

Natanson, Fulk

ADVANCED CALCULUS

THIRD EDITION

an introduction
to analysis

Advanced Calculus

AN INTRODUCTION TO ANALYSIS

Third Edition

WATSON FULKS

University of Colorado

JOHN WILEY & SONS

New York · Santa Barbara · Chichester · Brisbane · Toronto

Copyright © 1961, 1969, 1978, by John Wiley & Sons, Inc.

All rights reserved. Published simultaneously in Canada.

Reproduction or translation of any part of this work beyond that permitted by Sections 107 and 108 of the 1976 United States Copyright Act without the permission of the copyright owner is unlawful. Requests for permission or further information should be addressed to the Permissions Department, John Wiley & Sons.

Library of Congress Cataloging in Publication Data:

Fulks, Watson

Advanced calculus

Includes index.

1. Calculus 2. Mathematics analysis. I. Title.
QA303.F954 1978 515 78-5268
ISBN 0-471-02195-4

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

to the Student

In using this book, you should give first attention to the definitions, for they describe the terminology of mathematics; and you can no more hope to understand mathematics without learning the vocabulary than you can hope to understand any other foreign language without learning its vocabulary. As with any other language, mathematics grows partly by acquiring terms from neighboring languages (in this case primarily English), modifying their meanings, and appropriating them into itself. These form the technical terms of the subject. Since mathematics, in this country, is imbedded in the English language, it is important to distinguish between technical and nontechnical usage.

A word or phrase being defined is set in boldface type, as for example **uniform convergence**. Theorems which have names, such as the **fundamental theorem of calculus**, will have these names set in boldface. A triple-barred equal sign (\equiv) is used in two ways. It will mean identity, as for instance

$$x^2 - y^2 \equiv (x - y)(x + y).$$

It will also mean “defines” or “is defined by.” For example,

$$f(x) \equiv 2x^2 + 3 \quad \{-1 \leq x \leq 7\}$$

means that the function f is being defined in the designated interval. There

is, I believe, little cause for confusion arising from the ambiguous uses here indicated.

In the body of the text, occasional reference is made to exercises by number. For instance, parenthetical remarks such as (see Exercise B7) frequently occur. Unless a section number is indicated, the exercise in question will be found in the first set of exercises *following* the reference.

A few more words about study: in trying to understand a proof, you should look for the driving force behind it. You should try to analyze the proof and decide that a certain idea (or ideas) involved is critical, that the proof turns on this, and that the remaining parts are secondary calculations whose presence is clearly understood once the significant point is seen.

It is desirable to understand the role each proposition plays in the construction of a chapter. Which ones, you should ask yourself, are in the nature of preliminary or secondary results, which are of prime importance, and which are anticlimactic? The names—lemma, theorem, corollary—provide a rough rule of thumb, but by no means an accurate one.

Finally, after all this talk, let it be said that no amount of directions on how to study can begin to replace a real and growing interest in the material.

WATSON FULKS

to the Teacher

This book is designed to serve as an introduction to analysis. To this end I have made an effort to present analytical proofs backed by geometrical intuition, and to place a minimum of reliance on geometrical arguments. In fact, some stress is laid on this in the body of the text. I have not succeeded completely in avoiding some essential use of intuition in the proofs, but I have localized my serious transgressions to Chapter 12 wherein are located Green's, Gauss', and Stokes' theorems and some of their consequences.

Like most who stray from the narrow path of virtue, I seek to rationalize my action. My defense is simply that to avoid such geometrical arguments would involve more difficult work than I feel is advisable. In attempting to hold to a course between the heuristics of elementary calculus on the one hand, and the rigors of function theory and topology on the other, I have chosen to err in the direction of heuristics in Chapter 12. The motivation for this is, of course, that what I sacrifice in logic I hope to gain in pedagogy.

So much for apologies.

In this new edition, Part 1 of the text has been revised in the following ways. Continuity and differentiation have been separated, and all of the material on differentiation has been collected into a single chapter preceding integration. The chapter on integration has been expanded to include a section on discontinuous functions.

The middle part is concerned with calculus of several variables. The major changes from the second edition are in this part. The discussion of differentiation of a vector function of a vector variable has been modernized by essentially defining the derivative to be the Jacobian matrix. The general form of the chain rule is now given, as is the general form of the implicit transformation theorem.

In Part 3, on the theory of convergence (of sequences, series, and improper integrals) the changes are minor, being largely a matter of smoothing the exposition where it seemed appropriate.

The problems in each set are classified as A, B, or C Exercises. This classification is to be taken as an approximate grading of the difficulty of the problems. The A Exercises are straightforward applications of the theory, though the computations in some of them may be a little long. The B Exercises are somewhat deeper, and the C Exercises in general are intended to challenge the better students. The A Exercises have been largely reworked, and many new B and C Exercises have been added.

