WORKED PROBLEMS IN APPLIED MATHEMATICS

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Part 1

PROBLEMS

1

DERIVATION OF EQUATIONS AND FORMULATION OF PROBLEMS

Chapter 1 is devoted to problem material on the derivation of the equations of mathematical physics and the formulation of appropriate initial and boundary conditions. It also serves as a convenient place to list the basic equations appearing later in the book. Throughout, we assume that the reader is familiar with the physical laws underlying the mathematical formulation of the problems which arise in various branches of physics.

The chapter consists of three sections devoted in turn to problems of mechanics, heat conduction and the theory of electric and magnetic phenomena. Each section starts with the basic equations governing the corresponding set of problems, with appropriate references to sources where the derivations can be found. Special attention is devoted to the formulation of problems of electrodynamics, since this subject is inadequately covered in the available literature.[1]

1. Mechanics

This section contains problems on the derivation of equations of motion and formulation of initial and boundary conditions for vibrating strings, membranes, rods and plates, as well as some examples pertaining to the statics of deformable media. It will be assumed that the reader has already

[1] Those particularly interested in mathematical aspects of the formulation of physical problems can find relevant material in C5, G1, L1, P2, S1 and S13. (The reference scheme is explained in the Translator's Preface.)

encountered the basic equations in a first course on mathematical physics.[2] Thus we shall merely list the equations concisely, at the same time explaining the notation to be used in the book.

1. The equation of a vibrating string is

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = -\frac{q(x, t)}{T}, \qquad v = \sqrt{\frac{T}{\rho}},$$

where $u(x, t)$ is the displacement of the point of the string with abscissa x at the time t, $q(x, t)$ is the external load per unit length, T is the tension, and ρ is the linear density.

2. The equation for longitudinal oscillations of a rod of constant cross section is

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0, \qquad v = \sqrt{\frac{E}{\rho}},$$

where $u(x, t)$ is the displacement of the cross section of the rod with abscissa x at the time t, E is Young's modulus, and ρ is the density.

3. The equation for transverse oscillations of a rod (beam) is

$$\frac{\partial^4 u}{\partial x^4} + \frac{1}{a^4}\frac{\partial^2 u}{\partial t^2} = \frac{q(x, t)}{EJ}, \qquad a^2 = \sqrt{\frac{EJ}{\rho S}},$$

where $u(x, t)$ is the displacement of the points along the midline of the rod, $q(x, t)$ is the external load per unit length, E is Young's modulus, J is the moment of inertia of a transverse cross section, ρ is the density, and S is the cross-sectional area.

4. The equation of a vibrating membrane is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = -\frac{q(x, y, t)}{T}, \qquad v = \sqrt{\frac{T}{\rho}},$$

where $u(x, y, t)$ is the displacement of the point (x, y) of the membrane at the time t, $q(x, y, t)$ is the external load per unit area, T is the tension per unit length of the boundary of the membrane, and ρ is the surface density.

5. The equation for transverse oscillations of a thin elastic plate is

$$\Delta^2 u + \frac{1}{b^4}\frac{\partial^2 u}{\partial t^2} = \frac{q(x, y, t)}{D}, \qquad b^2 = \sqrt{\frac{D}{\rho h}},$$

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[2] See S6 (Vol. II), S14, T1 and T2. Concerning the derivation of the equations of vibrating plates, see T4.

where $u(x, t)$ is the displacement of the point (x, y) of the midplane of the plate at the time t, $q(x, y, t)$ is the density of the external load, D is the flexural rigidity, h is the thickness, ρ is the density, and

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the two-dimensional Laplacian operator.

The above equations lead to corresponding equations for static deflections, if we regard the external load q and the unknown displacement u as independent of the time t. For example, the equilibrium equation for the membrane is

6.
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{q(x, y)}{T},$$

the static deflection of the plate satisfies the equation

7.
$$\Delta^2 u = \frac{q(x, y)}{D},$$

and so on.

Among the other equations governing the statics of elastic bodies which will figure in this book, we cite the familiar equation

8.
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2,$$

for twisting of a prismatic rod, where $u(x, y)$ is the torsion function.

We now give some problems on the formulation of initial and boundary conditions for these equations, and also some problems on the derivation of other differential equations.

1. Describe the initial and boundary conditions for a vibrating string with fixed ends ($0 \leqslant x \leqslant l$), which is stretched at the point $x = c$ and time $t = 0$ to a height h, and then released without initial velocity.

Ans.

$$u\Big|_{t=0} = f(x) = \begin{cases} \dfrac{hx}{c}, & 0 \leqslant x \leqslant c, \\[2ex] \dfrac{h(l - x)}{l - c}, & c \leqslant x \leqslant l, \end{cases} \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0;$$

$$u\Big|_{x=0} = u\Big|_{x=l} = 0.$$

2. A concentrated load of mass m_0 is fastened at the point $x = c$ of a string $0 < x < l$ of length l. Find the equations describing vibrations of the string with arbitrary initial conditions, assuming that the ends of the string are fastened.

Ans.

$$u = \begin{cases} u_1, & 0 \leqslant x \leqslant c, \\ u_2, & c \leqslant x \leqslant l, \end{cases} \quad \frac{\partial^2 u_i}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u_i}{\partial t^2} = 0 \qquad (i = 1, 2),$$

with initial conditions

$$u|_{t=0} = f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x),$$

and boundary conditions

$$u_1|_{x=0} = u_2|_{x=l} = 0, \qquad u_1|_{x=c} = u_2|_{x=c}, \qquad \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial x}\right)_{x=c} = \frac{m_0}{T}\left.\frac{\partial^2 u}{\partial t^2}\right|_{x=c}.$$

3. Formulate initial and boundary conditions for the problem of longitudinal oscillations of a rod in the following special cases:

a) A rod of length l is clamped at the end $x = 0$ and stretched by a force F applied to the other end; at the time $t = 0$ the force is suddenly discontinued;

b) A tensile force $F(t)$ is applied at the time $t = 0$ to the end $x = l$ of a cantilever in equilibrium;

c) A cantilever clamped at the point $x = 0$, with a load of mass M_0 at the free end $x = l$, undergoes longitudinal oscillations subject to arbitrary initial conditions.

