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P. Harmand D. Werner W. Werner

M-Ideals in Banach Spaces and Banach Algebras

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Preface

The present notes centre around the notion of an M-ideal in a Banach space, introduced by E. M. Alfsen and E. G. Effros in their fundamental article "Structure in real Banach spaces" from 1972. The key idea of their paper was to study a Banach space by means of a collection of distinguished subspaces, namely its M-ideals. (For the definition of an M-ideal see Definition 1.1 of Chapter I.) Their approach was designed to encompass structure theories for C^*-algebras, ordered Banach spaces, L^1-preduals and spaces of affine functions on compact convex sets involving ideals of various sorts. But Alfsen and Effros defined the concepts of their M-structure theory solely in terms of the norm of the Banach space, deliberately neglecting any algebraic or order theoretic structure. Of course, they thus provided both a unified treatment of previous ideal theories by means of purely geometric notions and a wider range of applicability. Around the same time, the idea of an M-ideal appeared in T. Ando's work, although in a different context.

The existence of an M-ideal Y in a Banach space X indicates that the norm of X vaguely resembles a maximum norm (hence the letter M). The fact that Y is an M-ideal in X has a strong impact on both Y and X since there are a number of important properties shared by M-ideals, but not by arbitrary subspaces. This makes M-ideals an important tool in Banach space theory and allied disciplines such as approximation theory. In recent years this impact has been investigated quite closely, and in this book we have aimed at presenting those results of M-structure theory which are of interest in the general theory of Banach spaces, along with numerous examples of M-ideals for which they apply.

Our material is organised into six chapters as follows. Chapter I contains the basic definitions, examples and results. In particular we prove the fundamental theorem of Alfsen and Effros which characterises M-ideals by an intersection property of balls. In Chapter II we deal with some of the stunning properties of M-ideals, for example their proximinality. We also show that under mild restrictions M-ideals have to be complemented subspaces, a theorem due to Ando, Choi and Effros. The last section of Chapter II is devoted to an application of M-ideal methods to the classification of L^1-preduals. In Chapter III we investigate Banach spaces X which are M-ideals in their biduals. This geometric assumption has a number of consequences for the isomorphic structure of X. For instance, a Banach space has Pełczyński's properties (u) and (V) once it is an M-

ideal in its bidual; in particular there is the following dichotomy for those spaces X: a subspace of a quotient of X is either reflexive or else contains a complemented copy of c_0. Chapter IV sets out to study the dual situation of Banach spaces which are L-summands in their biduals. The results of this chapter have some possibly unexpected applications in harmonic analysis which we present in Section IV.4. Banach algebras are the subject matter of Chapter V. Here the connections between the notions of an M-ideal and an algebraic ideal are discussed in detail. The most far-reaching results can be proved for what we call "inner" M-ideals of unital Banach algebras. These can be characterised by having a certain kind of approximation of the identity. Luckily the M-ideals which are not inner seem to be the exception rather than the rule. The final Chapter VI presents descriptions of the M-ideals in various spaces of bounded linear operators. In particular we address the problem of which Banach spaces X have the property that the space of compact operators on X is an M-ideal in the space of bounded linear operators, a problem which has aroused a lot of interest since the appearance of the Alfsen-Effros paper. We give two characterisations of those spaces X, one of them following from our work in Chapter V, the other being due to N. Kalton.
Each chapter is accompanied by a "Notes and Remarks" section where we try to give precise references and due credits for the results presented in the main body of the text. There we also discuss additional material which is related to the topics of the chapter in question, but could not be included with complete proofs because of lack of space.
Only a few prerequisites are indepensable for reading this book. Needless to say, the cornerstones of linear functional analysis such as the Hahn-Banach, Krein-Milman, Krein-Smulian and open mapping theorems are used throughout these notes, often without explicitly mentioning them. We also assume the reader to be familiar with the basics of Banach algebra theory including the Gelfand-Naimark theorem representing a commutative unital C^*-algebra in the form $C(K)$, and with various special topics such as the representation of the extreme functionals on a $C(K)$-space as multiples of Dirac measures or the principle of local reflexivity (an explicit statement of which can be found in Theorem V.1.4). Other concepts that we need but are not so well-known will be recalled as required. For our notation we refer to the list of symbols.

