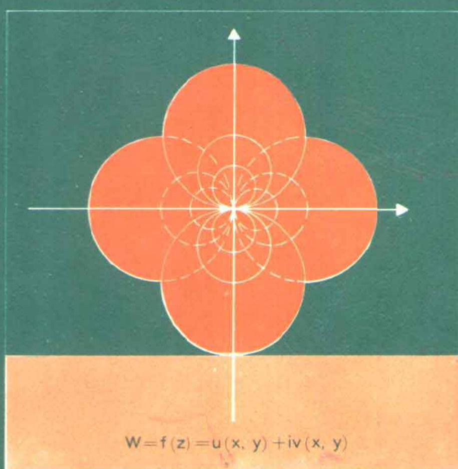
高等工程數學詳解

(1979最新版)第四版 第(三)冊

吳雄明譯著

ADVANCED ENGINEERING MATHEMATICS

Erwin Kreyszig Fourth Edition



久大書局

目　　錄

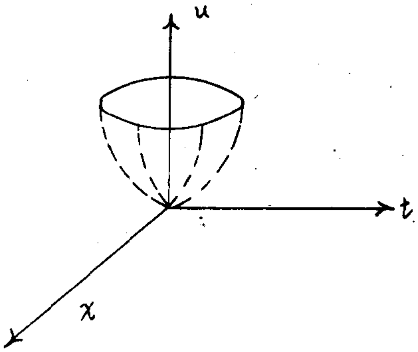
第十一章 偏微分方程

11-1 節 基本觀念

習題 11-1

對於在波方程式(1)中，一適合的常數 c 值，證明下列各函數均爲方程式(1)的解，並畫出這些函數在空間中的曲面。

1. $u = x^2 + t^2$



解：$\dfrac{\partial^2 u}{\partial t^2} = 2$，$\dfrac{\partial^2 u}{\partial x^2} = 2$

當 $c = \pm 1$ 時

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

曲面爲旋轉拋物面。

2. $u = x^2 + 9t^2$

解：$\dfrac{\partial^2 u}{\partial t^2} = 18$，$\dfrac{\partial^2 u}{\partial x^2} = 2$　　　橢圓拋物面

當 $c = \pm 3$ 時　　　（與第 1 題相類似）

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

，曲面爲橢圓拋物面。

3. $u = \cos t \sin x.$

解：$\dfrac{\partial^2 u}{\partial t^2} = -\cos t \sin x$，$\dfrac{\partial^2 u}{\partial x^2} = -\cos t \sin x.$

當 $c = \pm 1$ 時

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

曲面爲水波進行之波面。

4. $u=\sin t\ \sin x.$

解：$\dfrac{\partial^2 u}{\partial t^2}=-\sin t\ \sin x$ ， $\dfrac{\partial^2 u}{\partial x^2}=-\sin t\ \sin x.$

當 $c=\pm 1$ 時 ， $\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2}$

其曲面與第(3)題相似。

5. $u=\cos ct\ \sin x$

解：$\dfrac{\partial^2 u}{\partial t^2}=-c^2\cos ct\ \sin x$ ， $\dfrac{\partial^2 u}{\partial x^2}=-\cos ct\ \sin x$

$\therefore\ \dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2}$ ，其曲面與第(3)題相似。

6. $u=\sin wct\ \ \sin \omega x$

解：$\dfrac{\partial^2 u}{\partial t^2}=-c^2\omega^2\sin \omega ct\ \sin \omega x$ ，

$\dfrac{\partial^2 u}{\partial x^2}=-\omega^2\sin \omega ct\ \sin \omega x$

$\therefore\ \dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2}$ ，其曲面與第(3)題相似。

