V. M. STARZHINSKII

APPLIED METHODS IN THE THEORY OF NONLINEAR OSCILLATIONS

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PREFACE

Progress in the theory of nonlinear oscillations during the last decades was based chiefly on classical methods developed in the late 19th and early 20th centuries. This is illustrated by developments in the method of small parameter in monographs by Andronov, Vitt, and Khaikin [5], as well as Bulgakov [31], and Malkin [111a, b], developments in the method of averaging based on the Van der Pol method (Bogolyubov and Mitropolskii [22, 127d], Volosov and Morgunov [204]), by a new perturbation theory (Arnold [215]) based on classical perturbation methods, and by the Kamenkov V-function method [83, vol. II] based on fundamental results obtained by Lyapunov and Chetaev.

At the same time, new methods penetrated the theory of nonlinear oscillations: asymptotic methods developed by Bogolyubov, Krylov, and Mitropolskii [22, 102, 24, 127c], analytic functions methods introduced by Krasnoselskii [97a, b, 297a-d] and his school [98, 99], the method of point transformations developed by Andronov and Vitt [4, 5], and Neimark [137a, b], the stroboscopic method (Minorsky [125a, b, c]), the Gantmacher-Krein oscillation method [62], and the method of determining conditionally periodic motions introduced by Kolmogorov and Arnold [215, 286]. The idea of a new method is relative, of course, if we recall that Euler, Lagrange, and Laplace used averaging long before Van der Pol. This remark, however, is meant for future investigators.

Part One of the book is devoted to the combination of the Lyapunov, Poincaré, and averaging methods as applied to the analysis of oscillations in Lyapunov and nearly Lyapunov systems. A topic of interest is the investigation of oscillatory systems represented by analytic autonomous differential equations having no small parameters. The Lyapunov method of finding periodic solutions is known for the case of a conservative system (Lyapunov systems). The periodic solution obtained by means of the Lyapunov method depends, however, only on two constants of integration. Therefore it cannot in principle be a general solution for systems with more than two degrees of freedom; moreover, cases are known when the Lyapunov method fails. Chapter I, Section 1 discusses a transformation, outlined by Lyapunov, of an initial system whereby the system's order is lowered by two, a parameter equal to the square root of reduced energy is introduced, and the system becomes nonautonomous. If this parameter is sufficiently small, the methods of small parameter can be applied to the transformed quasilinear nonautonomous system.

This modification of the method proved effective in a number of problems, in particular, the problem of energy transfer. The first step is to determine the initial periodic mode and find its instability regions in the space of the system's parameters using the theory of parametric resonance [111b, 80]. The second step consists in determining the periodic modes that appear at critical values of the parameters and are, of course, distinct from the initial mode. This step uses the above-mentioned transformations and the Poincaré method of finding periodic solutions for nonautonomous systems. Other methods of small parameter can also be used with the transformed system, for instance, the method of averaging; in this case it becomes possible to carry out the third step of analysis, namely, investigation of the transient process, often referred to as energy transfer. All three steps are illustrated in Chapter III for the spring-loaded pendulum, pendulum subject to elastic suspension, and betatron oscillations of particles in cyclic accelerators with weak focusing. Note that the energy transfer problem is based on the general theory of oscillatory chains presented in Chapter II.

The next point is the application of perturbation theory (Chapter IV, Section 1). We assume that an unperturbed Lyapunov-type nonlinear autonomous system of order $2k + 2$ is perturbed by an analytic, and sufficiently small in norm, damping. A transformation of the perturbed system is carried out in which the unperturbed system is converted into a quasilinear nonautonomous system of order $2k$. Its solution is assumed known for a sufficiently small (compared to unity) square root of the initial value of the reduced energy of the system. In order to find the first- and higher-order corrections of the corresponding solution (i.e. with the same initial conditions) of the perturbed system, we must write a complete set of variational equations, that is, a sequence of nonhomogeneous systems of linear differential equations of order $2k + 1$ with variable coefficients. The complete system is given in operator form for the general finite-dimensional case of analytic perturbation theory. According to Poincaré, the integration of the complete system is reduced to quadratures provided a general integral of the unperturbed system is known.

The last section of the first part of the book treats oscillations in Lyapunov-type systems. We present here some of the results obtained by Nustrov [336a, b]; the table of contents gives a fair idea of the subjects discussed.

The second part of the book is also based on the results achieved in one of the classical methods developed in the years spanning the late 19th and early 20th centuries, the theory of normal forms (Poincaré, Lyapunov, Dulac, Siegel, Moser, Arnold, Pliss, and others).

Brjuno [238a-v] obtained general results in the theory of normal forms of nonlinear analytic autonomous systems of ordinary differential equations. The method was first introduced by Poincaré [149a]. The theory is applied in the second part of the book to analyze oscillations described by such equations.

Chapter V gives the elements of the theory of normal forms required to understand the material.

