UNIFORM SPACES

BY

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PREFACE

The subject matter of this book might be labelled fairly accurately *Intrinsic geometry of uniform spaces.* {For an impatient reader, this means elements (25%), dimension theory (40%), function spaces ($12\frac{1}{2}$%), and special topics in topology.} As the term "geometry" suggests, we shall not be concerned with applications to functional analysis and topological algebra. However, applications to topology and specializations to metric spaces are of central concern; in fact, these are the two pillars on which the general theory stands. This dictum brings up a second exclusion: the book is not much concerned with restatements of the basic definitions or generalizations of the fundamental concepts. These exclusions are matters of principle. A third exclusion is dictated mainly by the ignorance of the author, excused perhaps by the poverty of the literature, and at any rate violated in several places in the book: this is extrinsic (combinatorial and differential) geometry or topology.

More than 80% of the material is taken from published papers. The purpose of the notes and bibliography is not to itemize sources but to guide further reading, especially in connection with the exercises; so the following historical sketch serves also as the principal acknowledgement of sources.

The theory of uniform spaces was created in 1936 by Weil [W]. All the basic results, especially the existence of sufficiently many pseudometrics, are in Weil's monograph. However, Weil's original axiomatization is not at all convenient, and was soon succeeded by two other versions: the orthodox (Bourbaki [**Bo**]) and the heretical (Tukey [**T**]). The present author is a notorious heretic, and here advances the claim that in this book each system is used where it is most convenient, with the result that Tukey's system of uniform coverings is used nine-tenths of the time.

In the 1940's nothing of interest happened in uniform spaces. But three interesting things happened. Dieudonné [1] invented paracompactness and crystallized certain important metric methods in general topology, mainly the partition of unity. Stone [1] showed that all metrizable spaces are paracompact, and in doing so, established two important covering theorems whose effects are still spreading through uniform geometry. Working in another area, Eilenberg and MacLane defined the notions of category, functor, and naturality, and pointed out that their spirit is the spirit of Klein's Erlanger Programm and their reach is greater.

The organization of this book is largely assisted by a rudimentary version of the Klein-Eilenberg-MacLane program (outlined in a foreword to this book). We are interested in the single category of uniform spaces, two or three of its subcategories, and a handful of functors; but to consider them as instances of more general notions gives us a platform to stand on that is often welcome.

In 1952 Shirota [1] established the first deep theorem in uniform spaces, depending on theorems of Stone [1] and Ulam [1]. Except for reservations involving the axioms of set theory, the theorem is that every topological space admitting a complete uniformity is a closed subspace of a product of real lines. A more influential step was taken in 1952 by Efremovič [1] in creating proximity spaces. This initiated numerous significant Soviet contributions to uniform and proximity geometry (which are different but coincide in the all-important metric case), central among which is Smirnov's creation of uniform dimension theory (1956; Smirnov [4]). The methods of dimension theory for uniform and uniformizable spaces are of course mainly taken over from the classical dimension theory epitomized in the 1941 book of Hurewicz and Wallman [**HW**]. Classical methods were pushed a long way in our direction (1942-1955) by at least two authors not interested in uniform spaces: Lefschetz [**L**], Dowker [1; 2; 4; 5]. These methods—infinite coverings, sequential constructions—were brought into uniform spaces mainly by Isbell [1; 2; 3; 4] (from 1955).

Other developments in our subject in the 1950's do not really fall into a coherent pattern. What has been described above corresponds to Chapters I, II, IV and V of the book. Chapter III treats function spaces. The material is largely classical, with additions on injectivity and functorial questions from Isbell [5], and some new results of the same sort. The main results of Chapters VI (compactifications) and VIII (topological dimension theory) are no more recent than 1952 (the theorem Ind = dim of Katětov [2]).

The subject in Chapters VII and VIII is special features of fine spaces, i.e., spaces having the finest uniformity compatible with the topology. Chapter VII is as systematic a treatment of this topic as our present ignorance permits. Central results are Shirota's theorem (already mentioned) and Glicksberg's [2] 1959 theorem which determines in almost satisfactory terms when a product of fine spaces is fine. There is a connecting thread, a functor invented by Ginsburg—Isbell [1] to clarify Shirota's theorem, which serves at least to make the material look more like uniform geometry rather than plain topology. There are several new results in the chapter (VII. 1-2, 23, 25, 27-29, 31-34, 38); and a hitherto unpublished result of A. M. Gleason appears here for the first time. Gleason's theorem (VII. 19) extends previous results due mainly to Marczewski [1; 2] and Bokštein [1]. He communicated it to me after I had completed a draft of this book including the Marczewski and Bokštein theorems; I am grateful for his permission to use it in place of them.