證明下列函數為(1)式的解，其中 ν 與 ω 為任意二次可微分函數，在第7題中 $c=1$ 。

7. $u(x\cdot t)=\nu(x+t)+\omega(x-t)$

解：$\dfrac{\partial^2 u}{\partial t^2}=\dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial t}\right)=\dfrac{\partial}{\partial t}\left[\dfrac{\partial \nu(x+t)}{\partial t}+\dfrac{\partial \omega(x-t)}{\partial t}\right]$

$$=\frac{\partial}{\partial t}\left[\frac{\partial \nu(x+t)}{\partial(x+t)}\frac{\partial(x+t)}{\partial t}+\frac{\partial \omega(x-t)}{\partial(x-t)}\frac{\partial(x-t)}{\partial t}\right]$$

$$=\frac{\partial}{\partial t}\left[\frac{\partial \nu(x+t)}{\partial(x+t)}-\frac{\partial \omega(x-t)}{\partial(x-t)}\right]$$

$$=\frac{\partial^2 \nu(x+t)}{\partial(x+t)^2}+\frac{\partial^2 \omega(x-t)}{\partial(x-t)^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left[\frac{\partial \nu(x+t)}{\partial x}+\frac{\partial \omega(x-t)}{\partial t}\right]$$

$$=\frac{\partial}{\partial x}\left[\frac{\partial \nu(x+t)}{\partial(x+t)}\frac{\partial(x+t)}{\partial x}+\frac{\partial \omega(x-t)}{\partial(x-t)}\frac{\partial(x-t)}{\partial x}\right]$$

$$=\frac{\partial}{\partial x}\left[\frac{\partial \nu(x+t)}{\partial(x+t)}+\frac{\partial \omega(x-t)}{\partial(x-t)}\right]$$

$$=\frac{\partial^2 \nu(x+t)}{\partial(x+t)^2}+\frac{\partial^2 \omega(x-t)}{\partial(x-t)^2}$$

$$\therefore\quad \frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$$

8. $u(x,t)=\nu(x+ct)+\omega(x-ct)$

解：$$\frac{\partial^2 \mu}{\partial t^2}=\frac{\partial}{\partial t}\left[\frac{\partial \nu(x+ct)}{\partial t}+\frac{\partial \omega(x-ct)}{\partial t}\right]$$

$$=\frac{\partial}{\partial t}\left[\frac{\partial \nu(x+ct)}{\partial(x+ct)}\frac{\partial(x+ct)}{\partial t}+\frac{\partial \omega(x-ct)}{\partial(x-ct)}\frac{\partial(x-ct)}{\partial t}\right]$$

$$=\frac{\partial}{\partial t}\left[\frac{\partial \nu(x+ct)}{\partial(x+ct)}\cdot c+\frac{\partial \omega(x-ct)}{\partial(x-ct)}\cdot(-c)\right]$$

$$=c^2\frac{\partial^2 \nu(x+ct)}{\partial(x+ct)^2}+c^2\frac{\partial^2 \omega(x-ct)}{\partial(x-ct)^2}$$

$$=c^2\left[\frac{\partial^2 \nu(x+ct)}{\partial(x+ct)^2}+\frac{\partial^2 \omega(x-ct)}{\partial(x-ct)^2}\right]$$

同理　$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 \nu(x+ct)}{\partial(x+ct)^2}+\frac{\partial^2 \omega(x-ct)}{\partial(x-ct)^2}$$

$$\therefore\quad \frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

證明下列各函數熱傳方程式(2)的解

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$$

9. $u=e^{-t}\cos x.$

解：$\frac{\partial u}{\partial t}=-e^{-t}\cos x$　　$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial u}{\partial x})=-e^{-t}\cos x.$

當 $c=\pm 1$ 時，　$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$

10. $u=e^{-t}\sin x.$

解：$\frac{\partial u}{\partial t}=-e^{-t}\sin x$　　$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial u}{\partial x})=-e^{-t}\sin x.$

當 $c=\pm 1$ 時，　$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$

11. $u=e^{-\omega^2 c^2 t}\sin\omega x$

解：$\frac{\partial u}{\partial t}=-\omega^2 c^2 e^{-\omega^2 c^2 t}\sin\omega x$

$\frac{\partial^2 u}{x^2}=\frac{\partial}{\partial x}(\frac{\partial u}{\partial x})=-\omega^2 c^2 e^{-\omega^2 c^2 t}\sin\omega x$

