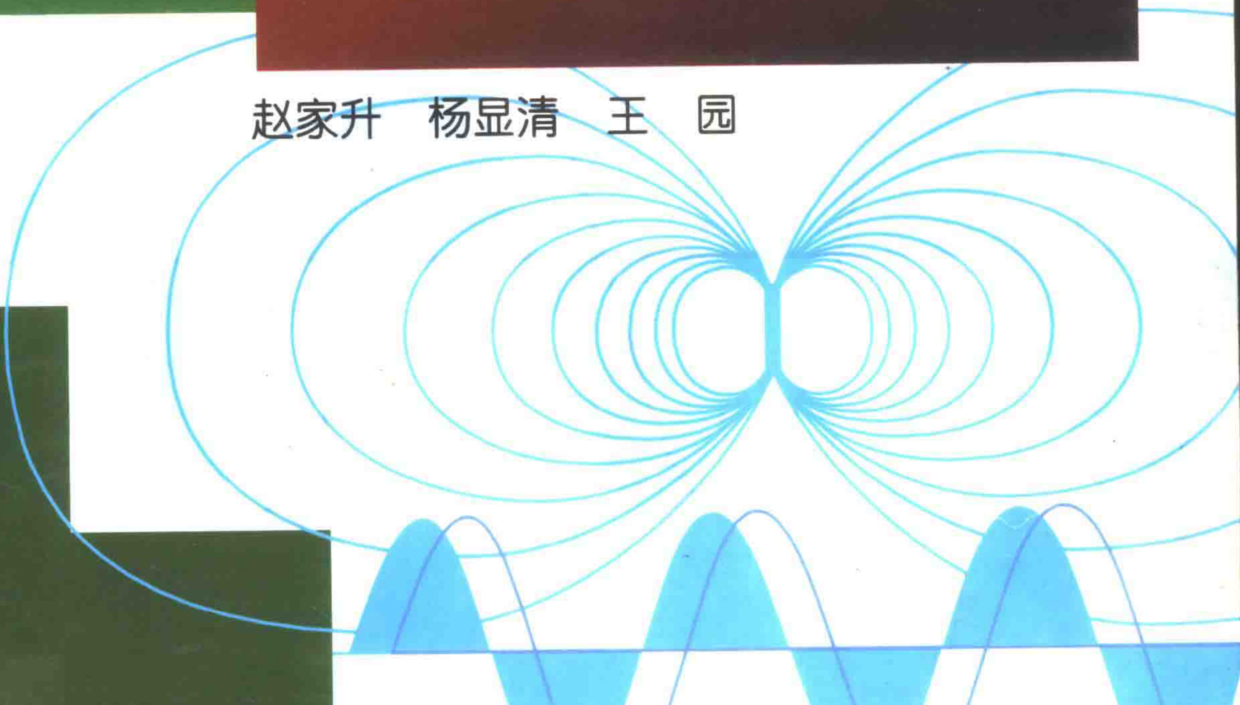
DIANCICHANG YU DIANCIBO JIETI ZHIDAO

电磁场与电磁波

解题指导

赵家升　杨显清　王　园



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内容简介

本书选编了378道题，内容包括矢量分析、麦克斯韦方程、平面电磁波、平面波的反射和透射、波导与谐振腔、传输线、天线、电磁波的其它论题、静态场、边值问题等十章。解题思路清晰、论证严密、方法得当、计算结果准确是本书的特点。

本书适合于学习“电磁场理论”、“电磁场与电磁波”等课程的学生作辅助教材，也可供考硕士研究生的读者以及教师和科技人员参考。

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赵家升　杨显清　王　园

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前　言

学习电磁场与电磁波理论，解题是重要的环节之一，也是学习中的难点所在。通过解题，能加深和巩固对基本理论的理解，能培养学生理论联系实际的能力，建立清晰的概念，掌握分析和计算技巧。本书就是着眼于解决这个既重要、又困难的问题而编写的，期望通过阅读本书能给学习者在解题思路和解题方法上带来一些示范和启迪。

本书对按电子部《1996—2000年全国电子信息类专业教材编审出版规划》出版的《电磁场与波》一书的习题共252道作了详细解答。包括矢量分析、麦克斯韦方程、平面电磁波、平面波的反射和透射、波导与谐振腔、传输线、天线、电磁波的其它论题、静态场以及边值问题等十章。另外，为了加深和加宽解题范围，以适应了解和掌握较复杂问题求解方法的要求，在第二、三、四、五、九、十章，增加了补充题共126道，也作了解答。这些补充题中的一部分选自历届硕士研究生相关科目的入学试题。

本书适合于学习"电磁场理论"、"电磁场与电磁波"等课程的学生作辅助教材，对报考电磁场与微波技术专业研究生的读者也是一本很好的参考书，也可供从事相关课程教学工作的教师参考。

本书第一、二、九、十章由杨显清副教授编写，第三、四、五、六、七章由王园副教授编写，第八章由赵家升教授编写。全书由赵家升教授审核定稿。本书的出版得到电子科技大学出版社、电子科技大学教务处教材科的支持；本书责任编辑杜倩也为本书付出了辛勤劳动，编者一并致以谢意。

限于编者水平，书中有不当之处，请不吝指正。

编　者
1999年3月于电子科技大学
微波工程系

目　　录

此为试读，需要完整PDF请访问：www.ertongbook.com

第一章 矢量分析

1-1 给定三个矢量 $\boldsymbol{A}$、$\boldsymbol{B}$ 和 $\boldsymbol{C}$ 如下：

$\boldsymbol{A}=\boldsymbol{a}_x+\boldsymbol{a}_y2-\boldsymbol{a}_z3$

$\boldsymbol{B}=-\boldsymbol{a}_x4+\boldsymbol{a}_z$

$\boldsymbol{C}=\boldsymbol{a}_x5-\boldsymbol{a}_z2$

求：(1)$\boldsymbol{a}_A$；(2)$|\boldsymbol{A}-\boldsymbol{B}|$；(3)$\boldsymbol{A}\cdot\boldsymbol{B}$；(4)$\theta_{AB}$；(5)$\boldsymbol{A}$ 在 $\boldsymbol{C}$ 方向上的分量；(6)$\boldsymbol{A}\times\boldsymbol{C}$；(7)$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$ 和$(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}$；(8)$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}$ 和 $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})$。

解：(1)$\boldsymbol{a}_A=\dfrac{\boldsymbol{A}}{|\boldsymbol{A}|}=\dfrac{\boldsymbol{a}_x+\boldsymbol{a}_y2-\boldsymbol{a}_z3}{\sqrt{1^2+2^2+(-3)^2}}=\boldsymbol{a}_x\dfrac{1}{\sqrt{14}}+\boldsymbol{a}_y\dfrac{2}{\sqrt{14}}-\boldsymbol{a}_z\dfrac{3}{\sqrt{14}}$

(2)$|\boldsymbol{A}-\boldsymbol{B}|=|(\boldsymbol{a}_x+\boldsymbol{a}_y2-\boldsymbol{a}_z3)-(-\boldsymbol{a}_x4+\boldsymbol{a}_z)|$

$=|\boldsymbol{a}_x5+\boldsymbol{a}_y2-\boldsymbol{a}_z4|=\sqrt{5^2+2^2+(-4)^2}=3\sqrt{5}$

(3)$\boldsymbol{A}\cdot\boldsymbol{B}=(\boldsymbol{a}_x+\boldsymbol{a}_y2-\boldsymbol{a}_z3)\cdot(-\boldsymbol{a}_x4+\boldsymbol{a}_z)=-7$