PREFACE

Most chapters are followed by exercises adding details to the theory (in some cases doing duty for proofs omitted in the text), starred exercises whose results will be used later in the text, occasional unsolved problems, and a major unsolved problem. The major unsolved problems are Problems A, B_1, B_2, B_3, C, D. Not all are precisely posed, but all describe areas in which there seems to be good reason to expect interesting results although the results now known are quite unsatisfactory. The appendix might reasonably be counted as another such problem, for it gives several characterizations of the line and draws attention to the plane.

A preliminary version of this book was prepared as a set of lecture notes at Purdue University in 1960. The work of writing it has been supported at Purdue by the Office of Naval Research and at the University of Washington by the National Science Foundation. I am indebted to Professors M. Henriksen and M. Jerison for helpful criticisms of the Purdue lecture notes. Professors E. Alfsen, H. Corson, J. de Groot, E. Hewitt, E. Michael, D. Scott, and J. Segal have contributed some criticisms and suggestions during the writing of the final version. Many blemishes surviving that far were caught and exposed by Professor P. E. Conner for the Editorial Committee. But none of my distinguished colleagues has assumed responsibility for the remaining errors, which are mine.

FOREWORD

Categories. A *concrete category* $\mathscr{C}$ is defined by defining a class $\mathfrak{D}$ of sets, called *objects* of $\mathscr{C}$, and for each ordered pair of objects (X, Y) a set $\mathrm{Map}(X, Y)$ of functions $f: X \to Y$, called *mappings*, such that

(a) The identity function on each object is a mapping;

(b) Every function which is a composition of mappings is a mapping.

The analysis of this definition presents some peculiar difficulties, because the class $\mathfrak{D}$ may be larger than any cardinal number, and is larger in all cases arising in this book. But the difficulties need not concern us here. All we need is an indication of what is superfluous in the definition, i.e., of when two concrete categories determine the same abstract category.

A *covariant functor* F: $\mathscr{C} \to \mathscr{D}$ is given when we are given two functions, F_0 and F_1, as follows. F_0 assigns to each object X of $\mathscr{C}$ an object $F_0(X)$ of $\mathscr{D}$. F_1 assigns to each mapping f: $X \to Y$ of $\mathscr{C}$ a mapping $F_1(f)$: $F_0(X) \to F_0(Y)$ of $\mathscr{D}$. Further,

(A) For each identity mapping 1_X of $\mathscr{C}$, $F_1(1_X) = 1_{F_0(X)}$;

(B) For every composed mapping gf of $\mathscr{C}$, $F_1(gf) = F_1(g)F_1(f)$.

Having noted the distinction between F_0 and F_1, we can ignore it for applications, writing $F_0(X)$ as $F(X)$, $F_1(f)$ as $F(f)$. The short notation defines a *composition* of covariant functors F: $\mathscr{C} \to \mathscr{D}$, G: $\mathscr{D} \to \mathscr{E}$ for us: $GF(X) = G(F(X))$, $GF(f) = G(F(f))$. Then an *isomorphism* is a covariant functor F: $\mathscr{C} \to \mathscr{D}$ for which there exists a covariant functor F^{-1}: $\mathscr{D} \to \mathscr{C}$ such that both FF^{-1} and $F^{-1}F$ are identity functors.

A *categorical property* is a predicate of categories $\mathfrak{P}$ such that if $\mathfrak{P}(\mathscr{C})$ and $\mathscr{C}$ is isomorphic with $\mathscr{D}$ then $\mathfrak{P}(\mathscr{D})$. Similarly we speak of categorical definitions, ideas, and so on.

The notion of a mapping f: $X \to Y$ having an inverse f^{-1}: $Y \to X$ is categorical. The defining conditions are just $ff^{-1} = 1_Y$, $f^{-1}f = 1_X$. A mapping having an inverse in $\mathscr{C}$ is called an *isomorphism* in $\mathscr{C}$.