$\therefore\ \frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$

證明下列各函數爲拉普氏方程式(3)的解，並畫出這些函數在空間的曲面圖。

12. $u=x^2-y^2$

解：$\frac{\partial^2 u}{\partial x^2}=2$　　$\frac{\partial^2 u}{\partial y^2}=-2$

$\therefore\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$　　爲雙曲拋物面。

13. $u=x^3-3xy^2$

解：$\frac{\partial^2 u}{\partial x^2}=6x$　　$\frac{\partial^2 u}{\partial y^2}=-6x$

$\therefore\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

14. $u=3x^2y-y^3$

解：$\frac{\partial^2 u}{\partial x^2}=6y \qquad \frac{\partial^2 u}{\partial y^2}=-6y$

$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

15. $u=e^x\cos y$

解：$\frac{\partial^2 u}{\partial x^2}=e^x\cos y \qquad \frac{\partial^2 u}{\partial y}=-e^x\cos y$

$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

16. $u=e^x\sin y$

解：$\frac{\partial^2 u}{\partial x^2}=e^x\sin y \qquad \frac{\partial^2 u}{\partial y^2}=-e^x\sin y$

$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

17. $u=ln(x^2+y^2)$

解：$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial u}{\partial x})=\frac{\partial}{\partial x}(\frac{2x}{x^2+y^2})=\frac{2}{x^2+y^2}+\frac{-4x^2}{(x^2+y^2)^2}$

$=\frac{2y^2-2x^2}{(x^2+y^2)^2}$

$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}(\frac{\partial u}{\partial y})=\frac{2x^2-2y^2}{(x^2+y^2)^2}$

$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

18. $u=\arctan(y/x)$

解：$\frac{\partial u}{\partial x}=\frac{-y}{x^2+y^2} \qquad \frac{\partial^2 u}{\partial x^2}=\frac{2xy}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y}=\frac{x}{x^2+y^2} \qquad \frac{\partial^2 u}{\partial y^2}=\frac{-2xy}{(x^2+y^2)}$

$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ ，在一固定 u 値令爲 u_0

則　$\tan u_0 = y/x$

∴在一固定 u_0 值時，平行於 xy 平面所截之圖形為一通過原點之直線，其斜率為 $\tan u_0$.

19. $u = \sin x \sinh y$.

解：$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(\cos x \sinh y) = -\sin x \sinh y$

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(\sin x \cosh y) = \sin x \sinh y$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

20. $u = \sin x \cosh y$.

解：$\frac{\partial^2 u}{\partial x^2} = -\sin x \cosh y$　　$\frac{\partial^2 u}{\partial y^2} = \sin x \cosh y$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

21. 證明函數 $u(x, y) = a \ln(x^2 + y^2) + b$ 滿足拉普拉斯方程式(3)，並求出常數 a, b 使合於在圓周 $x^2 + y^2 = 1$ 上 $u = 0$，及在圓周 $x^2 + y^2 = 4$ 上 $u = 3$ 之邊界條件，並畫出 u 所代表之一種曲面圖形。

解：(a) $\frac{\partial^2 u}{\partial x^2} = \frac{a(2y^2 - 2x^2)}{(x^2 + y^2)}$　　$\frac{\partial^2 u}{\partial y^2} = \frac{a(2x^2 - 2y^2)}{(x^2 + y^2)}$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(b) 若 $x^2 + y^2 = 1$　　$u = 0$

則　$a \ln 1 + b = 0 \Rightarrow b = 0$

若 $x^2 + y^2 = 4$，　$u = 3$

則　$a \ln 4 + b = 3$　　$\ln 4^a = 3$

$a = \frac{3}{2 \ln 2}$，即 $u = \frac{3}{2 \ln 2} \ln(x^2 + y^2)$

22. 證明 $u = 1/\sqrt{x^2 + y^2 + z^2}$ 是拉普拉斯方程式(5)的解。

解：$\dfrac{\partial^2 u}{\partial x^2}=\dfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$