Several people helped us with the manuscript in one way or the other. Thanks are due in particular to G. Godefroy, N. Kalton, Å. Lima, R. Payá, T. S. S. Rao, and A. Rodríguez-Palacios. Special mention must be made of E. Behrends who introduced us to the subject and suggested that the book be written. Last but not least we are very grateful to D. Yost for reading preliminary drafts of the manuscript; his criticism has been most invaluable.
A final piece of advice to those readers who would like to relax from M-ideals, M-summands and M-structure theory: we think it's a good idea to resort to [314].

Oldenburg, Berlin, Paderborn, December 1992

Contents

CHAPTER I

Basic theory of M-ideals

I.1 Fundamental properties

In this book we shall be concerned with decompositions of Banach spaces by means of projections satisfying certain norm conditions. The essential notions are contained in the following definition. We denote the annihilator of a subspace J of a Banach space X by $J^\perp = \{x^* \in X^* \mid x^*(y) = 0 \ \forall y \in J\}$.

Definition 1.1 *Let X be a real or complex Banach space.*

(a) *A linear projection P is called an M-*projection *if*

$$\|x\| = \max\{\|Px\|, \|x - Px\|\} \qquad \textit{for all } x \in X.$$

*A linear projection P is called an L-*projection *if*

$$\|x\| = \|Px\| + \|x - Px\| \qquad \textit{for all } x \in X.$$

(b) *A closed subspace $J \subset X$ is called an M-*summand *if it is the range of an M-projection. A closed subspace $J \subset X$ is called an L-*summand *if it is the range of an L-projection.*

(c) *A closed subspace $J \subset X$ is called an M-*ideal *if $J^\perp$ is an L-summand in X^*.*

Some comments on this definition are in order. First of all, every Banach space X contains the trivial M-summands $\{0\}$ and X. All the other M-summands will be called nontrivial. (Sometimes only trivial M-summands exist as will presently be shown.) The same remark applies to L-summands and M-ideals.

There is an obvious duality between L- and M-projections:

- P is an L-projection on X iff P^* is an M-projection on X^*.
- P is an M-projection on X iff P^* is an L-projection on X^*.

This remark yields the following characterisation of M-projections which will be useful in the sequel:

A projection $P \in L(X)$ is an M-projection if and only if

$$\|Px_1 + (Id - P)x_2\| \leq \max\{\|x_1\|, \|x_2\|\} \quad \text{for all } x_1, x_2 \in X. \tag{$*$}$$

In fact, $(*)$ means that the operator

$$(x_1, x_2) \mapsto Px_1 + (Id - P)x_2$$

from $X \oplus_\infty X$ to X is contractive whence its adjoint

$$x^* \mapsto (P^*x^*, (Id - P)^*x^*)$$

from X^* to $X^* \oplus_1 X^*$ is contractive. ($X \oplus_p Y$ denotes the direct sum of two Banach spaces, equipped with the ℓ^p-norm.) This means that P^* is an L-projection, and P must be an M-projection.

Turning to (b) let us note that there is only one M-projection P with $J = \operatorname{ran}(P)$ $(= \ker(Id - P))$ if J is an M-summand and only one L-projection P with $J = \operatorname{ran}(P)$ $(= \ker(Id - P))$ if J is an L-summand (cf. Proposition 1.2 below which contains a stronger statement). Consequently, there is a uniquely determined closed subspace $\widehat{J}$ such that

$$X = J \oplus_\infty \widehat{J}$$

resp.

$$X = J \oplus_1 \widehat{J}.$$

Then $\widehat{J}$ is called the complementary M- (resp. L-)summand. The duality of L- and M-projections may now be expressed as

- $X = J \oplus_\infty \widehat{J}$ iff $X^* = J^\perp \oplus_1 \widehat{J}^\perp$,
- $X = J \oplus_1 \widehat{J}$ iff $X^* = J^\perp \oplus_\infty \widehat{J}^\perp$.