(4)因 $\boldsymbol{A}\cdot\boldsymbol{B}=|\boldsymbol{A}||\boldsymbol{B}|\cos\theta_{AB}$

故 $\cos\theta_{AB}=\dfrac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|}=\dfrac{-7}{\sqrt{14}\times\sqrt{17}}=\dfrac{-7}{\sqrt{238}}$

$\theta_{AB}=\cos^{-1}\left(\dfrac{-7}{\sqrt{238}}\right)=116.98°$

(5)设 $\boldsymbol{A}$ 在 $\boldsymbol{C}$ 方向上的分量为 A_C，则

$|A_C|=|\boldsymbol{A}||\cos\theta_{AC}|=|\boldsymbol{A}|\dfrac{|\boldsymbol{A}\cdot\boldsymbol{C}|}{|\boldsymbol{A}||\boldsymbol{C}|}=\dfrac{|\boldsymbol{A}\cdot\boldsymbol{C}|}{|\boldsymbol{C}|}$

所以 $A_C=|A_C|\dfrac{\boldsymbol{C}}{|\boldsymbol{C}|}=\dfrac{|\boldsymbol{A}\cdot\boldsymbol{C}|}{|\boldsymbol{C}|^2}\boldsymbol{C}=\dfrac{11}{29}(\boldsymbol{a}_x5-\boldsymbol{a}_z2)$

(6)$\boldsymbol{A}\times\boldsymbol{C}=\begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ 1 & 2 & -3\\ 5 & 0 & -2\end{vmatrix}=-\boldsymbol{a}_x4-\boldsymbol{a}_y13-\boldsymbol{a}_z10$

(7)$\boldsymbol{B}\times\boldsymbol{C}=-\boldsymbol{a}_y3$

所以 $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{a}_x+\boldsymbol{a}_y2-\boldsymbol{a}_z3)\cdot(-\boldsymbol{a}_y3)=-6$

$\boldsymbol{A}\times\boldsymbol{B}=\boldsymbol{a}_x2+\boldsymbol{a}_y11+\boldsymbol{a}_z8$

所以 $(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}=(\boldsymbol{a}_x2+\boldsymbol{a}_y11+\boldsymbol{a}_z8)\cdot(\boldsymbol{a}_x5-\boldsymbol{a}_z2)=-6$

(8)$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}=(\boldsymbol{a}_x2+\boldsymbol{a}_y11+\boldsymbol{a}_z8)\times(\boldsymbol{a}_x5+\boldsymbol{a}_z2)$

$=-\boldsymbol{a}_x22+\boldsymbol{a}_y44-\boldsymbol{a}_z55$

$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{a}_x+\boldsymbol{a}_y2-\boldsymbol{a}_z3)\times(-\boldsymbol{a}_y3)=-\boldsymbol{a}_x9-\boldsymbol{a}_z3$

1-2 求标量场 $u=xy+yz+zx$ 在点 $M(1,2,3)$处沿矢径的方向导数。

解：在点 $M(1,2,3)$处

$\nabla u|_M=\left(\boldsymbol{a}_x\dfrac{\partial u}{\partial x}+\boldsymbol{a}_y\dfrac{\partial u}{\partial y}+\boldsymbol{a}_z\dfrac{\partial u}{\partial z}\right)_M$

$$=\boldsymbol{a}_x 5+\boldsymbol{a}_y 4+\boldsymbol{a}_z 3$$

点 $M(1,2,3)$ 处沿矢径方向的单位矢量

$$\boldsymbol{a}_l=\frac{\boldsymbol{a}_x+\boldsymbol{a}_y 2+\boldsymbol{a}_z 3}{\sqrt{1^2+2^2+3^2}}=\boldsymbol{a}_x\frac{1}{\sqrt{14}}+\boldsymbol{a}_y\frac{2}{\sqrt{14}}+\boldsymbol{a}_z\frac{3}{\sqrt{14}}$$

故所求方向导数为

$$\left.\frac{\partial u}{\partial l}\right|_M=(\nabla u\cdot\boldsymbol{a}_l)_M=\frac{22}{\sqrt{14}}$$

1-3 设 $\boldsymbol{r}=\boldsymbol{a}_x x+\boldsymbol{a}_y y+\boldsymbol{a}_z z, r=|\boldsymbol{r}|$。(1)求 $\nabla\left(\frac{1}{r}\right),\nabla\left(\frac{1}{r^3}\right),\nabla f(r)$;(2)证明 $\nabla(\boldsymbol{k}\cdot\boldsymbol{r})=\boldsymbol{k}$($\boldsymbol{k}$ 为一常矢量)。

解:(1) $\nabla\left(\frac{1}{r}\right)=\left(\frac{1}{r}\right)'\nabla r=-\frac{1}{r^2}\cdot\frac{\boldsymbol{r}}{r}=-\frac{\boldsymbol{r}}{r^3}$

$$\nabla\left(\frac{1}{r^3}\right)=\left(\frac{1}{r^3}\right)'\nabla r=-\frac{3}{r^4}\cdot\frac{\boldsymbol{r}}{r}=-\frac{3\boldsymbol{r}}{r^5}$$

$$\nabla f(r)=f'(r)\nabla r=f'(r)\frac{\boldsymbol{r}}{r}$$

(2) 设 $\boldsymbol{k}=\boldsymbol{a}_x k_x+\boldsymbol{a}_y k_y+\boldsymbol{a}_z k_z$,则

$$\boldsymbol{k}\cdot\boldsymbol{r}=k_x x+k_y y+k_z z$$

所以 $\nabla(\boldsymbol{k}\cdot\boldsymbol{r})=\left(\boldsymbol{a}_x\frac{\partial}{\partial x}+\boldsymbol{a}_y\frac{\partial}{\partial y}+\boldsymbol{a}_z\frac{\partial}{\partial z}\right)(k_x x+k_y y+k_z z)$

$$=\boldsymbol{a}_x k_x+\boldsymbol{a}_y k_y+\boldsymbol{a}_z k_z=\boldsymbol{k}$$

1-4 已知标量函数 $u=x^2+2y^2+3z^2+3x-2y-6z$。(1)求 ∇u;(2)在哪些点上 ∇u 等于0。

解:(1) $\nabla u=\boldsymbol{a}_x\frac{\partial u}{\partial x}+\boldsymbol{a}_y\frac{\partial u}{\partial y}+\boldsymbol{a}_z\frac{\partial u}{\partial z}$

$$=\boldsymbol{a}_x(2x+3)+\boldsymbol{a}_y(4y-2)+\boldsymbol{a}_z(6z-6)$$

(2) 令 $\nabla u=0$,则有 $2x+3=0,4y-2=0,6z-6=0$,所以

$$x=-\frac{3}{2},y=\frac{1}{2},z=1$$

即在点 $\left(-\frac{3}{2},\frac{1}{2},1\right)$ 处 $\nabla u=0$。

1-5 利用直角坐标系,证明 $\nabla(uv)=v\nabla u+u\nabla v$。

解:$\nabla(uv)=\boldsymbol{a}_x\frac{\partial}{\partial x}(uv)+\boldsymbol{a}_y\frac{\partial}{\partial y}(uv)+\boldsymbol{a}_z\frac{\partial}{\partial z}(uv)$

$$=\boldsymbol{a}_x\left(u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)+\boldsymbol{a}_y\left(u\frac{\partial v}{\partial y}+v\frac{\partial u}{\partial y}\right)+\boldsymbol{a}_z\left(u\frac{\partial v}{\partial z}+v\frac{\partial u}{\partial z}\right)$$

$$=u\left(\boldsymbol{a}_x\frac{\partial v}{\partial x}+\boldsymbol{a}_y\frac{\partial v}{\partial y}+\boldsymbol{a}_z\frac{\partial v}{\partial z}\right)+v\left(\boldsymbol{a}_x\frac{\partial u}{\partial x}+\boldsymbol{a}_y\frac{\partial u}{\partial y}+\boldsymbol{a}_z\frac{\partial u}{\partial z}\right)$$

$$=u\nabla v+v\nabla u$$

1-6 设 $\boldsymbol{A}=\boldsymbol{a}_x 4x+\boldsymbol{a}_y 2xy+\boldsymbol{a}_z z^2,\boldsymbol{B}=\boldsymbol{a}_x z+\boldsymbol{a}_y 2x^2+\boldsymbol{a}_z 3$,在点 $M(1,1,3)$ 处,求 $\nabla\cdot\boldsymbol{A}$,

