integrability conditions (7) or (10), respectively, to infer the conclusions of the MET. This has been done directly for white noise systems on compact manifolds by Carverhill [19] (without imposing any further assumptions). Later Kifer [28] has shown that white noise systems on compact manifolds satisfy much stronger integrability conditions.

Multiplicative ergodic theory becomes much more difficult when considering *infinite dimensional* RDS. Recall that for a finite dimensional linear deterministic system the Lyapunov exponents are precisely the real parts of the eigenvalues of A (for continuous time, $\dot{x} = Ax$) or the logarithms of the eigenvalues of A (for discrete time, $x_{n+1} = Ax_n$), respectively. Thus, the Lyapunov exponents are determined by the spectrum. The definition (see (8)) suggests that it is essential to have a 'well behaved' top part of the spectrum: isolated eigenvalues of finite multiplicity. Since spectra of infinite dimensional operators in general have a considerably more complicated structure than finite dimensional ones, it is clear that much less is to be expected for infinite dimensional RDS. For a survey on infinite dimensional systems see 4.3 below.

3 Random Dynamical Systems and Markov Processes

3.1 Two cultures

In the theory of RDS two well-established mathematical cultures meet, overlap, and sometimes collide:

- *Dynamical Systems*: the *flow* point of view. Typically $T = \mathbb{R}$ or $\mathbb{Z}$ unless mappings are non-invertible which typically happens only for discrete time. Invariance of a measure is defined as invariance with respect to the mappings of the system.

- *Markov processes, stochastic analysis*: Here time is almost exclusively $\mathbb{Z}^+$ or $\mathbb{R}^+$ (or part of it). Markov processes are defined and studied through their transition semigroups forward in time. The necessity to really *construct* stochastic processes with prescribed transition semigroups (their existence follows from Kolmogorov's theorem) created the theory of stochastic differential equations (SDE's) (which are really ODE's with white noise input). Continuous time is $\mathbb{R}^+$, and a filtration $\mathcal{F}_t$ (i. e., an increasing family of sub σ-algebras of $\mathcal{F}$) collects the information available at time t. Everything has to be adapted, i. e., $\mathcal{F}_t$-measurable. 'Invariant measure' in the Markov context means invariance with respect to the transition semigroup.

The door from Markov processes to dynamical systems was really opened around 1980, when several people (Elworthy [24], Bismut [12], Ikeda and Watanabe [25], Kunita [33]) realized that writing down an SDE for a Markov process means much more than originally thought of by the pioneers K. Itô et al. It means the construction of a *stochastic flow* (or, as we will see, of an RDS with independent increments) whose one-point motions are Markov with the prescribed transition semigroup or its generator, respectively.

Probabilists sometimes criticize moving from an SDE to an RDS as 'forgetting' some of the probabilistic structure of the original, e. g., the fact that coming from an SDE implies in particular that all n-point motions are Markov. We think that the contrary is true, as many contributions to this volume show.

First, the concept of an RDS allows to address completely *new questions on SDE's*, as for instance the problem of finding *all* invariant measures (and not only those solving the Fokker-Planck equation), the problem of random invariant manifolds, random normal forms etc.

Second, the underlying Markov structure gives rise to problems which do not make sense for deterministic dynamical systems, such as the interplay of measurability and adaptedness properties with dynamics, see Crauel [20], [21], [22].

There seems to be some need for describing the connection between RDS and Markov processes in some detail.

3.2 RDS with independent increments, Brownian RDS

An RDS $\varphi(t,\omega)$ over $(\Omega, \mathcal{F}, P, (\vartheta_t)_{t\in T})$ is said to have *independent increments* if for all $t_0 \le t_1 \le \ldots \le t_n$ the random variables

$$\varphi(t_1 - t_0, \vartheta_{t_0}\omega), \varphi(t_2 - t_1, \vartheta_{t_1}\omega), \ldots, \varphi(t_n - t_{n-1}, \vartheta_{t_{n-1}}\omega) \tag{11}$$

are independent. If, in addition, for $T = \mathbb{R}^+$ or $\mathbb{R}$ the map $t \mapsto \varphi(t,\omega)x$ is continuous for all $x \in X$ P-a. s., then the RDS or cocycle is said to be a *Brownian* RDS or cocycle.

Remarks (i) An RDS with independent increments automatically has *stationary* (time homogeneous) increments, as, by the ϑ_t invariance of P, $\varphi(h, \vartheta_t\omega) \stackrel{\mathcal{D}}{=} \varphi(h,\omega)$ for all $t \in T$.
(ii) If $\varphi(t,\omega)$ consists of invertible mappings then, by the cocycle property,

$$\varphi_{s,t}(\omega) := \varphi(t,\omega) \circ \varphi(s,\omega)^{-1} = \varphi(t-s, \vartheta_s\omega)$$

for $s \le t$, so (11) means that for $t_0 \le t_1 \le \ldots \le t_n$

$$\varphi_{t_0,t_1}, \varphi_{t_1,t_2}, \ldots, \varphi_{t_{n-1},t_n} \tag{12}$$

are independent.