The notion of a mapping f: $X \to Y$ being one-to-one is not categorical. However, it has an important categorical consequence. If f: $X \to Y$ is one-to-one then for any two mappings d: $W \to X$, e: $W \to X$, $fd = fe$ implies $d = e$. A mapping having this left cancellation property is called a *monomorphism*. Similarly a mapping f such that $gf = hf$ implies $g = h$ is called an *epimorphism*. A mapping $f: X \to X$ satisfying $ff = f$ is a *retraction*.

REMARK. If a retraction is either a monomorphism or an epimorphism then it is an identity.

A *contravariant functor* $F: \mathscr{C} \to \mathscr{D}$ assigns to each object X of $\mathscr{C}$ an object

$F(X)$ of $\mathscr{D}$ or to each mapping $f: X \to Y$ of $\mathscr{C}$ a mapping in the opposite direction $F(f): F(Y) \to F(X)$ of $\mathscr{D}$ satisfying condition (A) above and

(B*) For every composed mapping gf of $\mathscr{C}$, $F(gf) = F(f)F(g)$. Contravariant functors can be composed; but the composition of two contravariant functors is a covariant functor. In fact, functors of mixed variances can be composed, with an obvious rule for the variance of the composition.

A *duality* $F: \mathscr{C} \to \mathscr{D}$ is a contravariant functor admitting an inverse, which is a contravariant functor $F^{-1}: \mathscr{D} \to \mathscr{C}$ such that both FF^{-1} and $F^{-1}F$ are identity functors. It is a theorem that:

Every concrete category is the domain of a duality.

An interested reader may prove this, letting $F(X)$ be the set of all subsets of X and $F(f) = f^{-1}$.

The *principle of duality* says roughly that any categorical theorem θ for arbitrary categories implies another theorem θ^* for arbitrary categories. For example, if θ is a theorem about a single category $\mathscr{C}$, the statement of θ for $\mathscr{C}$ is equivalent to a statement about a category $\mathscr{D}$ related to $\mathscr{C}$ by a duality $F: \mathscr{C} \to \mathscr{D}$; that statement about D is θ^*, and it is true for arbitrary categories because every category is the range of a duality.

The theorem θ that a retraction f which is a monomorphism is an identity can illustrate duality. The statement θ^* is that if f is a mapping in $\mathscr{C}$, $F: \mathscr{C} \to \mathscr{D}$ is a duality, and $F(f)$ is a retraction and a monomorphism, then $F(f)$ is an identity. Using several translation lemmas we can simplify θ^* to the equivalent form: if f is a retraction and an epimorphism then f is an identity.

Note that we may have "dual problems" which are not equivalent to each other. A typical problem in a category $\mathscr{C}$ is, does $\mathscr{C}$ have the property $\mathfrak{P}$? If $\mathfrak{P}$ is categorical, there is a *dual property* $\mathfrak{P}^*$, and the given problem is equivalent to the problem does a category dual to $\mathscr{C}$ have the property $\mathfrak{P}^*$? By the *dual problem* we mean: Does $\mathscr{C}$ have the property $\mathfrak{P}^*$?

Finally, we need definitions of *subcategory, full subcategory,* and *functor of several variables*. A subcategory $\mathscr{D}$ of $\mathscr{C}$ is a category such that every object of $\mathscr{D}$ is an object of $\mathscr{C}$ and every mapping of $\mathscr{D}$ is a mapping of $\mathscr{C}$. $\mathscr{D}$ is a full subcategory if further, every mapping of $\mathscr{C}$ whose domain and range are objects of $\mathscr{D}$ is a mapping of $\mathscr{D}$.

For several variables we want the notion of a *product* of finitely many categories $\mathscr{C}_1, \cdots, \mathscr{C}_n$. The product is a category whose objects may be described as n-tuples $(X_1, \cdots, X_n)$, each X_i an object of $\mathscr{C}_i$. To represent these as sets the union $X_1 \cup \cdots \cup X_n$ would serve, if some care is taken about disjointness. The set $\mathrm{Map}((X_1, \cdots, X_n), (Y_1, \cdots, Y_n))$ is the product set $\mathrm{Map}(X_1, Y_1)$

$\times \cdots \times$ Map (X_n, Y_n). Again, the mappings $(f_1, \cdots, f_n)$ can be represented as functions on $X_1 \cup \cdots \cup X_n$, with $(f_1, \cdots, f_n) | X_i = f_i$.