It follows that M-summands are M-ideals and that the M-ideal J is an M-summand if and only if the L-summand complementary to $J^\perp$ is weak* closed. Let us note that the fact that J and $\widehat{J}$ are complementary L-summands in X means geometrically that B_X, the closed unit ball of X, is the convex hull of B_J and $B_{\widehat{J}}$.

As regards (c) the reader might wonder why we didn't introduce the notion of an "L-ideal", meaning a subspace whose annihilator is an M-summand. The reason is that such an "L-ideal" is automatically an L-summand (see Theorem 1.9 below). Note, however, that the expression "L-ideal" has occasionally appeared in the literature as a synonym of L-summand, e.g. in [11].

Proposition 1.2

(a) *If P is an M-projection on X and Q is a contractive projection on X satisfying* $\operatorname{ran}(P) = \operatorname{ran}(Q)$, *then* $P = Q$.

(b) *If P is an L-projection on X and Q is a contractive projection on X satisfying* $\ker(P) = \ker(Q)$, *then* $P = Q$.

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PROOF: We first prove (b). The decisive lever for our argument is that, for an L-summand J in X, there is for a given $x \in X$ one and only one best approximant y_0 in J, that is

$$\|x - y_0\| = \inf_{y \in J} \|x - y\|,$$

namely the image of x under the L-projection onto J. (In the language of approximation theory, L-summands are Chebyshev subspaces.) We apply this remark with $J = \ker(P)$. For $x \in X$ we have $x - Px \in \ker(P) = \ker(Q)$, hence

$$\begin{aligned} \|x - (x - Qx)\| &= \|Qx\| \\ &= \|Q(x - (x - Px))\| \\ &\leq \|Q\| \cdot \|Px\| \\ &\leq \|x - (x - Px)\|. \end{aligned}$$

This means that $x - Qx \in \ker(Q) = \ker(P)$ is at least as good an approximant to x in J as $x - Px$ which is the best one. From the uniqueness of the best approximant one deduces $Qx = Px$, thus $P = Q$, as claimed.

(a) follows from (b) since $\ker(P^*) = \operatorname{ran}(P)^\perp = \ker(Q^*)$. □

Corollary 1.3 *If an M-ideal is the range of a contractive projection Q, then it is an M-summand.*

PROOF: $\ker(Q^*) = J^\perp$. Thus, the L-projection with kernel $J^\perp$ is Q^* and hence weak* continuous. □

We now discuss some examples of M-ideals and M-summands.

Example 1.4(a) Let S be a locally compact Hausdorff space. Then $J \subset C_0(S)$ is an M-ideal if and only if there is a closed subset D of S such that

$$J = J_D := \{x \in C_0(S) \mid x(s) = 0 \quad \text{for all } s \in D\}.$$

It is an M-summand if and only if D is clopen (= closed and open).

PROOF: Obviously, $\mu \mapsto \chi_D \mu$ is the L-projection from $C_0(S)^* = M(S)$ onto $J_D^\perp$ so that J_D is an M-ideal if D is closed, and J_D is an M-summand if D is clopen. Suppose now that $J \subset C_0(S)$ is an M-ideal. In order to find the set D, we make use of the following elementary lemma which we state for future reference. The set of extreme points of a convex set C is denoted by $\operatorname{ex} C$.