3.3 RDS and Markov chains, discrete time $T = \mathbb{Z}^+$ or $\mathbb{Z}$

Case $T = \mathbb{Z}^+$: Here $\varphi(n,\omega) = \varphi(1, \vartheta^{n-1}\omega) \circ \ldots \circ \varphi(1,\omega)$, so the cocycle has independent increments if and only if $\varphi(1,\omega), \varphi(1,\vartheta\omega), \ldots$ are iid. We thus have a product of iid random mappings, i. e., a classical 'iterated function system'. The mapping

$$x \mapsto \varphi(n,\omega)x$$

defines a homogeneous Markov chain with transition kernel

$$P(x, B) = P\{\omega \mid \varphi(1,\omega)x \in B\}. \tag{13}$$

Putting $x_n = \varphi(n,\omega)x_0$ we have

$$x_{n+1} = \varphi(1, \vartheta^n\omega)x_n, \tag{14}$$

i. e., a stochastic difference equation generating the Markov chain.

Conversely, given a transition kernel $P(x, B)$ on X, we want to construct an RDS with independent increments over a dynamical system $(\Omega, \mathcal{F}, P, \vartheta)$, i. e., a cocycle $\varphi(n,\omega)$ with $(\varphi(1, \vartheta^n\omega))_{n\in\mathbb{Z}^+}$ iid, such that (13) holds. This question has been dealt with by Kifer [26] Section 1.1. It always has a positive answer as soon as X is a Borel subset of a Polish space and if we are content with a measurable mapping $(x,\omega) \mapsto \varphi(1,\omega)x$. If we want $x \mapsto \varphi(1,\omega)x$ to be continuous or homeomorphisms or smooth etc., a general answer to this representation problem is not known up to now (compare Kifer [26] p. 12).

Case $T = \mathbb{Z}$: Now $\varphi(n,\omega)$ is invertible, and the cocycle has independent increments if and only if $(\varphi(1, \vartheta^n\omega))_{n\in\mathbb{Z}}$ is iid. The mapping $x \mapsto \varphi(n,\omega)x$ defines a homogeneous Markov chain on all of $\mathbb{Z}$ starting at $x_0 = x$, and (14) can be inverted to give

$$x_n = \varphi(1, \vartheta^n\omega)^{-1}x_{n+1} = \varphi(-1, \vartheta^{-n}\omega)x_{n+1}.$$

We can now look at the *forward transition kernel*

$$P^+(x, B) = P\{\omega \mid \varphi(1,\omega)x \in B\}$$

and the *backward transition kernel*

$$P^-(x, B) = P\{\omega \mid \varphi(-1,\omega)x \in B\} = P\{\omega \mid \varphi(1,\omega)^{-1}x \in B\}.$$

Note that in general P^+ and P^- do not have the same invariant measures: a forward invariant measure ν^+ has to satisfy

$$\nu^+ = \int P^+(x, \cdot)\, d\nu^+(x) = E\varphi(1,\omega)\nu^+,$$

whereas a backward invariant measure ν^- is characterized by

$$\nu^- = \int P^-(x, \cdot)\, d\nu^-(x) = E\varphi(-1,\omega)\nu^- = E\varphi(1,\omega)^{-1}\nu^-.$$

How are measures $\nu^{\pm}$ related to invariant measures of the RDS, i. e., measures μ on $X \times \Omega$ whose disintegration satisfies $\varphi(1,\omega)\mu_\omega = \mu_{\vartheta\omega}$? For one sided time Ohno [37] has proved that if $\nu = \nu^+$ is an invariant measure for the forward transition kernel P^+, then $\mu = \nu \times P$ is invariant for the RDS. Conversely, if a product measure $\nu \times P$ is invariant for the RDS, then ν is invariant for P^+.

For two sided time, a product measure is invariant for the RDS if and only if ν is fixed under φ, i. e., $\varphi(1,\omega)\nu = \nu$ P-almost surely. If a measure $\nu^{\pm}$ is $P^{\pm}$ invariant then the measures μ^+ and μ^-, given by

$$\mu_\omega^{\pm} = \lim_{n\to\mp\infty} \varphi(n,\omega)^{-1}\nu^{\pm},$$

are invariant for the RDS (so-called *Markov* measures). They are the ones which 'remember' the Markov kernels $P^{\pm}$. This equally applies for continuous time $T = \mathbb{R}^+$ or $\mathbb{R}$. These questions have been studied systematically by Crauel [21].

However, typically an RDS has more invariant measures not coming from the Markov chain — and those measures are needed for a systematic study of the RDS.

3.4 RDS and Markov processes, continuous time $T = \mathbb{R}^+$ or $\mathbb{R}$

Here the situation is much nicer and its description more complete than in the discrete time case. Basic results are due to Baxendale [10] and Kunita [33], [34], [35]. We describe the situation conceptually, i. e., without stating all technical assumptions, and quote freely from the above sources. We mainly emphasize the RDS point of view.