$\nabla \cdot \boldsymbol{B}$和$\nabla \cdot (\boldsymbol{A}\times\boldsymbol{B})$。

解:$\nabla \cdot \boldsymbol{A}|_M=(4+2x+2z)_M=12$

$\nabla \cdot \boldsymbol{B}|_M=0$

$$\begin{aligned}\nabla \cdot (\boldsymbol{A}\times\boldsymbol{B})|_M &=\nabla \cdot [(\boldsymbol{a}_x 4x+\boldsymbol{a}_y 2xy+\boldsymbol{a}_z z^2)\times(\boldsymbol{a}_x z+\boldsymbol{a}_y 2x^2+\boldsymbol{a}_z 3)]_M \\ &=\nabla \cdot [\boldsymbol{a}_x(6xy-2x^2z^2)+\boldsymbol{a}_y(z^3-12x)+\boldsymbol{a}_z(8x^3-2xyz)]_M \\ &=(6y-4xz^2-2xy)_M=-32\end{aligned}$$

1-7 已知$\boldsymbol{A}=\boldsymbol{a}_x x^2yz+\boldsymbol{a}_y xy^2z+\boldsymbol{a}_z xyz^2$,求$\nabla \cdot \boldsymbol{A}$

解:
$$\begin{aligned}\nabla \cdot A &=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z} \\ &=2xyz+2xyz+2xyz=6xyz\end{aligned}$$

1-8 设$\boldsymbol{r}=\boldsymbol{a}_x x+\boldsymbol{a}_y y+\boldsymbol{a}_z z$,$r=|\boldsymbol{r}|$,$\boldsymbol{E}_0$和$\boldsymbol{k}$为常矢量。(1)求:$\nabla \cdot \boldsymbol{r}$,$\nabla \cdot \left(\frac{\boldsymbol{r}}{r}\right)$,$\nabla \cdot (r\boldsymbol{k})$,$\nabla \cdot [\boldsymbol{E}_0\sin(\boldsymbol{k}\cdot\boldsymbol{r})]$;(2)若$\nabla \cdot [f(r)\boldsymbol{r}]=0$,那么函数$f(r)$会有什么特点呢?

解:(1) $\nabla \cdot \boldsymbol{r}=\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}=3$

$$\nabla \cdot \left(\frac{\boldsymbol{r}}{r}\right)=\boldsymbol{r}\cdot\nabla\left(\frac{1}{r}\right)+\frac{1}{r}\nabla \cdot \boldsymbol{r}=-\frac{\boldsymbol{r}\cdot\boldsymbol{r}}{r^3}+\frac{3}{r}=\frac{2}{r}$$

$$\nabla \cdot (r\boldsymbol{k})=r\nabla \cdot \boldsymbol{k}+\boldsymbol{k}\cdot\nabla r=\boldsymbol{k}\cdot\frac{\boldsymbol{r}}{r}$$

$$\begin{aligned}\nabla \cdot [\boldsymbol{E}_0\sin(\boldsymbol{k}\cdot\boldsymbol{r})] &=\boldsymbol{E}_0\cdot\nabla[\sin(\boldsymbol{k}\cdot\boldsymbol{r})]+\nabla \cdot \boldsymbol{E}_0\sin(\boldsymbol{k}\cdot\boldsymbol{r}) \\ &=\boldsymbol{E}_0\cdot\cos(\boldsymbol{k}\cdot\boldsymbol{r})\nabla(\boldsymbol{k}\cdot\boldsymbol{r})=\boldsymbol{k}\cdot\boldsymbol{E}_0\cos(\boldsymbol{k}\cdot\boldsymbol{r})\end{aligned}$$

(2)
$$\begin{aligned}\nabla \cdot [f(r)\boldsymbol{r}] &=\boldsymbol{r}\cdot\nabla f(r)+f(r)\nabla \cdot \boldsymbol{r}=f'(r)\boldsymbol{r}\cdot\nabla r+3f(r) \\ &=f'(r)\boldsymbol{r}\cdot\frac{\boldsymbol{r}}{r}+3f(r)=f'(r)r+3f(r)\end{aligned}$$

于是得到 $f'(r)r+3f(r)=0$

解此微分方程得 $f(r)=\frac{C}{r^3}$ (C为任意常数)

1-9 已知矢量$\boldsymbol{E}=\boldsymbol{a}_x x^2+\boldsymbol{a}_y x^2y^2+\boldsymbol{a}_z 24x^2y^2z^3$,求:(1)$\nabla \cdot \boldsymbol{E}$对中心在原点的单位正方体$V\left(|x|\leqslant\frac{1}{2},|y|\leqslant\frac{1}{2},|z|\leqslant\frac{1}{2}\right)$的体积分$\int_V \nabla \cdot \boldsymbol{E}\mathrm{d}v$;(2)$\boldsymbol{E}$对正方体表面$S$的面积分$\oint_S \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S}$,并验证散度定理。

解:(1)$\nabla \cdot E=\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=2x+2x^2y+72x^2y^2z^2$

所以$\nabla \cdot \boldsymbol{E}$对中心在原点的单位正方体的体积分为

$$\begin{aligned}\int_V \nabla \cdot \boldsymbol{E}\mathrm{d}V &=\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}(2x+2x^2y+72x^2y^2z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &=\frac{1}{24}\end{aligned}$$

(2)$\boldsymbol{E}$对该正方体表面S的面积分为

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} 24x^2y^2\left(\frac{1}{2}\right)^3 \mathrm{d}x\mathrm{d}y - \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} 24x^2y^2\left(-\frac{1}{2}\right)^3 \mathrm{d}x\mathrm{d}y$$

$$+ \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[x \cdot \left(\frac{1}{2}\right)\right]^2 \mathrm{d}x\mathrm{d}z - \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[x\left(-\frac{1}{2}\right)\right]^2 \mathrm{d}x\mathrm{d}z$$

$$+ \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2}\right)^2 \mathrm{d}y\mathrm{d}z - \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(-\frac{1}{2}\right)^2 \mathrm{d}y\mathrm{d}z$$

$$= 2\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} 3x^2y^2 \mathrm{d}x\mathrm{d}y = \frac{1}{24}$$

由上面的结果知 $\int_V \nabla \cdot \boldsymbol{E} \mathrm{d}V = \oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{24}$，故验证了散度定理。

1-10 利用直角坐标系，证明：

$$\nabla \cdot (f\boldsymbol{A}) = f \nabla \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \nabla f$$

解： 设 $A = \boldsymbol{a}_x A_x + \boldsymbol{a}_y A_y + \boldsymbol{a}_z A_z$，则

$$f\boldsymbol{A} = \boldsymbol{a}_x f A_x + \boldsymbol{a}_y f A_y + \boldsymbol{a}_z f A_z$$