Lemma 1.5 *For $Z = J_1 \oplus_1 J_2$ we have (using the convention $\operatorname{ex} B_{\{0\}} = \emptyset$)*

$$\operatorname{ex} B_Z = \operatorname{ex} B_{J_1} \cup \operatorname{ex} B_{J_2}.$$

We shall apply Lemma 1.5 here with $Z = M(S)$, $J_1 = J^\perp$ and $J_2 =$ the complementary L-summand. Let

$$D = \{s \in S \mid \delta_s \in J^\perp\}.$$

Then D is closed, and by construction $J \subset J_D$. The inclusion $J_D \subset J$ follows from the Hahn-Banach and Krein-Milman theorems: By virtue of these results we only have to show $\operatorname{ex} B_{J^\perp} \subset J_D^\perp$, and this is true by Lemma 1.5 and by definition of D.
Finally, if $J = J_D$ is an M-summand, then the complementary M-summand is of the same form, $J_{\widehat{D}}$ say, consequently $D \cup \widehat{D} = S$ and D is clopen. □

In particular, c_0 is an M-ideal in $\ell^\infty = C(\beta\mathbb{N})$ (which could also be proved directly); the additional feature is that ℓ^∞ is the bidual of c_0. We shall study Banach spaces which are M-ideals in their biduals in detail in Chapter III.

Example 1.4(b) Let A be the disk algebra, that is the complex Banach space of continuous functions on the closed unit disk which are analytic in the open unit disk. It will be convenient to consider A (via boundary values) as a subspace of $C(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. We claim that J is a nontrivial M-ideal in A if and only if there is a closed subset $D \neq \emptyset$ of $\mathbb{T}$ with linear Lebesgue measure 0 such that[1]

$$J = J_D \cap A = \{x \in A \mid x(t) = 0 \quad \text{for all } t \in D\}.$$

PROOF: Note first that $J_D \cap A = \{0\}$ if D is a subset of $\mathbb{T}$ having positive linear measure, cf. e.g. [545, Theorem 17.18]. To see that an M-ideal J has the form $J_D \cap A$ one proceeds exactly as in Example 1.4(a); one only has to recall that

$$\operatorname{ex} B_{A^*} = \{\lambda \cdot \delta_t|_A \mid |\lambda| = 1,\ t \in \mathbb{T}\}. \tag{$*$}$$

[This amounts to saying that every $t \in \mathbb{T}$ is in the Choquet boundary of A, and a proof of this fact is contained in [240, p. 54ff.]; for an explicit statement see e.g. [587, p. 29]. Since this example will have some importance in the sequel, we would like to sketch a direct argument: The right hand side of $(*)$ is weak* closed and norming by the maximum modulus principle, therefore the Krein-Milman theorem (or rather its converse) implies "$\subset$". On the other hand, $\operatorname{ex} B_{A^*} \neq \emptyset$. So let us assume $p_0 := \lambda_0 \cdot \delta_{t_0}|_A \in \operatorname{ex} B_{A^*}$. (Here we use the fact that for $p \in \operatorname{ex} B_{X^*}$ and $X \subset Y$, p has an extension to some $q \in \operatorname{ex} B_{Y^*}$.) For $|\lambda| = 1,\ t \in \mathbb{T}$

$$(\Phi x)(s) := \frac{\lambda}{\lambda_0} \cdot x\Big(\frac{t}{t_0} s\Big)$$

defines an isometric isomorphism on A, hence $\lambda \cdot \delta_t|_A = \Phi^*(p_0) \in \operatorname{ex} B_{A^*}$.]
Now let $D \neq \emptyset$ be a closed subset of $\mathbb{T}$ of linear measure 0. We wish to find an L-projection from A^* onto $(J_D \cap A)^\perp$. A functional $p \in A^*$ may be represented as $p = \mu|_A$ for some measure $\mu \in M(\mathbb{T}) = C(\mathbb{T})^*$. Let $q = (\chi_D \mu)|_A$. Then the mapping $P : p \mapsto q$ is well-defined: If $p = \nu|_A$ is another representation, then $\nu - \mu$ annihilates A. The F. and M. Riesz theorem (cf. e.g. [545, Theorem 17.13]) implies that $\nu - \mu$ is absolutely continuous with respect to Lebesgue measure so that $\chi_D \mu = \chi_D \nu$ if D has measure 0. It

[1] Throughout, J_D will have the same meaning as in Example 1.4(a).

is easy to check that P is the required L-projection. Finally, the nontriviality follows from a theorem of Fatou [317, p. 80] which also shows that different D give rise to different M-ideals. □

We shall present a characterisation of M-ideals in a general function algebra in Theorem V.4.2. As regards this example, see also the abstract version of our approach in Corollary 1.19.