Case $T = \mathbb{R}^+$: Assume $X = \mathbb{R}^d$ (similar things hold on manifolds). Let $\varphi(t,\omega)$ be a Brownian RDS of homeomorphisms (or diffeomorphisms of some smoothness class). Then $(\varphi(t,\omega))_{t\in\mathbb{R}^+}$ is a Brownian motion with values in the group $\mathrm{Hom}(\mathbb{R}^d)$ (or $\mathrm{Diff}^*(\mathbb{R}^d)$ with a suitable $*$) in the sense of Baxendale [10], or

$$\varphi_{s,t}(\omega) = \varphi(t,\omega) \circ \varphi(s,\omega)^{-1},$$

$s,t \in \mathbb{R}^+$, is a temporally homogeneous Brownian flow in the sense of Kunita [35] p. 116.

By studying the infinitesimal mean

$$\lim_{h\searrow 0} \frac{1}{h} E(\varphi_{t,t+h}(\omega)x - x) = b(x)$$

and the infinitesimal covariance

$$\lim_{h\searrow 0} \frac{1}{h} E(\varphi_{t,t+h}(\omega)x - x)(\varphi_{t,t+h}(\omega)y - y)' = a(x,y),$$

Kunita constructs a vector field valued Brownian motion $(F(x,t,\omega))_{x\in\mathbb{R}^d, t\in\mathbb{R}^+}$, i. e., a continuous (in t) Gaussian process $(F(\cdot,t,\omega))_{t\in\mathbb{R}^+}$ with values in the space of vector fields on $\mathbb{R}^d$ (so $x \mapsto F(x,t,\omega)$ is a vector field), which has additively stationary independent increments and satisfies $F(\cdot,0,\omega) = 0$ (P-a. s.). The Brownian motion F is related to the

Brownian flow φ by $EF(x,t,\omega) = tb(x)$ and $\mathrm{cov}(F(x,t,\omega), F(y,s,\omega)) = \min\{t,s\}\, a(x,y)$. This implies that for all $s \in \mathbb{R}^+$

$$\varphi_{s,t}(\omega)x = x + \int_s^t F(\varphi_{s,u}(\omega)x, du, \omega), \quad t \in [s,\infty), \tag{15}$$

which has to be understood in the sense that (15) has a solution which coincides in distribution with the original Brownian flow φ. In short: The (forward) flow satisfies an Itô SDE driven by vector field valued Gaussian white noise. F is called the random infinitesimal generator of φ.

All n-point motions $(\varphi(t,\omega)x_1, \ldots, \varphi(t,\omega)x_n)$ are homogeneous Feller-Markov processes. In particular, $(\varphi(t,\omega)x)_{t\in\mathbb{R}^+}$ is a Markov process whose transition semigroup has generator

$$L = \sum_{i=1}^d b^i(x)\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^d a^{ij}(x,x)\frac{\partial^2}{\partial x^i \partial x^j}. \tag{16}$$

The backward flow $\varphi_{s,t}(\omega)^{-1} = \varphi(s,\omega)\circ\varphi(t,\omega)^{-1}$, $0 \le s \le t$, satisfies for each $t \in \mathbb{R}^+$ a backward Itô equation in $s \in [0,t]$,

$$\varphi_{s,t}(\omega)^{-1}x = x - \int_s^t \hat{F}(\varphi_{u,t}(\omega)^{-1}x, \hat{du}, \omega),$$

where

$$\hat{F}(x,t,\omega) = F(x,t,\omega) - t\,c(x), \qquad c_i(x) = \sum_{j=1}^d \frac{\partial a^{ij}}{\partial x^j}(x,y)\Big|_{y=x}, \tag{17}$$

and the backward integral $\int_s^t \hat{F}(\varphi_{u,t}(\omega)^{-1}x, \hat{du}, \omega)$ is formally defined by the same definition as the forward integral — the difference being the inverted measurability counting from t backward to s.

As usual, things get more symmetric if we use Stratonovich forward and backward integrals. Put

$$F^0(x,t,\omega) = F(x,t,\omega) - \frac{t}{2}c(x),$$

then F^0 is the forward as well as the backward Stratonovich infinitesimal generator of φ.

Conversely, given a temporally homogeneous $\mathcal{V}(\mathbb{R}^d)$ (= vector fields on $\mathbb{R}^d$) -valued Brownian motion F, we can write down the SDE (15) to generate a Brownian flow with generator F. We can easily construct a Brownian RDS describing the same object. Indeed, put

$$\begin{aligned}
\Omega &= \{\omega \mid \omega(0) = 0, \omega(\cdot) \in \mathcal{C}(\mathbb{R}^+, \mathcal{V}(\mathbb{R}^d))\} \\
\mathcal{F} &= \text{Borel field} \\
P &= \text{distribution of } F = \text{‘Wiener measure’} \\
\vartheta_t\omega(\cdot) &= \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}^+,
\end{aligned}$$