于是

$$\nabla \cdot (f\boldsymbol{A}) = \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z)$$

$$= f\frac{\partial A_x}{\partial x} + A_x\frac{\partial f}{\partial x} + f\frac{\partial A_y}{\partial y} + A_y\frac{\partial f}{\partial y} + f\frac{\partial A_z}{\partial z} + A_z\frac{\partial f}{\partial z}$$

$$= f\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + A_x\frac{\partial f}{\partial x} + A_y\frac{\partial f}{\partial y} + A_z\frac{\partial f}{\partial z}$$

$$= f\nabla \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \nabla f$$

1-11 求下列矢量场的旋度：

(1) $\boldsymbol{A} = \boldsymbol{a}_x 2x + \boldsymbol{a}_y y^2 + \boldsymbol{a}_z z^2$；

(2) $\boldsymbol{B} = \boldsymbol{a}_x yz + \boldsymbol{a}_y zx + \boldsymbol{a}_z xy$；

(3) $\boldsymbol{C} = \boldsymbol{a}_x (y^2+z^2) + \boldsymbol{a}_y (z^2+x^2) + \boldsymbol{a}_z (x^2+y^2)$。

解： (1)

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & y^2 & z^2 \end{vmatrix}$$

$$= \boldsymbol{a}_x\left(\frac{\partial z^2}{\partial y} - \frac{\partial y^2}{\partial z}\right) + \boldsymbol{a}_y\left(\frac{\partial 2x}{\partial z} - \frac{\partial z^2}{\partial x}\right) + \boldsymbol{a}_z\left(\frac{\partial y^2}{\partial x} - \frac{\partial 2x}{\partial y}\right)$$

$$= 0$$

(2)

$$\nabla \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \boldsymbol{a}_x\left(\frac{\partial xy}{\partial y} - \frac{\partial zx}{\partial z}\right) + \boldsymbol{a}_y\left(\frac{\partial yz}{\partial z} - \frac{\partial xy}{\partial x}\right) + \boldsymbol{a}_z\left(\frac{\partial zx}{\partial x} - \frac{\partial yz}{\partial y}\right)$$

$$= 0$$

(3) $\nabla\times\boldsymbol{C}=\begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y^2+z^2 & z^2+x^2 & x^2+y^2\end{vmatrix}$

$=\boldsymbol{a}_x\left[\dfrac{\partial(x^2+y^2)}{\partial y}-\dfrac{\partial(z^2+x^2)}{\partial z}\right]+\boldsymbol{a}_y\left(\dfrac{\partial(y^2+z^2)}{\partial z}-\dfrac{\partial(x^2+y^2)}{\partial x}\right]$

$+\boldsymbol{a}_z\left[\dfrac{\partial(z^2+x^2)}{\partial x}-\dfrac{\partial(y^2+z^2)}{\partial y}\right]$

$=\boldsymbol{a}_x 2(y-z)+\boldsymbol{a}_y 2(z-x)+\boldsymbol{a}_z 2(x-y)$

1-12 设 $\boldsymbol{r}=\boldsymbol{a}_x x+\boldsymbol{a}_y y+\boldsymbol{a}_z z, r=|\boldsymbol{r}|, \boldsymbol{E}_0$ 和 $\boldsymbol{k}$ 为常矢量。证明：

(1) $\nabla\times\boldsymbol{r}=0, \nabla\times\left(\dfrac{\boldsymbol{r}}{r}\right)=0, \nabla\times[f(r)\boldsymbol{r}]=0$;

(2) $\nabla\times\left(\dfrac{\boldsymbol{k}}{r}\right)=\dfrac{1}{r^3}(\boldsymbol{k}\times\boldsymbol{r}), \nabla\times[\boldsymbol{E}_0 e^{\boldsymbol{k}\cdot\boldsymbol{r}}]=\boldsymbol{k}\times\boldsymbol{E}_0 e^{\boldsymbol{k}\cdot\boldsymbol{r}}$。

解： (1) $\nabla\times\boldsymbol{r}=\boldsymbol{a}_x\left(\dfrac{\partial z}{\partial y}-\dfrac{\partial y}{\partial z}\right)+\boldsymbol{a}_y\left(\dfrac{\partial x}{\partial z}-\dfrac{\partial z}{\partial x}\right)+\boldsymbol{a}_z\left(\dfrac{\partial y}{\partial x}-\dfrac{\partial x}{\partial y}\right)=0$

$\nabla\times\left(\dfrac{\boldsymbol{r}}{r}\right)=\dfrac{1}{r}\nabla\times\boldsymbol{r}-\boldsymbol{r}\times\nabla\left(\dfrac{1}{r}\right)=0-\boldsymbol{r}\times\left(-\dfrac{\boldsymbol{r}}{r^3}\right)=0$

$\nabla\times[f(r)\boldsymbol{r}]=f(r)\nabla\times\boldsymbol{r}-\boldsymbol{r}\times\nabla f(r)$

$=0-\boldsymbol{r}\times[f'(r)\nabla r]=-f'(r)\boldsymbol{r}\times\dfrac{\boldsymbol{r}}{r}=0$

(2) $\nabla\times\left(\dfrac{\boldsymbol{k}}{r}\right)=\dfrac{1}{r}\nabla\times\boldsymbol{k}-\boldsymbol{k}\times\nabla\left(\dfrac{1}{r}\right)=0-\boldsymbol{k}\times\left(-\dfrac{\boldsymbol{r}}{r^3}\right)=\dfrac{1}{r^3}(\boldsymbol{k}\times\boldsymbol{r})$

$\nabla\times[\boldsymbol{E}_0 e^{\boldsymbol{k}\cdot\boldsymbol{r}}]=e^{\boldsymbol{k}\cdot\boldsymbol{r}}\nabla\times\boldsymbol{E}_0-\boldsymbol{E}_0\times\nabla e^{\boldsymbol{k}\cdot\boldsymbol{r}}$

$=0-\boldsymbol{E}_0\times[e^{\boldsymbol{k}\cdot\boldsymbol{r}}\nabla(\boldsymbol{k}\cdot\boldsymbol{r})]$

$=-\boldsymbol{E}_0\times(e^{\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{k})=\boldsymbol{k}\times\boldsymbol{E}_0 e^{\boldsymbol{k}\cdot\boldsymbol{r}}$

1-13 已知矢量 $\boldsymbol{A}=\boldsymbol{a}_x x+\boldsymbol{a}_y x^2+\boldsymbol{a}_z y^2 z$，求：(1) $\nabla\times\boldsymbol{A}$ 对正方形面积 $S(0\leqslant x\leqslant 2, 0\leqslant y\leqslant 2)$ 的积分 $\int_S\nabla\times\boldsymbol{A}\cdot d\boldsymbol{S}$；(2) $\boldsymbol{A}$ 对此面积 S 的边界 C 的线积分 $\oint_C\boldsymbol{A}\cdot d\boldsymbol{l}$，并由此验证斯托克斯公式。

解： (1) $\nabla\times\boldsymbol{A}=\boldsymbol{a}_x 2yz+\boldsymbol{a}_z 2x$

所以 $\int_S\nabla\times\boldsymbol{A}\cdot d\boldsymbol{S}=\int_0^2\int_0^2(\boldsymbol{a}_x 2yz+\boldsymbol{a}_z 2x)\cdot\boldsymbol{a}_z dxdy$