Example 1.4(c) Let K be a compact convex set in a Hausdorff locally convex topological vector space. As usual, $A(K)$ denotes the space of real-valued affine continuous functions on K. Let us recall the definition of a split face of K ([7, p. 133], [9]). A face F of K is called a split face if there is another face F' such that every $k \in K \backslash (F \cup F')$ has a *unique* representation

$$k = \lambda k_1 + (1 - \lambda) k_2 \quad \text{with } k_1 \in F,\ k_2 \in F',\ 0 < \lambda < 1.$$

It is known that every closed face of a simplex is a split face [7, p. 144].
Then J is an M-ideal in $A(K)$ if and only if there exists a closed split face F of K such that

$$J = J_F \cap A(K) = \{x \in A(K) \mid x(k) = 0 \quad \text{for all } k \in F\}.$$

The proof of this fact can be given along the lines of (b), the crucial step being the measure theoretic characterisation of closed split faces (see [7, Th. II.6.12]) which replaces the use of the F. and M. Riesz theorem.

Example 1.4(c) was one of the forerunners of the general M-ideal theory (cf. [9]). Another forerunner of the general theory is contained in the next example.

Example 1.4(d) In a C^*-algebra the M-ideals coincide with the closed two-sided ideals.

We shall present a proof of this fact in Theorem V.4.4. For the time being let us notice that in particular $K(H)$ is an M-ideal in $L(H)$ (where H denotes a Hilbert space) which was first shown by Dixmier [165] back in 1950. An independent proof of Dixmier's result will be given in Chapter VI. There we shall study the class of Banach spaces X for which $K(X)$ is an M-ideal in $L(X)$. Let us indicate at this point that this class contains all the spaces ℓ^p for $1 < p < \infty$ and c_0 as well as certain of their subspaces and quotient spaces (cf. Example VI.4.1, Corollary VI.4.20). Moreover, $K(X)$ is never an M-summand in $L(X)$ unless $\dim X < \infty$ (Proposition VI.4.3).

Example 1.4(e) Although there are certain similarities between algebraic ideals and M-ideals, there is one striking difference: Unlike the case of algebraic ideals the intersection of (infinitely many) M-ideals need not be an M-ideal. An example to this effect will be given in II.5.5. Also let us advertise now the description of M-ideals in G-spaces (Proposition II.5.2) and Theorem II.5.4 which are related to this phenomenon.

Here are some examples of L-summands.

Example 1.6(a) Consider $X = L^1(\mu)$. We assume that μ is localizable (e.g. σ-finite) so that $L^1(\mu)^* \cong L^\infty(\mu)$ in a natural fashion. Then the L-projections on $L^1(\mu)$ coincide with the characteristic projections $P_A(f) = \chi_A f$ for measurable sets A.

PROOF: Trivially, P_A is an L-projection. Conversely, for a given L-projection P the adjoint P^* is an M-projection on $L^\infty(\mu)$. Now $L^\infty(\mu)$ is a commutative unital C^*-algebra, and as such it may be represented as $C(K)$ by the Gelfand-Naimark theorem. (This representation is also possible in the real case.) Since the Gelfand transform is multiplicative, the idempotent elements in $L^\infty(\mu)$ (i.e., the measurable characteristic functions) correspond exactly to the idempotent elements in $C(K)$ (i.e., the continuous characteristic functions). Now Example 1.4(a) tells us that M-projections are characteristic projections. Thus the same is true for $L^\infty(\mu)$ so that P^*, and hence P, is a characteristic projection. □

