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Jacobi谱方法及其对奇异问题、无界区域问题和轴对称区域问题的应用

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Shanghai University Doctoral Dissertation (2000)

Jacobi Spectral Methods and Their Applications to Singular Problems, Unbounded Domains and Axisymmetric Domains

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Shanghai University Press

• Shanghai •

上 海 大 学

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答辩委员会对论文的评语

谱方法和拟谱方法是数值求解偏微分方程的主要工具之一。但通常的方法适用奇异问题。因此，奇异问题的谱与拟谱方法是一个前沿和困难问题。不仅具有理论意义，并且具有广阔的应用前景。

本文研究奇异问题的拟谱方法(某些无界区域问题和轴对称区域问题等)。它涉及函数逼近论和微分方程数值解等有关方面。结果是完整深入的。特别在 Jacobi 多项式函数逼近理论方面有创新。

答辩委员会认为作者基础扎实，有较强的科研工作能力。本文是一篇优秀的博士论文。经过投票一致通过答辩，并建议授予博士学位。

答辩委员会表决结果

经答辩委员会表决，一致同意通过博士论文答辩，并建议授予博士学位。

答辩委员会主席：姜礼尚

2000 年 3 月 16 日

摘 要

近十多年来，谱方法蓬勃地发展起来。它为数值求解偏微分方程提供了又一强有力的工具。其主要的优点是高精度，从而被广泛应用于计算流体力学、数值天气预报、化学反应数值模拟和生物学计算等领域。已有的谱方法多适应于有界区域上的非奇异问题，但许多实际问题是奇异问题或无界区域问题。这类问题的主要困难是解的奇异性和区域的无界性。因此，探索有效的高精度数值算法成为当前国际上谱方法研究的热点和难点。

本文将利用 Jacobi 多项式或以 Jacobi 多项式零点为节点的插值基函数来逼近奇异解，并建立有关的新的带权函数空间投影理论、Jacobi-Gauss 型求积和 Jacobi 插值逼近理论，这些构成了 Jacobi 谱方法和拟谱方法（包括一维和多维）的理论基础。这一方法被应用于奇异问题的数值解。同时，又能求解无界区域问题。事实上，通过适当的变量代换，我们可把无界区域问题转化成某些特定的退化系数的有界区域上的奇异问题。

另一方面，在极坐标或圆柱坐标下，轴对称区域问题也表现某类奇异问题。本文在建立 Jacobi 谱方法和拟谱方法数值分析理论的基础上，数值求解这三类问题。考虑了一维和二维的线性方程和非线性方程，并证明了相应的 Jacobi 谱格式和拟谱格式的稳定性和收敛性。同时，通过选取适当的 Jacobi 基函数，使离散格式得到的线性方程组的系数矩阵稀疏、对称。从而，使

Jacobi 谱方法在实际应用中更加有效。数值结果证实了理论分析的结果，也显示了这一方法的优越性。

本文针对奇异问题数值解进行研究，同时，为解决无界区域问题、轴对称区域问题及工程计算中众多困难问题提供了新的有效工具。Jacobi 谱方法和拟谱方法数值分析理论的研究极大地丰富了现有谱方法的理论基础，同时将发展泛函分析、数值逼近论和偏微分方程数值解的相关理论。

关键词 Jacobi 谱方法，Jacobi 拟谱方法，奇异问题，无界区域问题，轴对称区域问题

Abstract

In this dissertation, the theory of the Jacobi spectral methods and their applications to singular problems, unbounded domains and axisymmetric domains are studied. Some efficient algorithms to implement the Jacobi spectral methods are constructed. The main context of this dissertation consists of three parts.

Firstly, we establish the Jacobi interpolation approximations in weighted Sobolev spaces and certain Hilbert spaces. They are the theoretical foundation of the Jacobi pseudospectral methods. As indispensable tools, some weighted inverse inequalities and imbedding inequalities are given. Furthermore, various unusual Jacobi orthogonal projections are also included. The Jacobi pseudospectral methods are applied to numerical solutions of singular differential equations, differential equations in infinite intervals and in disc. Some numerical are presented to show the efficiency of these new approaches.

Secondly, we develop multiple-dimensional Jacobi polynomial approximations in non-isotropic Hilbert spaces. They are used to numerical solutions of multiple-dimensional singular partial differential equations with different singularities, such as degenerating coefficients, singular boundary values and singular source terms. They are also applied to unbounded domains, such as the whole plane, the half plane and some infinite straps. The Jacobi spectral methods are also available to problems in axisymmetric domains. We propose some efficient algorithms to implement the Jacobi spectral method. It makes this new approach more preferable.

On the other hand, we investigate multiple-dimensional, non-isotropic Jacobi interpolation approximations. In particular, we consider three hybrid interpolations, which are useful for problems in different unbounded domains. We take the nonlinear Klein-Gordon equation in an infinite strap and in a cylinder as examples to illustrate how to deal with nonlinear problems by using Jacobi pseudospectral methods.