$=\int_0^2\int_0^2 2x\,dx\,dy=8$

(2) $\oint_C\boldsymbol{A}\cdot d\boldsymbol{l}=\int_0^2\boldsymbol{A}\cdot\boldsymbol{a}_x|_{y=0}dx+\int_0^2\boldsymbol{A}\cdot(-\boldsymbol{a}_x)|_{y=2}dx$

$+\int_0^2\boldsymbol{A}\cdot\boldsymbol{a}_y|_{x=2}\cdot dy+\int_0^2\boldsymbol{A}\cdot(-\boldsymbol{a}_y)|_{x=0}dy$

$=\int_0^2 x\,dx-\int_0^2 x\,dx+\int_0^2 4\,dy-\int_0^2 0\,dy$

$=8$

由此可见 $\int_S\nabla\times\boldsymbol{A}\cdot d\boldsymbol{S}=\oint_C\boldsymbol{A}\cdot d\boldsymbol{l}=8$，故验证了斯托克斯公式。

1-14 利用直角坐标系，证明：

(1) $\nabla\times(f\boldsymbol{G})=f\nabla\times\boldsymbol{G}+\nabla f\times\boldsymbol{G}$；

(2) $\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{B}\cdot\nabla\times\boldsymbol{A}-\boldsymbol{A}\cdot\nabla\times\boldsymbol{B}$。

解： (1)
$$\begin{aligned}\nabla\times(f\boldsymbol{G})&=\boldsymbol{a}_x\left[\frac{\partial(fG_z)}{\partial y}-\frac{\partial(fG_y)}{\partial z}\right]+\boldsymbol{a}_y\left[\frac{\partial(fG_x)}{\partial z}-\frac{\partial(fG_z)}{\partial x}\right]\\&\quad+\boldsymbol{a}_z\left[\frac{\partial(fG_y)}{\partial x}-\frac{\partial(fG_x)}{\partial y}\right]\\&=f\left[\boldsymbol{a}_x\left(\frac{\partial G_z}{\partial y}-\frac{\partial G_y}{\partial z}\right)+\boldsymbol{a}_y\left(\frac{\partial G_x}{\partial z}-\frac{\partial G_z}{\partial x}\right)\right.\\&\quad\left.+\boldsymbol{a}_z\left(\frac{\partial G_y}{\partial x}-\frac{\partial G_x}{\partial y}\right)\right]+\left[\boldsymbol{a}_x\left(G_z\frac{\partial f}{\partial y}-G_y\frac{\partial f}{\partial z}\right)\right.\\&\quad\left.+\boldsymbol{a}_y\left(G_x\frac{\partial f}{\partial z}-G_z\frac{\partial f}{\partial x}\right)+\boldsymbol{a}_z\left(G_y\frac{\partial f}{\partial x}-G_x\frac{\partial f}{\partial y}\right)\right]\\&=f\nabla\times\boldsymbol{G}+\nabla f\times\boldsymbol{G}\end{aligned}$$

(2)
$$\begin{aligned}\boldsymbol{B}\cdot\nabla\times\boldsymbol{A}-\boldsymbol{A}\cdot\nabla\times\boldsymbol{B}&=\boldsymbol{B}\cdot\left[\boldsymbol{a}_x\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\boldsymbol{a}_y\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\right.\\&\quad\left.+\boldsymbol{a}_z\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\right]-\boldsymbol{A}\cdot\left[\boldsymbol{a}_x\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)\right.\\&\quad\left.+\boldsymbol{a}_y\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)+\boldsymbol{a}_z\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)\right]\\&=B_x\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+B_y\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\\&\quad+B_z\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-A_x\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)\\&\quad-A_y\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)-A_z\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)\\&=\frac{\partial}{\partial y}(B_xA_z-B_zA_x)+\frac{\partial}{\partial z}(B_yA_x-A_yB_x)\\&\quad+\frac{\partial}{\partial x}(B_zA_y-B_yA_z)\\&=\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})\end{aligned}$$

1-15 已知 $\boldsymbol{E}=\boldsymbol{a}_x(x^2+axz)+\boldsymbol{a}_y(xy^2+by)+\boldsymbol{a}_z(z-z^2+czx-2xyz)$ 试确定常数 a、b、c 使 $\boldsymbol{E}$ 为一无源场。

解： 要使得 $\boldsymbol{E}$ 为无源场，应有
$$\begin{aligned}\nabla\cdot\boldsymbol{E}&=(2x+az)+(2xy+b)+(1-2z+cx-2xy)\\&=(2+c)x+(a-2)z+(b+1)=0\end{aligned}$$

于是，得
$$a=2,\ b=-1,\ c=-2$$

1-16 设在圆柱坐标系中，$u=\left(1-\dfrac{a^2}{r^2}\right)r\cos\varphi$，求 ∇u 和 $\nabla^2 u$。

解： 在圆柱坐标系中
$$\nabla u=\boldsymbol{a}_r\frac{\partial u}{\partial r}+\boldsymbol{a}_\varphi\frac{1}{r}\frac{\partial u}{\partial\varphi}+\boldsymbol{a}_z\frac{\partial u}{\partial z}$$

此为试读，需要完整PDF请访问：www.ertongbook.com

$$=\boldsymbol{a}_r\frac{\partial}{\partial r}\left[\left(1-\frac{a^2}{r^2}\right)r\cos\varphi\right]+\boldsymbol{a}_\varphi\frac{1}{r}\frac{\partial}{\partial\varphi}\left[\left(1-\frac{a^2}{r^2}\right)r\cos\varphi\right]$$

$$=\boldsymbol{a}_r\left(1+\frac{a^2}{r^2}\right)\cos\varphi-\boldsymbol{a}_\varphi\left(1-\frac{a^2}{r^2}\right)\sin\varphi$$

$$\nabla^2u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}$$

$$=\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\left(1-\frac{a^2}{r^2}\right)r\cos\varphi\right]\right\}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\left[\left(1-\frac{a^2}{r^2}\right)r\cos\varphi\right]$$

$$=\frac{1}{r}\left(1-\frac{a^2}{r^2}\right)\cos\varphi-\frac{1}{r}\left(1-\frac{a^2}{r^2}\right)\cos\varphi=0$$

1-17 在圆柱坐标系中，求下列矢量的散度和旋度：

(1) $\boldsymbol{A}=\boldsymbol{a}_r r\cos^2\varphi+\boldsymbol{a}_\varphi r\sin\varphi$;

(2) $\boldsymbol{B}=\boldsymbol{a}_r(r^2-z\sin\varphi)+\boldsymbol{a}_\varphi r\cos\varphi+\boldsymbol{a}_z(z^2-r^2\sin\varphi\cos\varphi)$。

