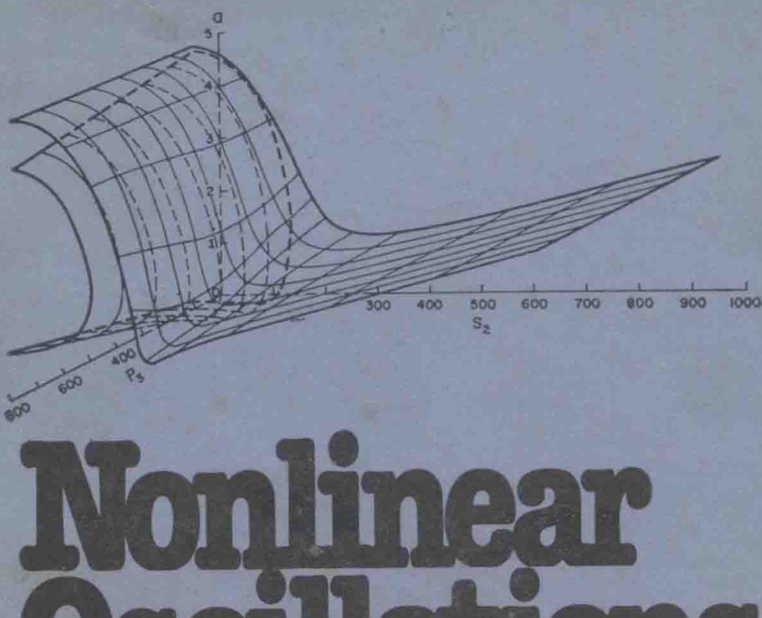
Ali Hasan Nayfeh
Dean T. Mook



Nonlinear Oscillations

A volume in Pure and Applied Mathematics:
A Wiley-Interscience Series of Texts, Monographs, and Tracts—Richard Courant, Founder; Lipman Bers, Peter Hilton, and Harry Hochstadt, Advisory Editors

NONLINEAR OSCILLATIONS

ALI HASAN NAYFEH

University Distinguished Professor

DEAN T. MOOK

Professor

Department of Engineering Science and Mechanics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia

A WILEY-INTERSCIENCE PUBLICATION

JOHN WILEY & SONS, New York • Chichester • Brisbane • Toronto

Library of Congress Cataloging in Publication Data:

Nayfeh, Ali Hasan, 1933–
Nonlinear oscillations.

(Pure and applied mathematics)
Includes bibliographical references and index.
1. System analysis. 2. Nonlinear theories.
3. Oscillations. I. Mook, Dean T., joint author.
II. Title.

QA402.N34   531   78-27102
ISBN 0-471-03555-6

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

PREFACE

Recently a large amount of research has been related to nonlinear systems having multidegrees of freedom, but hardly any of this can be found in the many existing books related to this general area. The previously published books emphasized, and some exclusively treated, systems having a single degree of freedom. These include the books of Krylov and Bogoliubov (1947); Minorsky (1947, 1962); Den Hartog (1947); Stoker (1950); McLachlan (1950); Hayashi (1953a, 1964); Timoshenko (1955); Cunningham (1958); Kauderer (1958); Lefschetz (1959); Malkin (1956); Bogoliubov and Mitropolsky (1961); Davis (1962); Struble (1962); Hale (1963); Butenin (1965); Mitropolsky (1965); Friedrichs (1965); Roseau (1966); Andronov, Vitt, and Khaikin (1966); Blaquière (1966); Siljak (1969), and Brauer and Nohel (1969). Exceptions are the books by Evan-Iwanowski (1976) and Hagedorn (1978), which treat multidegree-of-freedom systems. However, a number of recent developments have not been included. The primary purpose of this book is to fill this void.

Because this book is intended for classroom use as well as for a reference to researchers, it is nearly self-contained. Most of the first four chapters, which treat systems having a single degree of freedom, are concerned with introducing basic concepts and analytic methods, although some of the results in Chapter 4 related to multiharmonic excitations cannot be found elsewhere. In the remaining four chapters the concepts and methods are extended to systems having multidegrees of freedom.