$\dfrac{\partial^2 u}{\partial y^2}=\dfrac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}}$

$\dfrac{\partial^2 u}{\partial z^2}=\dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$

$\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$

與常微分方程式間之關係。

如果一個偏微分方程式僅含有其中某一自變數的導數，則我們可將其他自變數當作參數，而視為一常微分方程式求解，解下列方程式 $u=u(x,y)$.

23. $u_x=0$

解：$u_x=0 \qquad \dfrac{\partial u(x,y)}{\partial x}=0$

y 當作參數，則 $u_x=\dfrac{du(x\cdot y)}{dx}=0 \qquad u(x,y)=f(y)+c.$

24. $u_y=0$

解：$u_y=\dfrac{\partial u(x,y)}{\partial y}=0 \quad\Rightarrow\quad \dfrac{du(x,y)}{dy}=0$

$u(x,y)=f(x)+c$

25. $u_{xx}=0$

解：$u_{xx}=\dfrac{\partial^2 u(x,y)}{\partial x^2}=0 \quad\Rightarrow\quad \dfrac{d^2 u(x,y)}{dx^2}=0$

$u(x,y)=xf(y)+g(y)+c$

26. $u_{xx}+u=0$

解：令 $u(x,y)=ce^{mx}g(y)$ ，$u_{xx}=cm^2e^{mx}g(y)$

$cm^2e^{mx}g(y)+ce^{mx}g(y)=0 \Rightarrow m^2=1 \qquad m=\pm i$

$u(x,y)=ce^{ix}g(y)+ce^{-ix}g(y)$

$=(A\sin x+B\cos x)g(y)$

27. 解下列偏微分方程。

(a) $u_{xx}=0$　　(b) $u_{yy}=0$

解：$u_{xx}=0 \Rightarrow u(x,y)=xf(y)+g(y)+bx$

$u_{yy}=0 \Rightarrow u(x,y)=yF(x)+G(x)+cy$

$u_x=f(y)+b=yF_x(x)+G_x(x)$ ……………(1)

$u_y=F(x)+c=xf_y(y)+g_y(y)$ ……………(2)

由(1) $F_x(x)=a_1$　$F(x)=a_1x+a_2$，$G_x(x)=b$　$G(x)=bx$

由(2) $f_y(y)=b_1$　$f(y)=b_1y+b_2$，$g_y(y)=c$　$g(y)=cy$

$u(x,y)=x(b_1y+b_2)+cy=y(a_1x+a_2)+bx$

$a_1=b_1=a$　　$b_2=b$　　$a_2=c$

得　$u(x,y)=axy+bx+cy+d$

28. $u_x=0$　　$u_y=0$

解：$u_x=0$　　$u(x,y)=f(y)+a_1$

$u_y=0$　　$u(x,y)=g(x)+a_2$

$f(y)+a_1=g(x)+a_2$　　$f(y)=c_1$　$g(x)=c_2$

$u(x,y)=c$

29. $u_{xy}=0$　　$u_{xx}=0$　　$u_{yy}=0$

解：$u_{xx}=0 \Rightarrow u(x,y)=xf(y)+g(y)+a_1x$

$u_{yy}=0 \Rightarrow u(x,y)=yF(x)+G(x)+b_1y$

$u_{xy}=f_y(y)=F_x(x)=0$　　$f(y)=c_1$　　$F(x)=c_2$

$u(x,y)=c_1x+g(y)+a_1x=c_2y+G(x)+b_2y$

得　$u(x,y)=ax+bv+c$

30. $u_{xx}=0$，$u_{xy}=0$

解：$u_{xx}=0 \Rightarrow u(x,y)=xf(y)+g(y)+a_1x$

$u_{xy}=f_y(y)=0$　　$f(y)=c$

$u(x,y)=cx+g(y)+a_1x$

即　$u(x,y)=ax+g(y)+b$

令　$u_x=P$ 解下列各偏微分方程：

31. $u_xy=u_x$

解：$u_x=p \Rightarrow p_y=p$　得 $p=e^y f(x)$

$u_x = f(x)\ e^y$

$u = \int f(x)\, e^y\, dx + g(y) = F(x)\ e^y + g(y) \qquad F(x) = \int f(x)\, dx$