Most of the ideas and techniques in this dissertation can be used to explore other new spectral methods.

Key words Jacobi spectral methods, Jacobi pseudospectral methods, singular problems, differential equations in unbounded domains, differential equations in axisymmetric domains

CONTENTS

Chapter 1
Introduction

1.1 Motivation of the research

Spectral method employs global polynomials as trial functions for the discretization of the partial differential equations (PDEs). So it often provides very accurate approximations to the exact solutions with relatively less degrees of freedom. In the past two decades, the spectral method has gained increasing popularity in numerical simulations in many fields, such as fluid dynamics, quantum mechanics, numerical weather-prediction and so on. Nowadays, it has become one of the most important tools for numerical solutions of PDEs, and has been successfully used for computations in science and engineering, see, e.g., Gottlieb and Orsag [1], Canuto, Hussaini, Quarteroni and Zang [2], Bernardi and Maday [3], and Guo [4].

In contrast to finite difference and finite element methods, the fascinating merit of the spectral method is the high accuracy, i.e., the so called convergence of "infinite order". It means that the convergence rates of discrete solutions to exact solutions increase as the the regularity of the exact solutions increase. Unfortunately, this merit might be destroyed by some facts: (1) instability of nonlinear computations, (2) discontinuity of data, (3) unboundness of domains, (4) singularity of solutions. Some techniques have been proposed to overcome the first three difficulties. First of all, Kreiss and

Oliger[5], Gottlieb and Turkel[6], Kuo[7], Vandeven[8], Tadmor[9] and Guo[10] provided various filterings to weaken the instability in nonlinear computations. Next, Cai, Gottlieb and Shu[11, 12] developed certain essentially nonoscillatory approximations and one-side filters for fitting discontinuous data. In particular, Gottlieb, Shu, Solomonoff and Vandeven[13] and Gottlieb and Shu[14-17] recovered the spectral accuracy by using Gegenbauer approximation. On the other hand, Maday, Pernaud-Thomas and Vandeven[18], Funaro and Kavian[19], Iranzo and Falquès[20], Guo[21], Guo and Shen[22] used spectral methods associated with some orthogonal systems in unbounded domains. But so far, there is little work concerning spectral method for the singular problems.

The singularity of solutions could be caused by several factors, such as degenerating of coefficients, unboundness of data, corners of domains and so on. For instance, we consider a simple equation with perturbed ellipticity (see [25])

$$-(\rho_1(x)U'(x))' + \rho_0(x)U(x) = f(x), \quad x \in \Lambda = (-1,1) \qquad (1.1)$$

where $\rho_1(x) = (1-x)^\alpha(1+x)^\beta$ and $\rho_0(x) = (1-x)^\gamma(1+x)^\delta$ with $\alpha, \beta, \gamma, \delta \geq 0$. Problem (1.1) with suitable boundary value appear in many areas of applied mathematics and physics, such as transport processes (Ames [26]), thermal explosions (Chamber [27]), electrohydrodynamics (Keller [28]) and many others. There are some literatures on finite difference and finite element methods for numerical solutions of such problems, see, e.g., [29, 30, 31] and the references therein. However, a natural way for solving problem (1.1) numerically is to use the Jacobi approximations. The main idea is to fit singular solutions by Jacobi polynomials, to compare numerical solutions with some unusual orthogonal projections of exact solutions, and to measure the errors in certain Hilbert spaces which the exact solutions belong to. The advantages of this new approach are apparent. In fact, the Jacobi polynomials $J_l^{(\alpha,\beta)}(x)$ satisfy a singular Sturm-Liouville equation (see Chapter 2). Moreover, $J_l^{(\alpha,\beta)}(1) \sim l^\alpha$ and $J_l^{(\alpha,\beta)}(-1) \sim |l|^\beta$. These properties make it possible to fit singular solutions well. The Jacobi spectral method is also applicable to problems with singular boundary value, singular source term and other related problems.

The pioneering work was initialized by Guo Ben-yu in his publications [23] and [24] for one-dimensional singular problems. But in actual computations, the pseudospectral method is more preferable, since it saves work. In particular, it is easier to deal with nonlinear terms. But up to now, there is no paper concerning pseudospectral methods for such problems. One of the motivations of this dissertation is to develop pseudospectral methods for singular problems and other related problems.

A number of physical problems are set in unbounded domains. Some conditions at infinity are given by certain asymptotic behaviors of solutions. There are several ways for solving such problems. The simplest one is to restrict calculations to some bounded domains, impose certain artificial boundary conditions, and then resolve the approximate problems by the usual finite difference methods, finite element methods or spectral methods. However, this treatment causes additional errors. The second way is to use spectral methods associated with some orthogonal systems in unbounded domains, see, e.g., [18-22]. But this approach requires some quadratures over infinite intervals, which also cause errors. In fact, if we make some suitable variable transformations, then the original problems may become some singular problems in bounded