The description of the L-projections on $L^1(\mu)$ in terms of the measure space is a bit more involved if μ is arbitrary; we refer to [66, p. 58]. However, every space $L^1(\mu)$ is order isometric to a space $L^1(m)$ where m is (even strictly) localizable [560, p. 114]. Therefore, our initial restriction concerning μ is not a severe one, since we are dealing with Banach space properties of L^1 rather than properties of the underlying measure space.
It is worthwhile adopting the point of view of Banach lattice theory in this example. Using the notions of Banach lattice theory we have established the fact that the L-projections on an L^1-space (or an (AL)-space if one prefers) coincide with the band projections and that the L-summands coincide with the (projection) bands (which are the same as the order ideals in L^1); cf. [560, p. 113 and passim] for these matters. The advantage of this description is that it avoids explicit reference to the underlying measure space.
Incidentally, we have shown that the M-projections on $L^1(\mu)^*$ are weak* continuous. This is true for every dual Banach space, see Theorem 1.9 below.

Example 1.6(b) The Lebesgue decomposition $\mu = \mu_{\text{ac}} + \mu_{\text{sing}}$ with respect to a given probability measure furnishes another example of an L-decomposition. Note that neither of the L-summands in

$$C[0,1]^* = M[0,1] = L^1[0,1] \oplus_1 M_{\text{sing}}[0,1]$$

is weak* closed. (In fact, both of them are weak* dense.)

Example 1.6(c) In Chapter IV we shall study Banach spaces which are L-summands in their biduals. Prominent examples will be the L^1-spaces as well as their "noncommutative" counterparts, i.e., the preduals of von Neumann algebras.

The preceding examples have shown a variety of situations where M-ideals or L-summands arise in a natural way. Sometimes, however, it will also be of interest to have a criterion at hand in order to show that a given Banach space does not contain any nontrivial M-ideal or L-summand.

Proposition 1.7 *If X is smooth or strictly convex, then X contains no nontrivial M-ideal and no nontrivial L-summand.*

PROOF: Since points of norm one which are contained in a nontrivial L-summand never have a unique supporting hyperplane, we conclude that a smooth space cannot contain a nontrivial L-summand. On the other hand, if X is smooth and $x^* \in S_{X^*}$ attains its norm, then x^* is an extreme functional. Now the Bishop-Phelps theorem (see Theorem VI.1.9) yields that ex B_{X^*} is norm dense in S_{X^*}. By Lemma 1.5, X cannot contain a nontrivial M-ideal either. (Actually, we have shown the (by virtue of Proposition V.4.6) stronger statement that all the L-summands in the dual of a smooth space are trivial. However, we shall eventually encounter the dual of a smooth space which contains nontrivial M-ideals, namely L^∞/H^∞ (Remark IV.1.17).)
As for the case of strictly convex spaces, all the L-summands are trivial by Lemma 1.5, and the triviality of the M-ideals is a consequence of Corollary II.1.5 below. □

Proposition 1.7 applies in particular to the L^p-spaces for $1 < p < \infty$.
The following theorem describes a dichotomy concerning the existence of L-summands and M-ideals.

Theorem 1.8 *A complex Banach space or a real Banach space which is not isometric to $\ell^\infty(2)$ $(= (\mathbb{R}^2, \|\,.\,\|_\infty))$ cannot contain nontrivial M-ideals and nontrivial L-summands simultaneously.*

PROOF: Suppose that the real Banach space X admits nontrivial decompositions

$$\begin{aligned} X &= J \oplus_\infty \widehat{J} \\ &= Y \oplus_1 \widehat{Y}. \end{aligned}$$