I gratefully acknowledge the influence of a number of people on the changes incorporated in this new edition. The comments of the reviewers were particularly helpful and I take pleasure in recognizing them. In alphabetical order they are Professors N.J. DeLillo, Manhattan College; R.B. Guenther, Oregon State University; K.A. Heimes, Iowa State University; and M. Vuilleumier, Ohio State University.

I present no bibliography, though I must of course acknowledge the influence of many books, old and new, upon the shape this one has assumed.

WATSON FULKS

Contents

PART I	CALCULUS OF ONE VARIABLE	1
Chapter 1	The Number System	3
1.1	The Peano Axioms	3
1.2	Rational Numbers and Arithmetic	6
1.3	The Real Numbers: Completeness	9
1.4	Geometry and the Number System	13
1.5	Bounded Sets	15
1.6	Some Points of Logic	18
1.7	Absolute Value	19
Chapter 2	Functions, Sequences, and Limits	25
2.1	Mappings, Functions, and Sequences	25
2.2	Limits	30
2.3	Operations with Limits (Sequences)	37
2.4	Limits of Functions	44
2.5	Operations with Limits (Functions)	49
2.6	Monotone Sequences	53
2.7	Monotone Functions	56
		ix

Chapter 3	Continuity and More Limits	61
3.1	Continuity. Uniform Continuity	61
3.2	Operations with Continuous Functions	65
3.3	The Intermediate-Value Property	67
3.4	Inverse Functions	68
3.5	Cluster Points. Accumulation Points	74
3.6	The Cauchy Criterion	79
3.7	Limit Superior and Limit Inferior	84
3.8	Deeper Properties of Continuous Functions	89
Chapter 4	Differentiation	95
4.1	The Derivative. Chain Rule	95
4.2	The Mean-Value Theorem	101
4.3	The Cauchy Mean-Value Theorem	111
4.4	L'Hospital's Rule	113
4.5	Taylor's Formula with Remainder	120
4.6	Extreme Values	126
Chapter 5	Integration	131
5.1	Introduction	131
5.2	Preliminary Lemmas	133
5.3	The Riemann Integral	140
5.4	Properties of the Definite Integral	151
5.5	The Fundamental Theorem of Calculus	156
5.6	Further Properties of Integrals	160
5.7	Integrals of Discontinuous Functions	166
Chapter 6	The Elementary Transcendental Functions	175
6.1	The Logarithm	175
6.2	The Exponential Function	179
6.3	The Circular Functions	183

PART II VECTOR CALCULUS **197**

Chapter 7 Vectors and Curves **199**

7.1	Introduction and Definitions	199
7.2	Vector Multiplications	206
7.3	The Triple Products	214
7.4	Linear Independence. Bases. Orientation	219
7.5	Vector Analytic Geometry	223
7.6	Vector Spaces of Other Dimensions: E_n	225
7.7	Vector Functions. Curves	230
7.8	Rectifiable Curves and Arc Length	233
7.9	Differentiable Curves	237

Chapter 8 Functions of Several Variables. Limits and Continuity **249**

8.1	A Little Topology: Open and Closed Sets	249
8.2	A Little More Topology: Sequences, Cluster Values, Accumulation Points, Cauchy Criterion	254
8.3	Limits	260
8.4	Vector Functions of a Vector	264
8.5	Operations with Limits	268
8.6	Continuity	269
8.7	Geometrical Picture of a Function	274
8.8	Matrices and Linear Transformation	277

Chapter 9 Differentiable Functions **289**

9.1	Partial Derivatives	289
9.2	Differentiability. Total Differentials	299
9.3	The Derivative	307
9.4	The Gradient. The Del Operator. Directional Derivatives	315
9.5	The Chain Rule	321
9.6	The Mean Value Theorem and Taylor's Theorem for Several Variables	330
9.7	The Divergence and Curl of a Vector Field	333

Chapter 10	The Inversion Theorem	339
10.1	Transformations. Inverse Transformations	339
10.2	The Inversion Theorem	341
10.3	Implicit Functions	350
10.4	Global Inverses	358
10.5	Curvilinear Coordinates	360
10.6	Extreme Values	368
10.7	Extreme Values Under Constraints	372
Chapter 11	Multiple Integrals	379
11.1	Integrals Over Rectangles	379
11.2	Properties of the Integral. Classes of Integrable Functions	387
11.3	Iterated Integrals	389
11.4	Integration Over Regions. Area and Volume	394
Chapter 12	Line and Surface Integrals	405
12.1	Line Integrals. Potentials	405
12.2	Green's Theorem	417
12.3	Surfaces. Area	429
12.4	Surface Integrals. The Divergence Theorem	435
12.5	Stokes' Theorem. Orientable Surfaces	442
12.6	Some Physical Heuristics	449
12.7	Change of Variables in Multiple Integrals	451
PART III	THEORY OF CONVERGENCE	461
Chapter 13	Infinite Series	463
13.1	Convergence, Absolute and Conditional	463
13.2	Series with Nonnegative Terms: Comparison Tests	468
13.3	Series with Nonnegative Terms: Ratio and Root Tests. Remainders	475
13.4	Series with Variable Signs	480