32. $u_{xy} + u_x = 0$

解：∵ $u_x = p \qquad p_y + p = 0 \qquad \dfrac{dp}{dy} = -p$

得 $p = e^{-y} f(x)$

$u_x = f(x)\, e^{-y} \Rightarrow u(x,y) = \int f(x)\, e^{-y} dx + g(y)$

$= F(x)\ e^{-y} + g(y)$

33. $u_{xy} + u_x + x + y + 1 = 0$

解：$u_x = p$ ∴ $p_y + p + x + y + 1 = 0 \Rightarrow p = f(x)\, e^{-y} - x - y$

即 $u_x = f(x)\, e^{-y} - x - y$

$u = \int f(x)\, e^{-y}\, dx - \int x dx - \int y\, dx + c(y)$

$= B(x)\ e^{-y} - \dfrac{1}{2} x^2 - xy + c(y)$

34. 證明在 $z = \text{const}$ 下，若一平面 $z = z(x,y)$ 的水平曲線平行 x 軸，則 z 是微分方程式 $z_x = 0$ 的解，給予例題。

解：∵ $z(x,y) = c$ 且 $z(x,y)$ 的水平曲線平行 x 軸

∴ $z(x,y) = f(y) = c$

即 $z_x = 0$ ，例 $z = y$

35. 證明方程式 $yz_x - xz_y = 0$ 的解 $z = z(x,y)$，必代表旋轉曲面，給予例題。提示：令 $x = \gamma\cos\theta$，$y = \gamma\sin\theta$，證明方程式 $z_\theta = 0$.

解：令 $x = \gamma\cos\theta \qquad y = \gamma\sin\theta$

$z(x,y) = z(\gamma\cos\theta,\ \gamma\sin\theta) = z(\gamma,\theta)$

$z_\theta = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial \theta} = z_x(-\gamma\sin\theta) + z_y(\gamma\cos\theta)$

$= -yz_x + xz_y = -(yz_x - xz_y) = 0$

∴ $z(\gamma,\theta) = z(\gamma)$ 與 θ 無關

即 $z = (x,y)$ 代表一繞 z 軸旋轉之旋轉面

例 $z = x^2 + y^2 = \gamma^2$

11.3 習題

1. 振動繩索內基音之頻率與繩長、張力及每單位長之質量關係如何？

解：$f_n=\dfrac{\lambda_n}{2\pi}=\dfrac{cn\pi}{2\pi\ell}=\dfrac{cn}{2\ell}$　　　$c=\sqrt{\dfrac{T}{\rho}}$

$$\therefore\quad f_n=\frac{n}{2\ell}\sqrt{\frac{T}{\rho}}$$

求振動繩索的位移函數 $u(x,t)$（長度 $\ell=\pi$，兩端固定，且 $c^2=T/\rho=1$）並設最初的速度爲零，其位移如下。

2. $u(x,0)=0.01\sin x.$

解：$0.01\sin x=\sum\limits_{n=1}^{\infty}B_n\sin\dfrac{n\pi}{\ell}x$

$$B_n=\frac{2}{\ell}\int_0^{\ell}0.01\sin x\sin\frac{n\pi x}{\ell}dx$$

$$=\frac{2}{\pi}\int_0^{\pi}0.01\sin x\sin nx\,dx$$

$n=1$ 時 $B_n\neq 0$ 其均爲零.