This book emphasizes the physical aspects of the systems and consequently serves as a companion to *Perturbation Methods* by A. H. Nayfeh. Here many examples are worked out completely, in many cases the results are graphed, and the explanations are couched in physical terms.

An extensive bibliography is included. We attempted to reference every paper which appeared in an archive journal and related to the material in the book. However omissions are bound to occur, but none is intentional. Many exercises have been included at the end of each chapter except the first. These exercises progress in complexity, and many of them contain intermediate steps to help the reader. In fact, many of them would expand the state of the art if numerical results were computed. Some of these exercises provide further references.

We wish to thank Drs. D. T. Blackstock, M. P. Mortell, and B. R. Seymour for their valuable comments on Chapter 8 and Drs. J. E. Kaiser, Jr., and W. S. Saric for their valuable comments on Chapter 1. A special word of thanks goes to our children Samir (age 7), Tariq (age 10), and Mahir (age 11) Nayfeh and Art Mook (age 16) and to Patty Belcher and Tom Dunyak for their efforts in checking the references. Many of the figures were drawn by Chip Gilbert, Joe Mook, and Fredd Thrasher, and we wish to express our appreciation to them. We wish to thank Janet Bryant for her painstaking typing and retyping of the manuscript. Finally a word of appreciation goes to Indrek Wichman, Jerzy Klimkowski, Helen Reed, Albert Ten, and Ten Liu for proofreading portions of the manuscript.

Ali Hasan Nayfeh
Dean T. Mook

Blacksburg, Virginia
January 1979

CONTENTS

CHAPTER 1

Introduction

1.1. Preliminary Remarks

In this chapter we attempt to abstract the entire book. We introduce some of the nonlinear physical phenomena that are discussed in detail in subsequent chapters. The development of many of the results discussed here requires somewhat elaborate algebraic manipulations. Here we describe only the physical phenomena, leaving all the algebra to the subsequent chapters. The descriptions in this chapter are intended to give an overview of the whole book. Thus one might better see how a given topic fits into the overall picture by rereading portions or all of this chapter as one progresses through the rest of the book.

1.2. Conservative Single-Degree-of-Freedom Systems

In Chapter 2, free oscillations of conservative nonlinear systems are considered. Most of these systems are governed by equations having the general form

$$\ddot{u} + f(u) = 0 \tag{1.1}$$

Upon integrating, we obtain

$$\tfrac{1}{2}\dot{u}^2 = h - F(u) \tag{1.2}$$

where $F(u) = \int f\,du$ and h is a constant of integration. Referring to (1.1) and (1.2), we note that $f(u)$ is the (nonlinear) restoring force, $F(u)$ is the potential energy, $\frac{1}{2}\dot{u}^2$ is the kinetic energy, and h (which is determined by the initial conditions) is the total energy level per unit mass.

In the upper portion of Figure 1-1, the undulating line represents the potential energy, while the straight horizontal lines represent total energy levels. Each total energy level corresponds to a different motion, and the vertical distance between a given horizontal line and the undulating line represents the kinetic energy for that motion. Thus motion is possible only in those regions where the potential energy lies below the total energy level.

In the lower portion of Figure 1-1, the variation of $\dot{u}$ with u is shown. Such a graph is called a *phase plane*. For a given set of initial conditions (i.e., for a given