13.5	More Delicate Tests	483
13.6	Rearrangements	486
13.7	Improvement of Convergence	492

Chapter 14 Sequence and Series of Functions. Uniform Convergence **503**

14.1	Introduction	503
14.2	Uniform Convergence	504
14.3	Consequences of Uniform Convergence	510
14.4	Abel's and Dirichlet's Tests	521
14.5	A Theorem of Dini	525

Chapter 15 The Taylor Series **529**

15.1	Power Series. Interval of Convergence	529
15.2	Properties of Power Series	536
15.3	The Taylor and Maclaurin Series	543
15.4	The Arithmetic of Power Series	549
15.5	Substitution and Inversion	558
15.6	Complex Series	560
15.7	Real Analytic Functions	564

Chapter 16 Improper Integrals **567**

16.1	Improper Integrals. Conditional and Absolute Convergence	567
16.2	Improper Integrals with Nonnegative Integrands	576
16.3	The Cauchy Principal Value	579
16.4	An Alternation Test	581
16.5	Improper Multiple Integrals	584

Chapter 17 Integral Representations of Functions **591**

17.1	Introduction. Proper Integrals	591
17.2	Uniform Convergence	595
17.3	Consequences of Uniform Convergence	601

Chapter 18	Gamma and Beta Functions. Laplace's Method and Stirling's Formula	621
18.1	The Gamma Function	621
18.2	The Beta Function	625
18.3	Laplace's Method	629
18.4	Stirling's Formula	635
Chapter 19	Fourier Series	639
19.1	Introduction	639
19.2	The Class \mathcal{R}_2 . Approximation in the Mean. Bessel's Inequality	646
19.3	Some Useful Lemmas	650
19.4	Convergence Theorems	654
19.5	Differentiation and Integration. Uniform Convergence	664
19.6	Sine and Cosine Series. Change of Scale	669
19.7	Improvement of Convergence	673
19.8	The Fourier Integral	676
19.9	Function Spaces. Complete Orthonormal Sets	683
	Elementary Differentiation and Integration Formulas	691
	Answers, Hints, and Solutions	693
	Index	727

PART 1

CALCULUS OF ONE VARIABLE

The Number System

1.1 The Peano Axioms

Any study of mathematical analysis has its basis in the number system. It is therefore important for students of analysis to understand how the arithmetic can be developed from the natural numbers (another name for the positive integers). It is not our intention to carry out such a construction here; however, we do want to make some comments about the logical structure of that development.

A standard starting point in the development of the real numbers is a certain set of axioms that were first formulated by the Italian mathematician Peano. These axioms state that the natural numbers satisfy certain properties. From only these properties, making appropriate definitions as we proceed, we can develop all the usual rules of arithmetic. In the terminology of formal logic, we have a set of undefined objects that we choose to name the **natural numbers**, satisfying Peano's axioms. This means merely that the natural numbers are taken as the basic "atoms" of our mathematical system in terms of which we express the other mathematical

concepts, but which are themselves not expressed in more fundamental terms.

Perhaps an easier way to visualize the situation, since after all you cannot claim that you never heard of arithmetic or rational numbers, is this: We can verify that the positive integers are a system of objects that do satisfy the Peano axioms. We can now proceed to recapture all we know of arithmetic, by developing appropriate definitions and theorems from these axioms, using only those properties stated explicitly in the axioms.

The five Peano axioms follow.

Axiom 1. *1 is a natural number.*

Axiom 2. *To every natural number n there is associated in a unique way another natural number n' called the **successor** of n .*

Axiom 3. *The number 1 is not a successor of any natural number.*

Axiom 4. *If two natural numbers have equal successors, they are themselves equal. That is, if $n' = m'$, then $n = m$.*

Axiom 5. *Suppose that M is a collection or set of natural numbers with the properties*

- (i) *1 is in M ,*
- (ii) *n' is in M whenever n is in M .*

Then the collection M consists of all natural numbers.

Here equality is used in the sense of numerical identity; that is, $m = n$ means that m and n are symbols standing for the same number. Thus we take

- (i) $m = m$,
- (ii) $m = n$ implies that $n = m$,
- (iii) $m = n$, $n = k$ implies that $m = k$,

as part of the underlying logic and do not list these as part of our set of numerical axioms.