解： (1) $\nabla\cdot\boldsymbol{A}=\frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi}+\frac{\partial A_z}{\partial z}$

$$=\frac{1}{r}\frac{\partial}{\partial r}(r^2\cos^2\varphi)+\frac{1}{r}\frac{\partial}{\partial\varphi}(r\sin\varphi)$$

$$=2\cos^2\varphi+\cos\varphi$$

$$\nabla\times\boldsymbol{A}=\boldsymbol{a}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial\varphi}-\frac{\partial A_\varphi}{\partial z}\right)+\boldsymbol{a}_\varphi\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)+\boldsymbol{a}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)-\frac{1}{r}\frac{\partial A_r}{\partial\varphi}\right]$$

$$=\boldsymbol{a}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(r^2\sin\varphi)-\frac{1}{r}\frac{\partial}{\partial\varphi}(r\cos^2\varphi)\right]$$

$$=\boldsymbol{a}_z 2\sin\varphi(1+\cos\varphi)$$

(2) $\nabla\cdot\boldsymbol{B}=\frac{1}{r}\frac{\partial}{\partial r}(rB_r)+\frac{1}{r}\frac{\partial B_\varphi}{\partial\varphi}+\frac{\partial B_z}{\partial z}$

$$=\frac{1}{r}\frac{\partial}{\partial r}[r(r^2-z\sin\varphi)]+\frac{1}{r}\frac{\partial}{\partial\varphi}(r\cos\varphi)+\frac{\partial}{\partial z}(z^2-r^2\sin\varphi\cos\varphi)$$

$$=3r-\left(1+\frac{z}{r}\right)\sin\varphi+2z$$

$$\nabla\times\boldsymbol{B}=\boldsymbol{a}_r\left(\frac{1}{r}\frac{\partial B_z}{\partial\varphi}-\frac{\partial B_\varphi}{\partial z}\right)+\boldsymbol{a}_\varphi\left(\frac{\partial B_r}{\partial z}-\frac{\partial B_z}{\partial r}\right)+\boldsymbol{a}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(rB_\varphi)-\frac{1}{r}\frac{\partial B_r}{\partial\varphi}\right]$$

$$=\boldsymbol{a}_r\left[\frac{1}{r}\frac{\partial}{\partial\varphi}(z^2-r^2\sin\varphi\cos\varphi)\right]+\boldsymbol{a}_\varphi\left[\frac{\partial}{\partial z}(r^2-z\sin\varphi)\right.$$

$$\left.-\frac{\partial}{\partial r}(z^2-r^2\sin\varphi\cos\varphi)\right]+\boldsymbol{a}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(r^2\cos\varphi)-\frac{1}{r}\frac{\partial}{\partial\varphi}(r^2-z\sin\varphi)\right]$$

$$=-\boldsymbol{a}_r r\cos2\varphi+\boldsymbol{a}_\varphi(2r\cos\varphi-1)\sin\varphi+\boldsymbol{a}_z\left(2+\frac{z}{r}\right)\cos\varphi$$

1-18 给定矢量 $\boldsymbol{A}=\boldsymbol{a}_r a+\boldsymbol{a}_\varphi b+\boldsymbol{a}_z c$，其中 a、b、c 为常数。(1)问 $\boldsymbol{A}$ 是否为常矢量；(2)求 $\nabla\cdot\boldsymbol{A}$ 和 $\nabla\times\boldsymbol{A}$。

解： (1)由 $\boldsymbol{a}_r=\boldsymbol{a}_x\cos\varphi+\boldsymbol{a}_y\sin\varphi$，$\boldsymbol{a}_\varphi=-\boldsymbol{a}_x\sin\varphi+\boldsymbol{a}_y\cos\varphi$，得

$$\boldsymbol{A}=(\boldsymbol{a}_x\cos\varphi+\boldsymbol{a}_y\sin\varphi)a+(-\boldsymbol{a}_x\sin\varphi+\boldsymbol{a}_y\cos\varphi)b+\boldsymbol{a}_z c$$

$$=\boldsymbol{a}_x(a\cos\varphi-b\sin\varphi)+\boldsymbol{a}_y(a\sin\varphi+b\cos\varphi)+\boldsymbol{a}_z c$$

因此，$\boldsymbol{A}$ 不是常矢量。

$$(2)\nabla\cdot\boldsymbol{A}=\frac{1}{r}\frac{\partial}{\partial r}(ra)+\frac{1}{r}\frac{\partial b}{\partial\varphi}+\frac{\partial c}{\partial z}=\frac{a}{r}$$

$$\nabla\times\boldsymbol{A}=\boldsymbol{a}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(rb)-\frac{1}{r}\frac{\partial a}{\partial\varphi}\right]=\boldsymbol{a}_z\frac{b}{r}$$

1-19 求矢量 $\boldsymbol{A}=\boldsymbol{a}_x x^2+\boldsymbol{a}_y xy^2$ 沿圆周 $x^2+y^2=a^2$ 的线积分，并计算 $\nabla\times\boldsymbol{A}$ 对此圆面积的积分。

解： 矢量 $\boldsymbol{A}=\boldsymbol{a}_x x^2+\boldsymbol{a}_y xy^2$ 沿圆周 $x^2+y^2=a$ 的线积分

$$\oint_c\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}=\oint_c A_x\mathrm{d}x+A_y\mathrm{d}y=\oint_c x^2\mathrm{d}x+xy^2\mathrm{d}y$$

$$=\int_0^{2\pi}[a^2\cos^2\varphi\mathrm{d}(a\cos\varphi)+a\cos\varphi\cdot a^2\sin^2\varphi\,\mathrm{d}(a\sin\varphi)]$$

$$=\int_0^{2\pi}a^3\cos^2\varphi\mathrm{d}(\cos\varphi)+\int_0^{2\pi}a^4\cos^2\varphi\sin^2\varphi\,\mathrm{d}\varphi$$

$$=\frac{\pi a^4}{4}$$

又 $\nabla\times\boldsymbol{A}=\boldsymbol{a}_z\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)=\boldsymbol{a}_z y^2$

所以，$\nabla\times\boldsymbol{A}$ 对此圆面积的积分

$$\int_S\nabla\times\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}=\int_S y^2\mathrm{d}s=\int_0^a\int_0^{2\pi}r^2\sin^2\varphi\,r\mathrm{d}r\mathrm{d}\varphi$$

$$=\frac{\pi a^4}{4}$$

由上面的结果，有 $\oint_c\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}=\int_S\nabla\times\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}$，满足斯托克斯定理。

1-20 在 $r=5$，$z=0$ 和 $z=4$ 围成的圆柱形区域上，对矢量 $\boldsymbol{A}=\boldsymbol{a}_r r^2+\boldsymbol{a}_z 2z$ 验证散度定理。

解：
$$\oint_S\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}=\oint_S(\boldsymbol{a}_r r^2+\boldsymbol{a}_z 2z)\cdot(\boldsymbol{a}_r r\mathrm{d}\varphi\,\mathrm{d}z+\boldsymbol{a}_\varphi\mathrm{d}r\mathrm{d}z+\boldsymbol{a}_z r\mathrm{d}r\mathrm{d}\varphi)$$

$$=\int_0^4\int_0^{2\pi}r^3|_{r=5}\mathrm{d}\varphi\,\mathrm{d}z+\int_0^5\int_0^{2\pi}(2zr)|_{z=4}\mathrm{d}r\mathrm{d}\varphi$$

$$-\int_0^5\int_0^{2\pi}(2zr)|_{z=0}\mathrm{d}r\mathrm{d}\varphi$$

$$=1000\pi+200\pi=1200\pi$$

$$\int_V\nabla\cdot\boldsymbol{A}\mathrm{d}V=\int_V\left[\frac{1}{r}\frac{\partial}{\partial r}(r\cdot r^2)+\frac{\partial}{\partial z}(2z)\right]\mathrm{d}V$$

$$=\int_V(3r+2)\mathrm{d}V=\int_0^4\int_0^5\int_0^{2\pi}(3r+2)r\mathrm{d}\varphi\mathrm{d}r\mathrm{d}z$$

$$=1200\pi$$

由上述结果可知 $\int_V\nabla\cdot\boldsymbol{A}\mathrm{d}V=\oint_S\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}$，验证了散度定理。