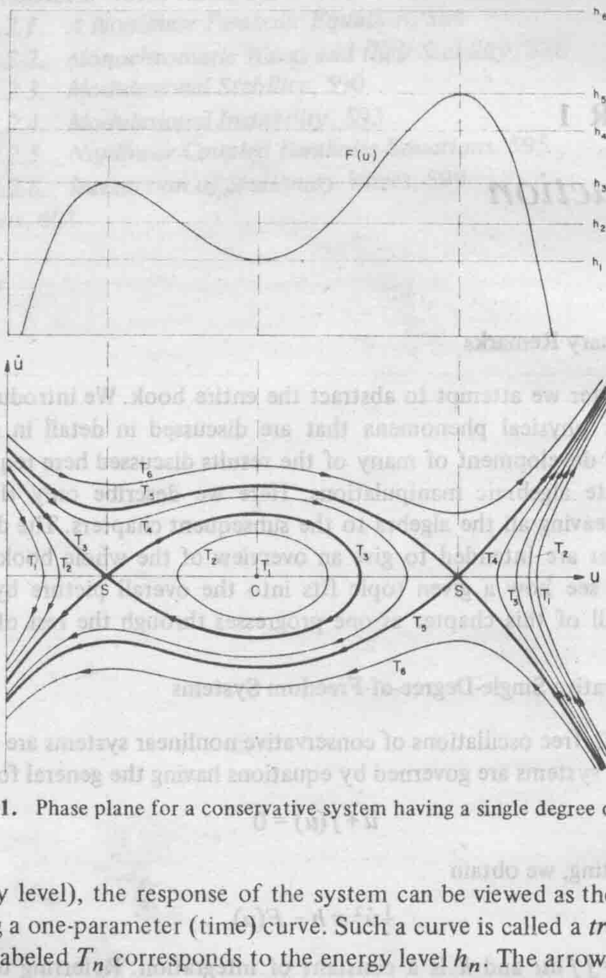


Figure 1-1. Phase plane for a conservative system having a single degree of freedom.

total energy level), the response of the system can be viewed as the motion of a point along a one-parameter (time) curve. Such a curve is called a *trajectory*. The trajectory labeled T_n corresponds to the energy level h_n. The arrows indicate the direction in which the point representing the motion moves as time increases.

The points labeled S are called *saddle points* or *cols*, and the one labeled C is called a *center*. Saddle points and centers correspond to extrema of the potential energy and hence they are *equilibrium points*. Saddle points correspond to maxima while centers correspond to minima of the potential energy. The trajectories that intersect at the saddle points (T_3 and T_5 in Figure 1-1) are called *separatrices*. They are the heavy lines. The point representing the motion moves toward S along two of the separatrices and away from S along the other two. If the representative point is displaced a small distance away from S, there are three possibilities. First, the point can be placed exactly on an inward-bound separatrix, and hence it approaches S as time increases. Second, it can be placed on a closed trajectory, and at times it is far away from S, though it periodically passes close to S. (Here we assume that the equilibrium points are isolated.) Third, it can be placed on an open trajectory, and hence it approaches infinity as time increases. Because the representative point does not stay close to S for all small displacements, the motion is said to be unstable in the neighborhood of a saddle point (i.e., an equilibrium point corresponding to a maximum of the potential energy is unstable).

In the neighborhood of the center, the trajectories are closed, and hence the response is periodic (though not necessarily harmonic). Thus if the motion is displaced slightly from a center, the representative point will always move on a closed trajectory which surrounds the center and stay close to it. (Again we assume that the equilibrium points are isolated.) Thus the motion is said to be stable in the neighborhood of a center (i.e., an equilibrium point corresponding to a minimum of the potential energy is stable). An examination of these closed trajectories shows that the period is a function of the amplitude of the motion. In general, these trajectories do not extend the same distances to the right and the left of the center; thus the midpoint of the motion shifts away from the static center as the amplitude increases. This shift is often called *drift* or *steady-streaming*.

Several analytical methods are introduced and subsequently used to provide approximate expressions for the response. These methods treat small, but finite, periodic motions in the neighborhood of a center. For various examples, the approximate and exact values of the periods are compared.

1.3. Nonconservative Single-Degree-of-Freedom Systems

In Chapter 3, free oscillations of nonconservative systems are introduced. Examples of positive damping due to dry friction (Coulomb damping), viscous effects, form drag, radiation, and hysteresis are presented; examples of negative damping are also included.