In Chapter VI we single out the class of problems in which the normal form has the simplest form as given by the Poincaré theorem and in which the general solution of the Cauchy problem can be obtained at each step of the approximation efficiently. This class covers damped oscillatory systems (asymptotically stable in linear approximation) with analytic nonlinearities of the general type. The results are illustrated by two examples of mechanical systems with one and two degrees of freedom.

In the next chapter we consider third-order systems with two pure imaginary and one negative (Chapter VII, Section 1) or vanishing (Chapter VII, Section 2) eigenvalues of the linear part. Chapter VII, Section 1 concludes with an investigation of oscillations in electromechanical systems with "one and a half" degrees of freedom.

Finally, normal forms and resonances are studied in analytic fourth-order (Chapter VIII, Section 1) and sixth-order (Chapter VIII, Section 4) autonomous systems with two and three pairs, respectively, of various pure imaginary (in general, nonconservative) eigenvalues of the matrix of the linear part. The Cauchy problem is solved in the general case with quadratic terms included. The results derived from the Molchanov and Bibikov-Pliss stability criteria are discussed for third-power normal forms. Two methods are suggested for constructing the Lyapunov function for the case of conservative systems: a direct method, and one based on Chetaev's linear combination of integrals obtained by means of third-power normal forms. The results are applied to the Ishlinskii problem concerning the motion of the gyroscopic frame of a sensor element of a gyroscopic compass (Chapter VIII, Section 2).

In the first approximation, the two parts of the book are independent.

The book is aimed at engineers with a strong mathematical background, scientists working in mechanics and applied mathematics, and undergraduate and postgraduate students of Applied Physics and Physics and Mathematics departments.

The book is based on a course of lectures presented by the author to engineering students at the Mechanics and Mathematics Department of Moscow University in 1956-1976.

If the author has been successful in giving the reader an insight into the theory of oscillations and stability, he owes this primarily to the late Boris V. Bulgakov and Nikolai G. Chetaev.

The formulas within each subsection of the text are numbered without citing the section number. If a formula of another section is cited, the number of this section is added to the formula number; if the formula cited belongs to a different chapter, the number of this chapter is written in front of the section number and is separated by a comma. The same rule holds when sections and subsections are cited.

V. Starzhinskii

CONTENTS

OSCILLATIONS IN LYAPUNOV SYSTEMS

* * *

CHAPTER I

INTRODUCTION

§ 1. Transformation of Lyapunov Systems

The order of Lyapunov systems ([108a], §§ 33-45; [111a], Ch. IV; [111b], Ch. VII) can be reduced by two by using the energy integral and choosing, following Lyapunov, the polar angle in the plane of critical variables as the independent variable. The transformed system [371e-g, j, n, s, t] is nonautonomous and includes a parameter equal to the square root of the reduced constant energy. If this parameter is sufficiently small compared to unity, the Poincaré method ([188a], vol. I, Ch. III) of determining periodic solutions of nonautonomous systems (see Section 2 and Chapter III of this book) can be applied to the transformed system. The application of the Poincaré method is of special interest when the Lyapunov method ([108a], §§ 34-45; [111a], §§ 26-29; [111b], Ch. VII, §§ 1-4) cannot be applied to finding periodic solutions of the initial system.

In general, however, other methods of small parameter, for example, the method of averaging [150, 22, 127d, 204, 66a], can be applied to a transformed system. Since this allows us to not only determine periodic solutions but examine a broader range of problems, such as transient processes and so on, the usefulness of transforming Lyapunov systems becomes apparent. This aspect of the problem is discussed in Chapter III.

1.1. General case [371 e-g, j, s]. We consider a system of Lyapunov differential equations

$$\frac{dx}{dt} = -\lambda y + X(x, y, x_1, \ldots, x_n),$$

$$\frac{dy}{dt} = \lambda x + Y(x, y, x_1, \ldots, x_n),$$

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + X_s(x, y, x_1, \ldots, x_n) \qquad (1.1)$$

$$(s = 1, \ldots, n).$$

Here λ, p_{sr} are real constants, and $X, Y, X_1, \ldots, X_n$ are real analytic functions of $x, y, x_1, \ldots, x_n$ whose expansions begin with terms of order not lower than two. Lyapunov proved the following theorem ([108a], § 42): *if*

(a) *the equation* $\det \| p_{sr} - \delta_{sr}\varkappa \| = 0$ *has no roots of the type* $m\lambda i$ $(m = 0, \pm 1, \pm 2, \ldots, i = \sqrt{-1})$, *and*

(b) *it is possible to find power series in an arbitrary constant c satisfying system* (1.1)

$$\begin{aligned} x &= cx^{(1)} + c^2x^{(2)} + \ldots, \\ y &= cy^{(1)} + c^2y^{(2)} + \ldots, \\ x_s &= cx_s^{(1)} + c^2x_s^{(2)} + \ldots \quad (s = 1, \ldots, n), \end{aligned} \tag{1.2}$$

where $x^{(k)}, y^{(k)}, x_1^{(k)}, \ldots, x_n^{(k)}$ $(k = 1, 2, \ldots)$ *are periodic functions of t with the same period, and* $x^{(k)}(t_0) = y^{(k)}(t_0) = 0$ *for* $k > 1$, *then the series found are absolutely convergent if c remains below a certain limit and for these values of c the series are a periodic solution of the initial system* (1.1).