Our aim is to show that under this assumption Y must be one-dimensional. Once this is achieved we conclude by symmetry that $\widehat{Y}$ must be one-dimensional as well so that X is isometric to $\ell^1(2)$ $(\cong \ell^\infty(2))$. If X is a complex Banach space, then the above reasoning applies to the underlying real space and yields that X is $\mathbb{R}$-isometric to $\ell^\infty_{\mathbb{R}}(2)$ which of course is impossible.
Let us now present the details of the proof that $\dim(Y) = 1$. We denote by P the M-projection onto J and by π the L-projection onto Y. We first claim:

$$J \cap Y = \{0\} \tag{1}$$

Assume to the contrary that there exists some $u \in J \cap Y$ with $\|u\| = 1$. Let $x \in \widehat{J}$, $\|x\| = 1$. We shall show $x \in Y$: Since $\|u \pm x\| = 1$ we have

$$\begin{aligned} 2 &= \|u + x\| + \|u - x\| \\ &= \|u + \pi(x)\| + \|x - \pi(x)\| + \|u - \pi(x)\| + \|\pi(x) - x\| \\ &\geq 2 \cdot \|u\| + 2 \cdot \|x - \pi(x)\| \end{aligned}$$

so that

$$x = \pi(x) \in Y.$$

This shows $\widehat{J} \subset Y$ and in particular $\widehat{J} \cap Y \neq \{0\}$. Thus we may repeat the same argument with the duo $\widehat{J}$ & Y to obtain $J \subset Y$ as well. Consequently $Y = X$ in contrast to our assumption that the L-decomposition is nontrivial. Therefore (1) holds.

To show that Y is one-dimensional we again argue by contradiction. Suppose there exists a two-dimensional subspace Y_0 of Y. By (1), $P_{|Y_0}$ is injective so that $J_0 := P(Y_0)$ is two-dimensional, too. By Mazur's theorem (e.g. [318, p. 171]) (or since every convex function on $\mathbb{R}$ has a point of differentiability) J_0 contains a smooth point z, i.e., $\|z\| = 1$ and

$$\ell(x) := \lim_{h\to 0} \frac{1}{h}\Big(\|z + hx\| - 1\Big)$$

exists for all $x \in J_0$. Since J_0 is supposed to be two-dimensional, we can find some $x \in J_0$, $\|x\| = 1$, such that

$$\lim_{h\to 0} \frac{1}{h}\Big(\|z + hx\| - 1\Big) = 0. \tag{2}$$

Let us write $z = \dfrac{Py}{\|Py\|}$ for some $y \in Y_0$, $\|y\| = 1$. We next claim:

$$\lim_{h\to 0} \frac{1}{h}\Big(\|y + hx\| - 1\Big) = 0 \tag{3}$$

To prove this we note

$$\|y + hx\| = \max\{\|Py + hx\|, \|y - Py\|\}$$

since P is an M-projection and $x \in J$. Also note

$$1 = \|y\| = \max\{\|Py\|, \|y - Py\|\}.$$

Thus, if $\|Py\| < 1$, we have for sufficiently small h

$$\|y + hx\| = \|y - Py\| = 1,$$

and (3) follows immediately. If $\|Py\| = 1$ (hence $z = Py$) and $\|y - Py\| < 1$, we have for sufficiently small h

$$\|y + hx\| = \|Py + hx\| = \|z + hx\|,$$

and (3) follows from (2). It is left to consider the case where $1 = \|Py\| = \|y - Py\|$. In this case we have

$$\|y + hx\| = \max\{\|z + hx\|, 1\},$$

and again (3) follows from (2).
We can now conclude the proof as follows. For $h > 0$ we have

$$\|y + hx\| = \|y + h\pi(x)\| + h\|x - \pi(x)\|,$$

$$\|y - hx\| = \|y - h\pi(x)\| + h\|x - \pi(x)\|.$$

Observing

$$\|y + h\pi(x)\| + \|y - h\pi(x)\| \geq 2 \cdot \|y\| = 2$$

we obtain from the above equations

$$\frac{1}{h}\Big(\|y + hx\| - 1\Big) + \frac{1}{h}\Big(\|y - hx\| - 1\Big) \geq 2 \cdot \|x - \pi(x)\|,$$

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