In Figure 1-2, a typical phase plane is shown. This one describes the oscillations of a simple pendulum under the action of viscous damping. Depending on the initial conditions, the pendulum may execute several complete revolutions before the oscillatory motion begins. The trajectories spiral into points that correspond to the straight-down position of the pendulum. These points are called *foci*. The straight-up positions correspond to the saddle points in the phase plane. And as in the case of conservative systems, the trajectories that pass through the saddle points are called separatrices.

The concept of a limit cycle is introduced. As an example, we consider Rayleigh's or van der Pol's equation:

$$\ddot{u} + \omega_0^2 u = \epsilon(\dot{u} - \tfrac{1}{3}\dot{u}^3) \tag{1.3}$$

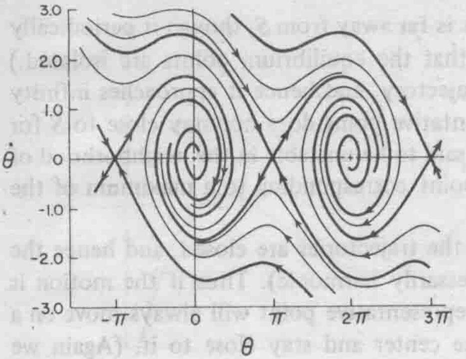


Figure 1-2. Phase plane for a simple pendulum with viscous damping.

We regard the right-hand side of (1.3) as a damping term and note that its influence depends on the amplitude of the motion. When the amplitude of the motion is small, $\frac{1}{3}\dot{u}^3$ is small compared with $\dot{u}$ and the "damping" force has the same sign as the velocity (negative damping); thus the response grows. When the

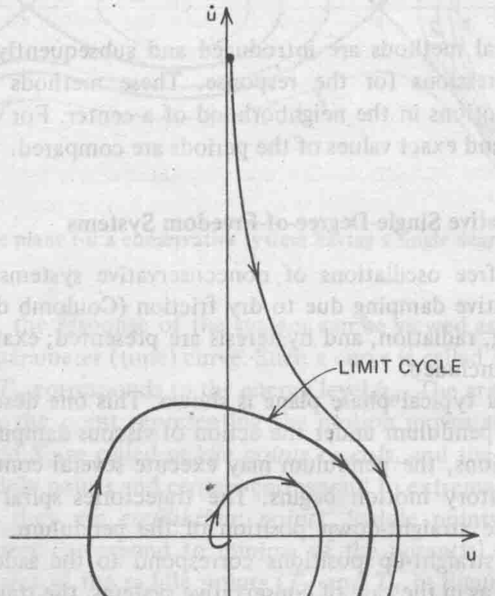


Figure 1-3. Phase plane for van der Pol's equation ($\epsilon = 0.1$).

amplitude is large, $\frac{1}{3}\dot{u}^3$ is large compared with $\dot{u}$ and the damping force has the opposite sign of the velocity (positive damping); thus the motion decays. This behavior of growth when the amplitude is small and decay when the ampitude is large suggests that somewhere in between there exists a motion whose amplitude