A *pure covariant functor* on $\mathscr{C}_1, \cdots, \mathscr{C}_n$ is a covariant functor defined on the product $\mathscr{C}_1 \times \cdots \times \mathscr{C}_n$. A functor on $\mathscr{C}_1, \cdots, \mathscr{C}_n$, *covariant in* the set I of indices and *contravariant in* the remaining indices, is a function F on $\mathscr{C}_1 \times \cdots \times \mathscr{C}_n$ to a category $\mathscr{D}$, taking objects to objects, mappings to mappings, identities to identities; taking mappings $f = \{f_i\}: \{X_i\} \rightarrow \{Y_i\}$ to mappings $F(f): F(\{Z_i\}) \rightarrow F(\{W_i\})$, where for $i \in I$, $Z_i = X_i$ and $W_i = Y_i$, but for $i \notin I$, $Z_i = Y_i$ and $W_i = X_i$; and preserving the composition operation defined in the product category by $g \circ f = h$, where $h_i = g_i f_i$ for $i \in I$, $h_i = f_i g_i$ otherwise.

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CHAPTER I

FUNDAMENTAL CONCEPTS

Some of the notions of the theory of uniform spaces are familiar from metric spaces. It is possible to found the entire theory on the notion of a set with a family of pseudometrics called a gage. We shall not do this here, but we shall lead in gradually from metric notions to the idea which will be the foundation of the development: the uniform coverings.

Metric uniform spaces. Recall that a *pseudometric* d on a set X is a real-valued function on $X \times X$ satisfying $d(x, y) = d(y, x) \geqq 0$ and $d(x, z) + d(z, y) \geqq d(x, y)$, for all x, y, z in X. It is called a *metric* if it separates points, i.e., $x \neq y$ implies $d(x, y) > 0$. A metric space consists of a set X with a metric d on X. Commonly we refer to "the metric space X" and use the letter d freely for the distance in any metric space.

Recall that a function $f: X \to Y$, where X and Y are metric spaces, is called *uniformly continuous* if for each $\epsilon > 0$ there is $\delta > 0$ such that whenever $d(x, x') < \delta$ in X, $d(f(x), f(x')) < \epsilon$. Every uniformly continuous function is continuous, but the converse is not true.

A covering $\mathscr{U}$ of X is called a *uniform covering* provided there is a positive number ϵ such that every subset of X of diameter less than ϵ is a subset of some element of $\mathscr{U}$. Such an ϵ is called a *Lebesgue number* for $\mathscr{U}$.

1. *A covering $\mathscr{U}$ of a metric space X is uniform if and only if there is $\delta > 0$ such that for each point x in X, the δ-neighborhood of x is contained in some element of $\mathscr{U}$.*

Proof. If $\mathscr{U}$ is uniform with Lebesgue number ϵ, then every $(\epsilon/2)$-neighborhood of a point is contained in an element of $\mathscr{U}$. Conversely, if elements of $\mathscr{U}$ contain all δ-neighborhoods then δ is a Lebesgue number for $\mathscr{U}$.

2. *A function $f: X \to Y$ is uniformly continuous if and only if for every uniform covering $\mathscr{V}$ of Y there is a uniform covering $\mathscr{U}$ of X such that, for each element U of $\mathscr{U}$, $f(U)$ is contained in some element of $\mathscr{V}$.*

PROOF. Suppose f is uniformly continuous and $\mathscr{V}$ is a covering of Y having Lebesgue number ϵ. If δ is such that any two points of X at distance less than δ have images at distance less than ϵ, then the collection of $(\delta/2)$-neighborhoods of all points of X is the required uniform covering $\mathscr{U}$.

Conversely, suppose the condition on coverings is satisfied and $\epsilon > 0$. Let $\mathscr{V}$ be the covering of Y consisting of all sets of diameter $< \epsilon$, and $\mathscr{U}$ a uniform covering of X each element of which has its image contained in a single element of $\mathscr{V}$. If δ is a Lebesgue number for $\mathscr{U}$, then any two points of X at distance $< \delta$ have images at distance $< \epsilon$.

A uniformly continuous function $f: X \to Y$ is called a *uniform equivalence* if f is one-to-one and onto and the inverse function $f^{-1}: Y \to X$ is also uniformly continuous.

3. *If $f: X \to Y$ is a uniform equivalence, then a collection $\{U_\alpha\}$ of subsets of X is a uniform covering if and only if the collection $\{f(U_\alpha)\}$ is a uniform covering of Y. The converse is also true.*

The proof is left as an exercise.