$B_1=0.01$　　$\because$

$\because\ \dfrac{\partial u}{\partial t}=0$　　　$\therefore\ B_n^*=0$

得　$u(x,t)=0.01\cos\lambda_1 t\sin x$

$\lambda_n=\dfrac{c\pi n}{\ell}$　　　$\lambda_1=\dfrac{\pi}{\pi}=1$

$$u(x,t)=0.01\cos t\sin x$$

3. $f(x)=k\sin 2x.$

解：$B_n=\dfrac{2}{\ell}\displaystyle\int_0^{\ell}f(x)\sin\dfrac{n\pi x}{\ell}dx.$

$$B_n=\frac{2}{\pi}\int_0^{\pi}k\sin 2x\sin nx\,dx$$

除 $n=2$ 外 所有 $B_n=0$

$$B_2=\kappa\frac{2}{\pi}\int_0^\pi \sin^2 2x\,dx=\kappa$$

B_n^* 均爲零 $\lambda_n=cn\pi/\ell=cn\pi/\pi=n.$

$$u(x,t)=\sum B_n\cos\lambda_n t\sin\frac{n\pi x}{\ell}$$

$$=\kappa\cos\lambda_2 t\sin 2x=\kappa\cos 2t\sin 2x$$

4. $f(x)=\kappa(\sin x+\sin 3x)$

解：$B_n=\dfrac{2}{\ell}\displaystyle\int_0^\ell f(x)\sin\frac{n\pi x}{\ell}dx$

$$B_n=\frac{2}{\pi}\int_0^\pi \kappa(\sin x+\sin 3x)\sin nx\,dx$$

除 $n=1$，$n=3$ 外 其他 $B_n=0$

$B_1=\kappa$ $B_3=\kappa$

B_n^* 均爲零 $\lambda_n=cn\pi/\ell=cn\pi/\pi=n$

$$u(x,t)=\sum B_n\cos\lambda_n t\sin\frac{n\pi x}{\ell}$$

$$=\kappa\cos\lambda_1 t\sin x+\kappa\cos\lambda_3 t\sin 3x$$

$$=\kappa\cos t\sin x+\kappa\cos 3t\sin 3x$$

5. $u(x,0)=f(x)=\begin{cases}\dfrac{\kappa x}{a} & 0<x<a\\ \kappa\dfrac{(\pi-x)}{(\pi-a)} & a<x<\pi\end{cases}$

解：$B_n=\dfrac{2}{\ell}\displaystyle\int_0^\ell f(x)\sin\frac{n\pi x}{\ell}dx$

$$=\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$$

$$=\frac{2}{\pi}\int_0^a\frac{\kappa x}{a}\sin nx\,dx+\int_a^\pi\frac{\kappa}{a-\pi}(x-\pi)\sin nx\,dx$$

$$B_{n1}=\frac{2}{\pi}\int_0^a \frac{\kappa x}{a}\sin nx\,dx$$

$$=\frac{2\kappa}{\pi a}\left[-\left(\frac{x}{n}(\cos nx)\right)\Big|_0^a+\frac{1}{n}\int_0^a \cos nx\,dx\right]$$

$$=\frac{2\kappa}{\pi a}\left[-\frac{a}{n}\cos na+\frac{1}{n^2}\sin na\right]$$

$$B_{n2}=\frac{2}{\pi}\int_a^\pi \kappa\left(\frac{x-\pi}{a-\pi}\right)\sin nx\,dx$$

$$=\frac{2\kappa}{\pi(a-\pi)}\int_a^\pi (x-\pi)\sin nx\,dx$$

$$=\frac{2\kappa}{\pi(a-\pi)}\left[(x-\pi)\left(-\frac{1}{n}\cos nx\right)\Big|_a^\pi+\frac{1}{n}\int_a^\pi \cos nx\,dx\right]$$

$$=\frac{2\kappa}{\pi(a-\pi)}\left[\frac{a-\pi}{n}\cos na-\frac{1}{n^2}\sin na\right]$$