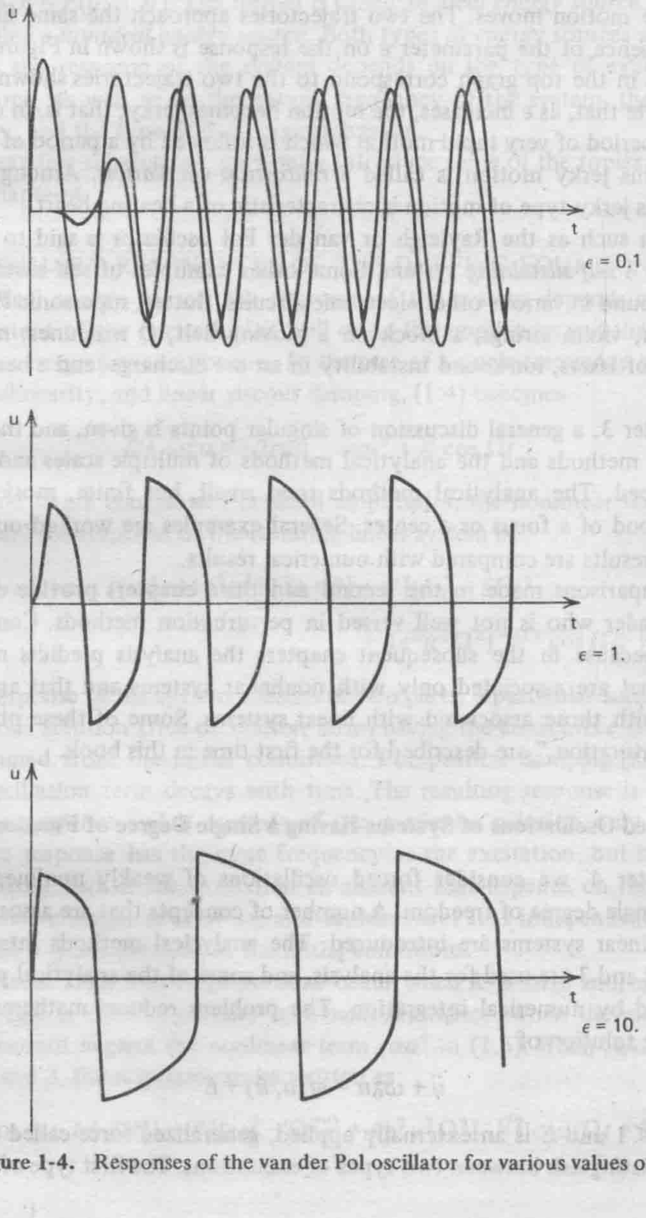


Figure 1-4. Responses of the van der Pol oscillator for various values of ϵ.

neither grows nor decays. This is the case, and the motion is said to approach a *limit cycle*.

In Figure 1-3, a phase plane for the van der Pol equation is shown. There are two trajectories. One begins well outside the limit cycle, while the other begins near the origin. Again the arrows indicate the direction in which the point representing the motion moves. The two trajectories approach the same limit cycle.

The influence of the parameter ϵ on the response is shown in Figure 1-4. The two curves in the top graph correspond to the two trajectories shown in Figure 1-3. We note that, as ϵ increases, the motion becomes jerky; that is, in each cycle there is a period of very rapid motion which is followed by a period of very slow motion. This jerky motion is called a *relaxation oscillation*. Among other examples, this jerky type of motion is characteristic of a beating heart.

A system such as the Rayleigh or van der Pol oscillator is said to be a *self-exciting* or a *self-sustaining* system. Some other examples of self-sustaining systems are found in various other electronic circuits, flutter, supersonic flow past a liquid film, violin strings, a block on a moving belt, Q machines, multimode operation of lasers, ion-sound instability in an arc discharge, and a beam-plasma system.

In Chapter 3, a general discussion of singular points is given, and then various qualitative methods and the analytical methods of multiple scales and averaging are described. The analytical methods treat small, but finite, motions in the neighborhood of a focus or a center. Several examples are worked out, and the analytical results are compared with numerical results.

The comparisons made in the second and third chapters provide confidence for the reader who is not well versed in perturbation methods. Confidence is essential because in the subsequent chapters the analysis predicts many phenomena that are associated only with nonlinear systems and that are in sharp contrast with those associated with linear systems. Some of these phenomena, such as "saturation," are described for the first time in this book.

1.4. Forced Oscillations of Systems Having a Single Degree of Freedom

In Chapter 4, we consider forced oscillations of weakly nonlinear systems having a single degree of freedom. A number of concepts that are associated only with nonlinear systems are introduced. The analytical methods introduced in Chapters 2 and 3 are used for the analysis, and some of the analytical predictions are verified by numerical integration. The problem reduces mathematically to finding the solution of

$$\ddot{u} + \omega_0^2 u = \epsilon f(u, \dot{u}) + E \tag{1.4}$$

where $\epsilon \ll 1$ and E is an externally applied, generalized force called the *excitation*. We distinguish between two types of excitations. The first type of excitation