Finally, we may define a *metric uniform space* as a set X together with a family μ of coverings of X such that for at least one distance function d on X, μ is precisely the family of all uniform coverings of the metric space (X, d). The preceding remarks and results show that every metric space determines uniquely a metric uniform space; that two different distance functions d, e, on the same set X, determine the same uniform space if and only if the identity mapping is a uniform equivalence between (X, d) and (X, e); and further, if we are given the metric uniform spaces (X, μ) and (Y, ν) and a function $f: X \to Y$, we can determine whether f is uniformly continuous by I.2, without knowing or constructing any specific distance functions.

This summary treatment of metric uniform spaces will not be used in developing the general theory. It is included in justification; we propose to define such terms as "uniformly continuous" and "completion", and we ought to show that the notions are true generalizations of the familiar notions for metric spaces. The remaining details in this showing will be swept into exercises or omitted.

Another important point is that, while we lose the distance function in passing from metric space to metric uniform space, we do not lose the topology. For example,

4. *For metric spaces X and Y, a function f: $X \to Y$ is continuous if and only if for each point x in X, for each uniform covering $\mathscr{V}$ of Y, there exist an element V of $\mathscr{V}$ and a uniform covering $\mathscr{U}$ of X such that for every element U of $\mathscr{U}$ which contains x, $f(U) \subset V$.*

Another important point: the metric uniform space has *more* structure than the topological space. There may be two homeomorphic metric uniform spaces which are not uniformly equivalent; and more.

5. *There is an uncountable family of countable discrete metric spaces, no two of which are uniformly equivalent to each other.*

PROOF. From topology we know that there is an uncountable family of compact subspaces of the plane, no two of which are homeomorphic with each other.

(Recall the construction. For any increasing sequence of positive integers, $n_1 < n_2 < \cdots$, begin with the segment from $(0, 0)$ to $(0, 1)$ in the plane and at each point $(0, 2^{-i})$ attach n_i short whiskers.)

Let $\{C_\alpha\}$ be such a family of spaces in the plane. In each C_α select a countable dense subset $\{p_n^\alpha\}$. Let the coordinates of p_n^α be (x_n^α, y_n^α). Let X_α be the subset of three-space consisting of all points $(x_n^\alpha, y_n^\alpha, 1/m)$, with $m \geqq n$. Then X_α is a countable metric space. It is discrete since each of its points is above the horizontal coordinate plane and for each $\epsilon > 0$ there are only finitely many points of X_α with third coordinate greater than ϵ. Of course, X_α is not closed in three-space; its closure Y_α consists of X_α and a copy of C_α. Moreover, Y_α is the completion of X_α. Now suppose $f: X_\alpha \to X_\beta$ is a uniform equivalence. In particular, f maps X_α uniformly continuously into the complete space Y_β. Hence f has a unique uniformly continuous extension $g: Y_\alpha \to Y_\beta$. The same reasoning applies to $f^{-1}: X_\beta \to X_\alpha$, which must have a uniformly continuous extension $h: Y_\beta \to Y_\alpha$.

Finally, consider the composed mapping hg of Y_α into itself. On the dense set X_α, hg coincides with the identity mapping i. But for any two continuous mappings $p: A \to B, q: A \to B$, where A and B are Hausdorff spaces, the set of all a in A such that $p(a) = q(a)$ is a closed set. Hence $hg: Y_\alpha \to Y_\alpha$ is the identity. Similarly $gh: Y_\beta \to Y_\beta$ is the identity. But then g maps C_α homeomorphically onto C_β, which is absurd. It follows that X_α and X_β cannot be uniformly equivalent.

This result illustrates a useful rule of thumb in the theory of uniform spaces: *All counterexamples are discrete.*

Uniformities and preuniformities. We shall need a little of the terminology of the theory of quasi-ordered sets. Moreover, it will be convenient to use slightly nonstandard terminology; so every reader should note carefully the following definitions.

A set S is said to be *quasi-ordered* by a relation $<$ if $<$ is transitive. A subset Q of S is *cofinal* in S if for each element s of S there exists an element q of Q such that $q < s$. Q is *residual* in S if whenever $q \in Q$ and $r < q$ in S, $r \in Q$. Q is *antiresidual* if for $q \in Q$ and $r > q$ in S, r is in Q. The quasi-ordered set S is *directed* if for any p and q in S there exists s in S satisfying $s < p$ and $s < q$. Such an s may be called a *common successor* of p and q.