1-21 在圆柱坐标系中，点 M 的位置由 $\left(4,\frac{2}{3}\pi,3\right)$ 给出。求该点：(1)在直角坐标系中

的坐标;(2)在球坐标系中的坐标。

解: (1)在直角坐标系中该点的坐标

$$x=\rho\cos\varphi=4\cos\frac{2\pi}{3}=-4\times\frac{1}{2}=-2$$

$$y=\rho\sin\varphi=4\sin\frac{2\pi}{3}=4\times\frac{\sqrt{3}}{2}=2\sqrt{3}$$

$$z=3$$

故在直角坐标系中的位置为$(-2,2\sqrt{3},3)$。

(2)在球坐标系中

$$r=\sqrt{4^2+3^2}=5$$

$$\theta=\cos^{-1}\frac{z}{r}=\cos^{-1}\frac{3}{5}=53.1°$$

$$\varphi=\frac{2\pi}{3}=120°$$

故在球坐标系中的位置为$(5,53.1°,120°)$。

1-22 用球坐标表示的场$\boldsymbol{E}=\boldsymbol{a}_r\dfrac{25}{r^2}$。(1)求在点$(-3,4,-5)$处的$|\boldsymbol{E}|$和$E_x$;(2)求$\boldsymbol{E}$与矢量$\boldsymbol{A}=\boldsymbol{a}_x2-\boldsymbol{a}_y2+\boldsymbol{a}_z$的夹角。

解: (1)在点$(-3,4,-5)$处,$r=\sqrt{(-3)^2+4^2+(-5)^2}=5\sqrt{2}$,所以

$$|\boldsymbol{E}|=\frac{25}{r^2}=\frac{25}{(5\sqrt{2})^2}=0.5$$

$$E_x=|\boldsymbol{E}|\cos\theta_{rx}=|\boldsymbol{E}|\cdot\frac{x}{r}=0.5\times\left(\frac{-3}{5\sqrt{2}}\right)=-0.212$$

(2)在点$(-3,4,-5)$处,$\boldsymbol{a}_r=\dfrac{\boldsymbol{r}}{r}=\boldsymbol{a}_x\left(-\dfrac{3}{5\sqrt{2}}\right)+\boldsymbol{a}_y\dfrac{4}{5\sqrt{2}}+\boldsymbol{a}_z\left(-\dfrac{5}{5\sqrt{2}}\right)$

故

$$\cos\theta_{EA}=-\frac{\boldsymbol{E}\cdot\boldsymbol{A}}{|\boldsymbol{E}|\cdot|\boldsymbol{A}|}=\frac{-1.344}{0.5\times3}$$

故

$$\theta_{EA}=\cos^{-1}\left(\frac{-1.344}{1.5}\right)=\pi-\cos^{-1}0.896=153.36°$$

1-23 给定矢量$\boldsymbol{A}=\boldsymbol{a}_ra+\boldsymbol{a}_\theta b+\boldsymbol{a}_\varphi c$,其中$a$、$b$、$c$为常数。(1)问$\boldsymbol{A}$是否为常矢量?(2)计算$\nabla\cdot\boldsymbol{A}$和$\nabla\times\boldsymbol{A}$。

解: (1)在球坐标系中

$$\boldsymbol{a}_r=\boldsymbol{a}_x\sin\theta\cos\varphi+\boldsymbol{a}_y\sin\theta\sin\varphi+\boldsymbol{a}_z\cos\theta$$

$$\boldsymbol{a}_\theta=\boldsymbol{a}_x\cos\theta\cos\varphi+\boldsymbol{a}_y\cos\theta\sin\varphi-\boldsymbol{a}_z\sin\theta$$

$$\boldsymbol{a}_\varphi=-\boldsymbol{a}_x\sin\varphi+\boldsymbol{a}_y\cos\varphi$$

所以

$$\begin{aligned}\boldsymbol{A}&=\boldsymbol{a}_ra+\boldsymbol{a}_\theta b+\boldsymbol{a}_\varphi c\\&=\boldsymbol{a}_x(a\sin\theta\cos\varphi+b\cos\theta\cos\varphi-c\sin\varphi)\\&\quad+\boldsymbol{a}_y(a\sin\theta\sin\varphi+b\cos\theta\sin\varphi+c\cos\varphi)\\&\quad+\boldsymbol{a}_z(a\cos\theta-b\sin\theta)\end{aligned}$$

故 $\boldsymbol{A}$ 不是常矢量。

$$(2)\nabla\cdot\boldsymbol{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2a)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(b\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial c}{\partial\varphi}$$

$$=\frac{2a}{r}+\frac{b}{r}\mathrm{ctg}\theta$$

$$\nabla\times\boldsymbol{A}=\boldsymbol{a}_r\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}c\sin\theta)-\frac{\partial b}{\partial\varphi}\right]+\boldsymbol{a}_\theta\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a}{\partial\varphi}-\frac{\partial}{\partial r}(rc)\right]$$

$$+\boldsymbol{a}_\varphi\frac{1}{r}\left[\frac{\partial}{\partial r}(rb)-\frac{\partial a}{\partial\theta}\right]$$

$$=\boldsymbol{a}_r\frac{c}{r}\mathrm{ctg}\theta-\boldsymbol{a}_\theta\frac{c}{r}+\boldsymbol{a}_\varphi\frac{b}{r}$$

1-24 在球坐标系中，已知 $u=\left(c_1r^2+\frac{c_2}{r^3}\right)\sin2\theta\cos\varphi$，其中 c_1、c_2 为常数，求 ∇u。

解： $$\nabla u=\boldsymbol{a}_r\frac{\partial u}{\partial r}+\boldsymbol{a}_\theta\frac{1}{r}\frac{\partial u}{\partial\theta}+\boldsymbol{a}_\varphi\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi}$$

$$=\boldsymbol{a}_r\left(2c_1r-\frac{3c_2}{r^4}\right)\sin2\theta\cos\varphi$$

$$+\boldsymbol{a}_\theta\left(2c_1r+\frac{2c_2}{r^4}\right)\cos2\theta\cos\varphi$$

$$-\boldsymbol{a}_\varphi\left(2c_1r+\frac{2c_2}{r^4}\right)\cos\theta\sin\varphi$$

1-25 在球坐标系中，求下列矢量的散度和旋度：

(1) $\boldsymbol{E}=\boldsymbol{a}_r\frac{2\cos\theta}{r^3}+\boldsymbol{a}_\theta\frac{\sin\theta}{r^3}$；

(2) $\boldsymbol{E}=\boldsymbol{a}_rr^2+\boldsymbol{a}_\theta\left(\frac{1}{r^3}-r\right)r\cos\theta$。

解： $$(1)\nabla\cdot\boldsymbol{E}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{2\cos\theta}{r^3}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\sin\theta}{r^3}\right)$$

$$=-\frac{2\cos\theta}{r^4}+\frac{2\cos\theta}{r^4}=0$$

$$\nabla\times\boldsymbol{E}=\boldsymbol{a}_\varphi\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\cdot\frac{\sin\theta}{r^3}\right)-\frac{\partial}{\partial\theta}\left(\frac{2\cos\theta}{r^3}\right)\right]$$

$$=\boldsymbol{a}_\varphi\frac{1}{r}\left[-\frac{2\sin\theta}{r^3}+\frac{2\sin\theta}{r^3}\right]=0$$

$$(2)\nabla\cdot\boldsymbol{E}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cdot r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\cdot\left(\frac{1}{r^3}-r\right)r\cos\theta\right]$$

$$=4r+\left(\frac{1}{r^3}-r\right)\frac{\cos2\theta}{\sin\theta}$$

$$\nabla\times\boldsymbol{E}=\boldsymbol{a}_\varphi\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{1}{r^3}-r\right)r\cos\theta\right]=-\boldsymbol{a}_\varphi\left(\frac{1}{r^3}+3r\right)\cos\theta$$