$$B_n=B_{n1}+B_{n2}=\frac{2\kappa}{\pi}\left[-\frac{\cos na}{n}+\frac{\sin na}{an^2}+\frac{\cos na}{n}-\frac{\sin na}{n^2(a-\pi)}\right]=\frac{2\kappa}{a(\pi-a)}\frac{\sin na}{n^2}$$

$$B_n^*=0$$

$$u(x,t)=\sum_{n=1}^{\infty}B_n\cos\lambda_n t\sin nx \qquad \because \lambda_n=n$$

$$u(x,t)=\sum_{n=1}^{\infty}B_n\cos nt\sin nx.$$

6. $$u(x,0)=f(x)=\begin{cases}\dfrac{x}{10} & 0<x<\dfrac{\pi}{4}\\[2mm] \dfrac{1}{20}(\pi-2x) & \dfrac{\pi}{4}<x<\dfrac{3}{4}\pi\\[2mm] \dfrac{1}{10}(x-\pi) & \dfrac{3}{4}\pi<x<\pi\end{cases}$$

解：$B_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$$=\frac{2}{\pi}\left[\int_0^{\pi/4}\frac{x}{10}\sin nx\,dx+\int_{\pi/4}^{3\pi/4}\frac{1}{20}(\pi-2x)\sin nx\,dx\right.$$

$$\left.+\int_{3\pi/4}^{\pi}\frac{1}{10}(x-\pi)\sin nx\,dx\right]$$

$$B_{n1}=\frac{2}{\pi}\int_0^{\pi/4}\frac{x}{10}\sin nx\,dx$$

$$=\frac{1}{5\pi}\left[-\frac{\pi}{4n}\cos\frac{n\pi}{4}+\frac{1}{n^2}\sin\frac{n\pi}{4}\right]$$

$$B_{n2}=\frac{2}{\pi}\int_{\pi/4}^{3\pi/4}\frac{1}{20}(\pi-2x)\sin nx\,dx$$

$$=\frac{1}{10\pi}\left[\frac{\pi}{2n}\cos\frac{3n\pi}{4}+\frac{\pi}{2n}\cos\frac{n\pi}{4}-\frac{2}{n^2}\sin\frac{3n\pi}{4}\right.$$

$$\left.+\frac{2}{n^2}\sin\frac{n\pi}{4}\right]$$

$$B_{n3}=\frac{2}{\pi}\int_{3\pi/4}^{\pi}\frac{1}{10}(x-\pi)\sin nx\,dx$$

$$=\frac{1}{5\pi}\left[-\frac{\pi}{4n}\cos\frac{3n\pi}{4}-\frac{1}{n^2}\sin\frac{3n\pi}{4}\right]$$

$$B_n=B_{n1}+B_{n2}+B_{n3}=\frac{-4}{5\pi}\frac{1}{n^2}\left[\cos\frac{n\pi}{2}\sin\frac{n\pi}{4}\right]$$

$$B_n^*=0$$

$$\therefore u(x,t)=\sum_{n=1}^{\infty}B_n\cos\lambda_n+\sin nx$$

$$=\sum_{n=1}^{\infty}-\frac{4}{5\pi}\frac{1}{n^2}\left[\cos\frac{n\pi}{2}\sin\frac{n\pi}{4}\right]\cos nt\sin nx$$

7. $u(x,0)=f(x)=\begin{cases} 0 & 0<x<\frac{\pi}{4} \\ \frac{4\kappa}{\pi}(x-\frac{\pi}{4}) & \frac{\pi}{4}<x<\frac{\pi}{2} \\ \frac{4\kappa}{\pi}(\frac{3\pi}{4}-x) & \frac{\pi}{2}<x<\frac{3}{4}\pi \\ 0 & \frac{3\pi}{4}<x<\pi \end{cases}$