One important example of a quasi-ordered set is any family S of subsets of a given set X, ordered by inclusion $\subset$. In this case a common successor of p and q is a subset of their intersection.

For another example consider coverings of X. A covering $\{U_\alpha\}$ is called a *refinement* of a covering $\{V_\beta\}$ if each U_α is a subset of at least one V_β. We write $\{U_\alpha\} < \{V_\beta\}$, and note that $<$ is a quasi-ordering.

Inclusion is actually a partial ordering, i.e., besides being transitive it is reflexive and anti-symmetric. Note that refinement is also reflexive, but not anti-symmetric. Two coverings $\mathcal{U}$, $\mathcal{V}$, may be equivalent in the sense that $\mathcal{U} < \mathcal{V}$ and $\mathcal{V} < \mathcal{U}$.

Any two coverings $\{U_\alpha\}, \{V_\beta\}$, of a set have a coarsest common refinement, the covering $\{U_\alpha \cap V_\beta\}$. It may be denoted by $\{U_\alpha\} \wedge \{V_\beta\}$. Of course, there are usually other coverings equivalent to this one.

If $\mathcal{U}$ is a covering of X and A is a subset of X, the *star* $\operatorname{St}(A, \mathcal{U})$ of A with respect to $\mathcal{U}$ is the union of all elements of $\mathcal{U}$ which have a nonempty intersection with A. The collection $\{\operatorname{St}(U, \mathcal{U}) : U \in \mathcal{U}\}$ is a covering and is called $\mathcal{U}^*$, the *star* of $\mathcal{U}$. If $\mathcal{U}^*$ is a refinement of $\mathcal{V}$, $\mathcal{U}$ is called a *star-refinement* of $\mathcal{V}$, and one writes $\mathcal{U} <^* \mathcal{V}$. The relation $<^*$ is again a quasi-ordering, generally not reflexive.

In any quasi-ordered set, a *filter* is a directed antiresidual subset. A *filter base* is a cofinal subset of a filter, i.e., a directed set. In a partially ordered family of sets, ordered by inclusion, a *proper filter* is a filter which does not have the empty set as an element.

Now we come to the main definitions. A *preuniformity* μ on a set X is a family of coverings of X which forms a filter with respect to $<^*$. A *uniformity* μ on X is a preuniformity such that for any two points, x, y, of X, there is a covering $\mathcal{U}$ in μ, no element of which contains both x and y. A *uniform space* μX is a set X with a uniformity μ on X. The elements of μ are called *uniform coverings*.

6. *A family μ of coverings is a preuniformity if and only if* (i) *for $\mathcal{U}$ and $\mathcal{V}$ in μ, $\mathcal{U} \wedge \mathcal{V}$ is in μ;* (ii) *for $\mathcal{U} < \mathcal{V}$ and $\mathcal{U}$ in μ, $\mathcal{V}$ is in μ; and* (iii) *every element of μ has a star-refinement in μ*

7. *For any two points, x, y, of a uniform space, there is a uniform covering $\mathcal{U}$ such that $\operatorname{St}(x, \mathcal{U})$ and $\operatorname{St}(y, \mathcal{U})$ are disjoint.*

In some of the literature, what we call a uniform space is called a separated uniform space, and a set with any preuniformity on it is called a uniform space. We shall use the term "separate" mainly in the following (customary) sense: a family $\{f_\alpha\}$ of functions with the same domain X but possibly different ranges separates points provided $x \neq y$ in X implies that for some α, $f_\alpha(x) \neq f_\alpha(y)$. Other uses of the word are introduced in places, particularly in Chapter VI.

The uniformities or preuniformities on any set form a partially ordered set under inclusion. The preuniformities, like the topologies, form a complete lattice. (This is a corollary of Proposition 9 below.) Evidently, any preuniformity which contains a uniformity is a uniformity. As with topologies, a preuniformity μ containing a preuniformity ν is said to be *finer* than ν. The usage of the terms "strong" and "weak" is not standardized, and we shall avoid them as far as possible.

The weak uniformity induced by a family of functions is too useful to be avoided, and fortunately there has been little or no terminological confusion here. We have

8. THEOREM. *For any family* $\{f_\alpha\}$ *of functions on a set* X *into various uniform spaces, there is a coarsest preuniformity on* X *including all the inverse images of uniform coverings under these functions. If the functions separate points, then this preuniformity is a uniformity.*

This uniformity is the *weak uniformity* induced by the family $\{f_\alpha\}$. The proof of Theorem 8 is not difficult, but we shall take some time marking out important ideas in it.

