Watson Fulks

ADVANCED CALCULUS

THIRD EDITION

an introduction to analysis

Advanced Calculus

AN INTRODUCTION TO ANALYSIS

Third Edition

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to the Student

In using this book, you should give first attention to the definitions, for they describe the terminology of mathematics; and you can no more hope to understand mathematics without learning the vocabulary than you can hope to understand any other foreign language without learning its vocabulary. As with any other language, mathematics grows partly by acquiring terms from neighboring languages (in this case primarily English), modifying their meanings, and appropriating them into itself. These form the technical terms of the subject. Since mathematics, in this country, is imbedded in the English language, it is important to distinguish between technical and nontechnical usage.

A word or phrase being defined is set in boldface type, as for example **uniform convergence**. Theorems which have names, such as the **fundamental theorem of calculus**, will have these names set in boldface. A triple-barred equal sign ($\equiv$) is used in two ways. It will mean identity, as for instance

$$x^2 - y^2 \equiv (x-y)(x+y).$$

It will also mean "defines" or "is defined by." For example,

$$f(x) \equiv 2x^2 + 3 \qquad \{-1 \leqslant x \leqslant 7\}$$

means that the function f is being defined in the designated interval. There is, I believe, little cause for confusion arising from the ambiguous uses here indicated.

In the body of the text, occasional reference is made to exercises by number. For instance, parenthetical remarks such as (see Exercise B7) frequently occur. Unless a section number is indicated, the exercise in question will be found in the first set of exercises *following* the reference.

A few more words about study: in trying to understand a proof, you should look for the driving force behind it. You should try to analyze the proof and decide that a certain idea (or ideas) involved is critical, that the proof turns on this, and that the remaining parts are secondary calculations whose presence is clearly understood once the significant point is seen.

It is desirable to understand the role each proposition plays in the construction of a chapter. Which ones, you should ask yourself, are in the nature of preliminary or secondary results, which are of prime importance, and which are anticlimactic? The names—lemma, theorem, corollary—provide a rough rule of thumb, but by no means an accurate one.

Finally, after all this talk, let it be said that no amount of directions on how to study can begin to replace a real and growing interest in the material.

WATSON FULKS

to the Teacher

This book is designed to serve as an introduction to analysis. To this end I have made an effort to present analytical proofs backed by geometrical intuition, and to place a minimum of reliance on geometrical arguments. In fact, some stress is laid on this in the body of the text. I have not succeeded completely in avoiding some essential use of intuition in the proofs, but I have localized my serious transgressions to Chapter 12 wherein are located Green's, Gauss', and Stokes' theorems and some of their consequences.

Like most who stray from the narrow path of virtue, I seek to rationalize my action. My defense is simply that to avoid such geometrical arguments would involve more difficult work than I feel is advisable. In attempting to hold to a course between the heuristics of elementary calculus on the one hand, and the rigors of function theory and topology on the other, I have chosen to err in the direction of heuristics in Chapter 12. The motivation for this is, of course, that what I sacrifice in logic I hope to gain in pedagogy.

So much for apologies.

In this new edition, Part 1 of the text has been revised in the following ways. Continuity and differentiation have been separated, and all of the material on differentiation has been collected into a single chapter preceding integration. The chapter on integration has been expanded to include a section on discontinuous functions.

The middle part is concerned with calculus of several variables. The major changes from the second edition are in this part. The discussion of differentiation of a vector function of a vector variable has been modernized by essentially defining the derivative to be the Jacobian matrix. The general form of the chain rule is now given, as is the general form of the implicit transformation theorem.

In Part 3, on the theory of convergence (of sequences, series, and improper integrals) the changes are minor, being largely a matter of smoothing the exposition where it seemed appropriate.

The problems in each set are classified as A, B, or C Exercises. This classification is to be taken as an approximate grading of the difficulty of the problems. The A Exercises are straightforward applications of the theory, though the computations in some of them may be a little long. The B Exercises are somewhat deeper, and the C Exercises in general are intended to challenge the better students. The A Exercises have been largely reworked, and many new B and C Exercises have been added.

I gratefully acknowledge the influence of a number of people on the changes incorporated in this new edition. The comments of the reviewers were particularly helpful and I take pleasure in recognizing them. In alphabetical order they are Professors N.J. DeLillo, Manhattan College; R.B. Guenther, Oregon State University; K.A. Heimes, Iowa State University; and M. Vuilleumier, Ohio State University.

I present no bibliography, though I must of course acknowledge the influence of many books, old and new, upon the shape this one has assumed.

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Contents

PART 1

CALCULUS OF ONE VARIABLE

CHAPTER 1

The Number System

1.1 The Peano Axioms

Any study of mathematical analysis has its basis in the number system. It is therefore important for students of analysis to understand how the arithmetic can be developed from the natural numbers (another name for the positive integers). It is not our intention to carry out such a construction here; however, we do want to make some comments about the logical structure of that development.

A standard starting point in the development of the real numbers is a certain set of axioms that were first formulated by the Italian mathematician Peano. These axioms state that the natural numbers satisfy certain properties. From only these properties, making appropriate definitions as we proceed, we can develop all the usual rules of arithmetic. In the terminology of formal logic, we have a set of undefined objects that we choose to name the **natural numbers**, satisfying Peano's axioms. This means merely that the natural numbers are taken as the basic "atoms" of our mathematical system in terms of which we express the other mathematical concepts, but which are themselves not expressed in more fundamental terms.

Perhaps an easier way to visualize the situation, since after all you cannot claim that you never heard of arithmetic or rational numbers, is this: We can verify that the positive integers are a system of objects that do satisfy the Peano axioms. We can now proceed to recapture all we know of arithmetic, by developing appropriate definitions and theorems from these axioms, using only those properties stated explicitly in the axioms.

The five Peano axioms follow.

Axiom 1. *1 is a natural number.*

Axiom 2. *To every natural number n there is associated in a unique way another natural number n' called the* **successor** *of n.*

Axiom 3. *The number 1 is not a successor of any natural number.*

Axiom 4. *If two natural numbers have equal successors, they are themselves equal. That is, if $n' = m'$, then $n = m$.*

Axiom 5. *Suppose that M is a collection or set of natural numbers with the properties*

(i) *1 is in M,*
(ii) *n' is in M whenever n is in M.*

Then the collection M consists of all natural numbers.

Here equality is used in the sense of numerical identity; that is, $m = n$ means that m and n are symbols standing for the same number. Thus we take

(i) $m = m$,
(ii) $m = n$ implies that $n = m$,
(iii) $m = n$, $n = k$ implies that $m = k$,

as part of the underlying logic and do not list these as part of our set of numerical axioms.