1-26 有三个矢量

$\boldsymbol{A}=\boldsymbol{a}_r\sin\theta\cos\varphi+\boldsymbol{a}_\theta\cos\theta\cos\varphi-\boldsymbol{a}_\varphi\sin\varphi$（球坐标系）

$\boldsymbol{B}=\boldsymbol{a}_rz^2\sin\varphi+\boldsymbol{a}_\varphi z^2\cos\varphi+\boldsymbol{a}_z2rz\sin\varphi$（圆柱坐标系）

$\boldsymbol{C}=\boldsymbol{a}_x(3y^2-2x)+\boldsymbol{a}_yx^2+\boldsymbol{a}_z2z$（直角坐标系）

问：(1)哪些矢量可表示为一个标量的梯度？

(2)哪些矢量可表示为一个矢量的旋度？

解： (1)
$$\nabla\times\boldsymbol{A}=\boldsymbol{a}_r\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(-\sin\theta\cos\varphi)-\frac{\partial}{\partial\varphi}(\cos\theta\cos\varphi)\right]+\boldsymbol{a}_\theta\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}(\sin\theta\cos\varphi)-\frac{\partial}{\partial r}(-r\sin\varphi)\right]+\boldsymbol{a}_\varphi\frac{1}{r}\left[\frac{\partial}{\partial r}(r\cos\theta\cos\varphi)-\frac{\partial}{\partial\theta}(\sin\theta\cos\varphi)\right]=0$$

$$\nabla\times\boldsymbol{B}=\boldsymbol{a}_r\left[\frac{1}{r}\frac{\partial}{\partial\varphi}(2rz\sin\varphi)-\frac{\partial}{\partial z}(z^2\cos\varphi)\right]+\boldsymbol{a}_\varphi\left[\frac{\partial}{\partial z}(z^2\sin\varphi)-\frac{\partial}{\partial r}(2rz\sin\varphi)\right]+\boldsymbol{a}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(rz^2\cos\varphi)-\frac{1}{r}\frac{\partial}{\partial\varphi}(z^2\sin\varphi)\right]=0$$

$$\nabla\times\boldsymbol{C}=\boldsymbol{a}_x\left[\frac{\partial}{\partial y}(2z)-\frac{\partial}{\partial z}(x^2)\right]+\boldsymbol{a}_y\left[\frac{\partial}{\partial z}(3y^2-2x)-\frac{\partial}{\partial x}(2z)\right]+\boldsymbol{a}_z\left[\frac{\partial}{\partial x}(x^2)-\frac{\partial}{\partial y}(3y^2-2x)\right]=\boldsymbol{a}_z(2x-6y)$$

所以，矢量 $\boldsymbol{A}$ 和矢量 $\boldsymbol{B}$ 可表示为一个标量的梯度。

(2)
$$\nabla\cdot\boldsymbol{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sin\theta\cos\varphi)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta\cos\varphi)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(-\sin\varphi)=\frac{2}{r}\sin\theta\cos\varphi+\frac{\cos2\theta\cdot\cos\varphi}{r\sin\theta}-\frac{\cos\varphi}{r\sin\theta}=0$$

$$\nabla\cdot\boldsymbol{B}=\frac{1}{r}\frac{\partial}{\partial r}(rz^2\sin\varphi)+\frac{1}{r}\frac{\partial}{\partial\varphi}(z^2\cos\varphi)+\frac{\partial}{\partial z}(2rz\sin\varphi)=\frac{1}{r}z^2\sin\varphi-\frac{1}{r}z^2\sin\varphi+2r\sin\varphi=2r\sin\varphi$$

$$\nabla\cdot\boldsymbol{C}=\frac{\partial}{\partial x}(3y^2-2x)+\frac{\partial}{\partial y}(x^2)+\frac{\partial}{\partial z}(2z)=-2+2=0$$

所以，矢量 $\boldsymbol{A}$ 和矢量 $\boldsymbol{C}$ 可以表示为一个矢量的旋度。

第二章 麦克斯韦方程

2-1 证明：通过任意闭合曲面的传导电流与位移电流之和等于零。

解： 将麦克斯韦方程

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{J} + \boldsymbol{J}_d$$

两边取散度，得

$$\nabla \cdot (\boldsymbol{J} + \boldsymbol{J}_d) = \nabla \cdot (\nabla \times \boldsymbol{H}) = 0$$

将上式对任意体积 V 积分，并利用散度定理，即得

$$\oint_S (\boldsymbol{J} + \boldsymbol{J}_d) \cdot \mathrm{d}\boldsymbol{S} = \int_V \nabla \cdot (\boldsymbol{J} + \boldsymbol{J}_d) \mathrm{d}V = 0$$

2-2 有一种典型的金属导体，电导率 $\sigma = 5 \times 10^7 \mathrm{S/m}$，相对介电常数 $\varepsilon_r = 1$。若导体中的传导电流密度为

$$\boldsymbol{J} = \boldsymbol{a}_x 10^6 \sin[117.1(3.22t - z)] \quad \mathrm{A/m^2}$$

求位移电流密度 $\boldsymbol{J}_d$。

解： 由欧姆定律 $\boldsymbol{J} = \sigma \boldsymbol{E}$，得

$$\boldsymbol{E} = \frac{\boldsymbol{J}}{\sigma} = \boldsymbol{a}_x 0.02 \sin[117.1(3.22t - z)] \quad \mathrm{V/m}$$

所以 $$\boldsymbol{J}_d = \frac{\partial \boldsymbol{D}}{\partial t} = \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{a}_x 6.68 \times 10^{-11} \cos[117.1(3.22t - z)] \quad \mathrm{A/m^2}$$

2-3 当电场 $\boldsymbol{E} = \boldsymbol{a}_x E_0 \cos\omega t \, \mathrm{V/m}$，$\omega = 1000 \mathrm{rad/s}$ 时，计算下列媒质中传导电流密度与位移电流密度的振幅之比：

(1)铜：$\sigma = 5.7 \times 10^7 \mathrm{S/m}$，$\varepsilon_r = 1$；

(2)蒸馏水：$\sigma = 2 \times 10^{-4} \mathrm{S/m}$，$\varepsilon_r = 80$；

(3)聚苯乙烯：$\sigma = 10^{-16} \mathrm{S/m}$，$\varepsilon_r = 2.53$。

解： 因为 $\boldsymbol{J} = \sigma \boldsymbol{E} = \boldsymbol{a}_x \sigma E_0 \cos\omega t \quad \mathrm{A/m^2}$

$$\boldsymbol{J}_d = \varepsilon_r \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = -\boldsymbol{a}_x \varepsilon_r \varepsilon_0 \omega E_0 \sin\omega t \quad \mathrm{A/m^2}$$

所以，传导电流密度与位移电流密度的振幅之比为

$$\frac{J_m}{J_{dm}} = \frac{\sigma E_0}{\varepsilon_r \varepsilon_0 \omega E_0} = \frac{\sigma}{\varepsilon_r \varepsilon_0 \omega}$$

(1)铜：$\sigma = 5.7 \times 10^7 \mathrm{S/m}$，$\varepsilon_r = 1$，故得

$$\left(\frac{J_m}{J_{dm}}\right)_{\text{铜}} = \frac{5.7 \times 10^7}{8.854 \times 10^{-12} \times 1000} = 0.64 \times 10^{16}$$

(2)蒸馏水：$\sigma = 2 \times 10^{-4} \mathrm{S/m}$，$\varepsilon_r = 80$，故得

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