解：$B_n=\frac{2}{\pi}\int_0^{\pi} f(x)\ \sin\ nxdx$

$$=\frac{2}{\pi}[\int_{\pi/4}^{\pi/2}\frac{4\kappa}{\pi}(x-\frac{\pi}{4})\sin nxdx$$

$$+\int_{\pi/2}^{3\pi/4}\frac{4\kappa}{\pi}(\frac{3\pi}{4}-x)\sin nxdx]$$

$$B_{n1}=\frac{2}{\pi}\int_{\pi/4}^{\pi/2}\frac{4\kappa}{\pi}(x-\frac{\pi}{4})\sin\ nxdx$$

$$=\frac{8\kappa}{\pi^2}[-\frac{\pi}{4n}\cos\frac{n\pi}{2}+\frac{1}{n^2}(\sin\frac{n\pi}{2}-\sin\frac{n\pi}{4})]$$

$$B_{n2}=\frac{2}{\pi}\int_{\pi/2}^{3\pi/4}\frac{4\kappa}{\pi}(\frac{3\pi}{4}-x)\sin\ nxdx$$

$$=\frac{8\kappa}{\pi^2}[\frac{\pi}{4n}\cos\frac{n\pi}{2}-\frac{1}{n^2}\sin\frac{3}{4}n\pi+\frac{1}{n^2}\sin\frac{n\pi}{2}]$$

$$B_n=B_{n1}+B_{n2}=\frac{8\kappa}{\pi^2}[\frac{2}{n^2}\sin\frac{n\pi}{2}-\frac{1}{n^2}\sin\frac{n\pi}{4}-\frac{1}{n^2}\sin\frac{3n\pi}{4}]$$

$$B_1=\frac{8\kappa}{\pi^2}[2-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}]=\frac{8\kappa}{\pi^2}(2-\sqrt{2})$$

$$B_2=\frac{8\kappa}{\pi^2}[-\frac{1}{4}+\frac{1}{4}]=0$$

$$B_3=-\frac{8\kappa}{\pi^2}(\frac{2}{9}-\frac{\sqrt{2}}{9})$$

$$B_4 = 0$$

$$\vdots$$

$$\therefore\ u(x,t)=\sum_{n=1}^{\infty} B_n \cos\lambda_n t\ \sin nx$$

$$=\frac{8\kappa}{\pi^2}[(2-\sqrt{2})\cos t\ \sin x$$

$$-\frac{1}{9}(2+\sqrt{2})\cos 3t\ \sin 3x+\cdots\cdots]$$

8. $u(x,0)=f(x)=0.01x(\pi-x)$

解：$B_n=\frac{2}{\pi}\int_0^{\pi}0.01\,x(\pi-x)\sin nx\,dx$

$$=\frac{0.02}{\pi}[-\frac{x(\pi-x)}{n}\cos nx\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}(\pi-2x)\cos nx\,dx]$$

$$=\frac{0.02}{\pi}\cdot\frac{1}{n}[\frac{\pi-2x}{n}\sin nx\Big|_0^{\pi}+\frac{2}{n}\int_0^{\pi}\sin nx\,dx]$$

$$=\frac{-0.04}{n^3\pi}[(-1)^n-1]$$

$$\therefore u(x,t)=\sum_{n=1}^{\infty}B_n\cos\lambda_n t\sin nx$$

$$=\frac{0.08}{\pi}[\cos t\ \sin x+\frac{1}{3^3}\cos 3t\sin 3x$$

$$+\frac{1}{5^3}\cos 5t\sin 5x+\cdots\cdots]$$

9. $u(x,0)=f(x)=0.01x(\pi^2-x^2)$

解：$B_n=\frac{2}{\pi}\int_0^{\pi}0.01x(\pi^2-x^2)\sin nx\,dx$

$$=\frac{2}{\pi}\times 0.01[\int_0^{\pi}x(\pi^2-x^2)\sin nx\,dx]$$

$$=\frac{0.02}{\pi}[-\frac{x\pi^2-x^3}{n}\cos nx\Big|_0^{n}+\frac{1}{n}\int_0^{\pi}(\pi^2-3x^2)\cos nx\,dx]$$