Let us analyze the cases in which at least one of these conditions is violated and the Lyapunov theorem consequently does not hold. If condition (a) is violated, we have the "resonant" case discussed by Ryabov [355a]. Condition (b) is violated if the expansions of $X, Y, X_1, \ldots, X_n$ do not contain the terms x^ν and y^ν $(\nu = 2, 3, \ldots)$. In the latter case each coefficient of each of series (1.2) will be identically zero at each step of the calculation.

In Chapter III we demonstrate, however, that even in these cases it is possible to find periodic solutions of system (1.1) provided they exist. With a view to the remarks made in the introduction to this section, we shall consider a transformation of a Lyapunov system not bound, in general, by conditions (a) and (b). We assume in what follows that system (1.1) possesses a first integral,* which is inevitably an analytic function of $x, y, x_1, \ldots, x_n$ ([108a], § 38; [111a], § 25; [111b], Ch. VII, § 1) of the type

$$H = x^2 + y^2 + W(x_1, \ldots, x_n) + S_3(x, y, x_1, \ldots, x_n) = \mu^2 \quad (\mu > 0), \tag{1.3}$$

where W is a quadratic form. The Lyapunov substitution

$$\begin{aligned} x &= \rho\cos\vartheta, \\ y &= \rho\sin\vartheta, \\ x_s &= \rho z_s \quad (s = 1, \ldots, n) \end{aligned} \tag{1.4}$$

* This is included in the definition of a Lyapunov system.

transforms system (1.1) and the first integral (1.3) to the form

$$\frac{d\rho}{dt} = \rho^2 R(\rho, \vartheta, z),$$

$$\frac{d\vartheta}{dt} = \lambda + \rho\Theta(\rho, \vartheta, z),$$

$$\frac{dz_s}{dt} = p_{s1}z_1 + \cdots + p_{sn}z_n + \rho Z_s(\rho, \vartheta, z) \quad (s = 1, \ldots, n), \tag{1.5}$$

$$\rho^2 [1 + W(z) + \rho S(\rho, \vartheta, z)] = \mu^2. \tag{1.6}$$

Here R, Θ, $Z_1, \ldots, Z_n$, and S are analytic functions of the variables ρ, $z_1, \ldots, z_n$ in some neighbourhood of zero values whose expansions in powers of ρ, in general, begin with zero-power terms; the coefficients of power series in ρ, $z_1, \ldots, z_n$ are periodic functions of ϑ with period 2π that are polynomials with respect to $\cos\vartheta$ and $\sin\vartheta$

$$R = \rho^{-2} [X(\rho\cos\vartheta, \rho\sin\vartheta, \rho z)\cos\vartheta + Y(\rho\cos\vartheta, \rho\sin\vartheta, \rho z)\sin\vartheta],$$

$$\Theta = \rho^{-2} [-X\sin\vartheta + Y\cos\vartheta],$$

$$Z_s = \rho^{-2} X_s(\rho\cos\vartheta, \rho\sin\vartheta, \rho z) - z_s R(\rho, \vartheta, z) \quad (s = 1, \ldots, n),$$

$$S = \rho^{-3} S_3(\rho\cos\vartheta, \rho\sin\vartheta, \rho z) \tag{1.7}$$

and z is a vector with the components $z_1, \ldots, z_n$.

We assume now that $1 + W > 0$ in (1.6). This holds true for all values of $z_1, \ldots, z_n$ if W is a positive-definite quadratic form (this is true for the energy integral) and for sufficiently small values of $z_1, \ldots, z_n$ in the general case. We solve equation (1.6) with respect to ρ for one selected branch of the analytic function; specifically, we presume only the arithmetic value of the root:

$$\rho = (1 + W)^{-1/2} \mu \left\{ 1 - \frac{1}{2}(1 + W)^{-3/2} S(0, \vartheta, z)\mu + \left[\frac{5}{8}(1 + W)^{-3} S^2(0, \vartheta, z) - \frac{1}{2}(1 + W)^{-2} S'_\rho(0, \vartheta, z)\right]\mu^2 \right\} + O(\mu^4). \tag{1.8}$$

Assuming μ to be sufficiently small, we introduce phase time ϑ, by dividing the last n equations (1.5) by the second

$$\frac{dz_s}{d\vartheta} = \frac{p_{s1}z_1 + \ldots + p_{sn}z_n + \rho Z_s(\rho, \vartheta, z)}{\lambda + \rho\Theta(\rho, \vartheta, z)} \quad (s = 1, \ldots, n).$$