A *basis* for a uniformity μ is a filter base for μ considered as a filter of coverings; and similarly for a preuniformity. A *sub-basis* for a uniformity or preuniformity is a family of coverings whose finite intersections form a basis. Now a family ν of coverings which satisfies condition (iii) of Proposition 6, every covering in ν has a starrefinement in ν, is called a *normal family*. It is convenient and customary to use the term *normal sequence* for something more special than a sequence which is a normal family: specifically, for a sequence of coverings $\mathcal{U}^n$ such that $\mathcal{U}^{n+1} <^* \mathcal{U}^n$ for each n.

9. *Every normal family of coverings is a sub-basis for a preuniformity.*

PROOF. The required preuniformity is the family of all coverings which can be refined by finite intersections of coverings from the given family, which automatically satisfies (i) and (ii) of I.6. For (iii) we need only observe that if $\mathcal{V}^i <^* \mathcal{U}^i$ for $i = 1, \cdots, n$, then $\mathcal{V}^1 \wedge \cdots \wedge \mathcal{V}^n <^* \mathcal{U}^1 \wedge \cdots \wedge \mathcal{U}^n$.

PROOF OF THEOREM 8. The operation $f_\alpha^{\cdot}$ on coverings preserves starrefinements; so the inverse images of uniform coverings form a normal family. Then this is a sub-basis for a preuniformity μ, which is the coarsest possible.

We should note that a sub-basis need not be a normal family; of course, a basis must.

There is a complement to Proposition 9. Every union of normal families of coverings is a normal family; so every family λ of coverings contains a largest normal subfamily μ. The coverings in μ are called *normal in* λ. A slightly broader notion is generally more useful. A covering $\mathscr{U}$ is said to be *normal with respect to* λ provided $\mathscr{U}$ belongs to the smallest anti-residual family containing λ and is normal in it; that is, provided $\mathscr{U}$ is the first term of some normal sequence of coverings each of which has a refinement in λ. We note that if $\mathscr{U}$ is known to belong to some preuniformity contained in λ, it must be normal with respect to λ. The converse is true if λ is a filter with respect to $<$.

10. *If λ is a family of coverings of a set which forms a filter under refinement, then there is a finest preuniformity contained in λ, and it consists of all coverings normal with respect to λ.*

Proof. It suffices to show that the family μ of all coverings normal with respect to λ is a preuniformity, for we have already noted that every preuniformity contained in λ is contained in μ. We use Proposition 6. Conditions (ii) and (iii) are obviously satisfied. It remains only to note that if $\mathscr{U}$ and $\mathscr{V}$ are in μ, there are normal sequences $\{\mathscr{U}^n\}$ and $\{\mathscr{V}^n\}$ in the filter λ with $\mathscr{U}=\mathscr{U}^1$, $\mathscr{V}=\mathscr{V}^1$; and $\{\mathscr{U}^n \wedge \mathscr{V}^n\}$ is a normal sequence.

Uniform topology and uniform continuity. The *uniform topology* of a uniform space X is defined as follows. A subset N of X is a neighborhood of a point x of N if for some uniform covering $\mathscr{U}$, N contains $\mathrm{St}(x, \mathscr{U})$. N is open if it is a neighborhood of each of its points.

We wish to prove

11. Theorem. *Every uniform space is a completely regular Hausdorff space in the uniform topology.*

The construction required for this proof can be made to yield another important theorem; so we put it off for a while. Observe here that the uniform space X is at least a T_1 space; for X is an open set (obvious), any union of open sets is open (obvious), the intersection of any two open sets is open (easy exercise), and a point is closed (proof follows). For any point x, for any point $y \neq x$, by definition there must be a uniform covering $\mathscr{U}$, no element of which contains both x and y. Then $\mathrm{St}(y, \mathscr{U}) \subset X - \{x\}$. Thus $X-\{x\}$ is a neighborhood of each of its points, and the point x is closed.

For the rest of the proof we shall need real-valued continuous functions. We may as well construct uniformly continuous functions, since it is no harder. A function f on a uniform space X to a uniform space Y is called *uniformly continuous* if for every uniform covering $\mathscr{V}$ of Y there is a uniform