Ans.

$$a)\ u|_{t=0} = \frac{Fx}{ES}, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \qquad u|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0;$$

$$b)\ u|_{t=0} = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \qquad u|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=l} = \frac{F(t)}{ES};$$

$$c)\ u|_{t=0} = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad u|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=l} = -\frac{M_0}{ES}\left.\frac{\partial^2 u}{\partial t^2}\right|_{x=l}.$$

4. Derive the differential equation for longitudinal oscillations of a thin rod of variable cross section $S = S(x)$. As an example, derive the equation for oscillations of a conical rod.

Ans.

$$\frac{1}{S(x)}\frac{\partial}{\partial x}\left[S(x)\frac{\partial u}{\partial x}\right] - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0, \qquad v = \sqrt{\frac{E}{\varrho}}.$$

5. Derive the equation for torsional oscillations of a shaft of circular cross section.

Ans.

$$\frac{\partial^2\theta}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2\theta}{\partial t^2} = 0,$$

where $\theta(x, t)$ is the angular displacement of the cross section x relative to the equilibrium position, $v = \sqrt{G/\rho}$, ρ is the density, and G is the shear modulus.

Hint. The torque at the cross section x is given by the expression

$$M = GJ\frac{\partial\theta}{\partial x},$$

where J is the polar moment of inertia of a cross section of the shaft.

6. Formulate initial and boundary conditions for the problem of torsional oscillations of a shaft of circular cross section and length l, where the end $x = 0$ is clamped and a disk-shaped mass with moment of inertia J_0 is attached to the other end. At the time $t = 0$, the disk is rotated through a given angle α and then released without initial velocity.

Ans.

$$\theta\big|_{t=0} = \alpha\frac{x}{l}, \qquad \frac{\partial\theta}{\partial t}\bigg|_{t=0} = 0,$$

$$\theta\big|_{x=0} = 0, \qquad \frac{\partial\theta}{\partial x}\bigg|_{x=l} = -\frac{J_0}{GJ}\frac{\partial^2\theta}{\partial t^2}\bigg|_{x=l}.$$

7. A cantilever of length l is clamped at one end $x = 0$ and loaded by a force F at the other end. At the time $t = 0$, the action of the force is suddenly discontinued. Formulate initial and boundary conditions for the corresponding oscillations.

Ans. Initial conditions

$$u\big|_{t=0} = \frac{F}{6EJ}(3lx^2 - x^3), \qquad \frac{\partial u}{\partial t}\bigg|_{t=0} = 0,$$

and boundary conditions

$$u\big|_{x=0} = \frac{\partial u}{\partial x}\bigg|_{x=0} = 0, \qquad \frac{\partial^2 u}{\partial x^2}\bigg|_{x=l} = \frac{\partial^3 u}{\partial x^3}\bigg|_{x=l} = 0.$$

8. Describe initial and boundary conditions for the problem of free oscillations of a disk-shaped plate with clamped edge, whose initial deformation is due to a concentrated force F applied at the center of the disk.

Ans.

$$u\big|_{t=0} = \frac{Fr^2}{8\pi D}\ln\frac{r}{a} + \frac{F}{16\pi D}(a^2 - r^2), \qquad \frac{\partial u}{\partial t}\bigg|_{t=0} = 0;$$

$$u\big|_{r=a} = 0, \qquad \frac{\partial u}{\partial r}\bigg|_{r=a} = 0.$$

Hint. To determine the static deflection due to the concentrated force, consider the force as the limiting case of a load of density $F/\pi\varepsilon^2$ uniformly distributed over a small disk of radius ε.

9. Show that the problem of the deflection of a plate with a simply supported polygonal boundary reduces to the solution of Poisson's equation

$$\Delta w = f(x, y),$$

with boundary condition $w|_\Gamma = 0$ (f is a known function).

Hint. Note that in the present case, the boundary conditions on the supported edge can be written in the form $u|_\Gamma = 0$, $\Delta u|_\Gamma = 0$.

10. Show that the velocity potential for the three-dimensional flow of an ideal incompressible fluid containing no sources is described by Laplace's equation

$$\Delta u = 0.$$

Hint. Use the condition

$$\int_S \mathbf{v} \cdot \mathbf{n} \, dS = 0$$

($\mathbf{v}$ is the vector describing the velocity of fluid particles at a given point, S is an arbitrary closed surface inside the flow, and $\mathbf{n}$ is the exterior normal to the surface S) and the condition

$$\mathbf{v} = -\text{grad } u$$

for potential flow.

11. Formulate mathematically the problem of the flow of an ideal fluid past an object bounded by a surface S, where fluid emanates from a point source of strength m located at a point M_0 in the region exterior to S.

Ans. The problem reduces to finding a solution of the equation

$$\Delta u = 0$$

which is regular (i.e., has no singularities) in the region exterior to S, except at the point M_0. In a neighborhood of M_0,

$$u = \frac{m}{4\pi\rho \, |MM_0|} + \text{a regular function}$$

where M is a point near M_0 and ρ is the density of the fluid ($|MM_0|$ denotes the distance between M and M_0). The desired function u must satisfy the boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_S = 0$$

and the condition

$$u = O(R^{-1}), \qquad R \to \infty$$

at infinity.

2. Heat Conduction

As proved in courses on mathematical physics (see S1, T1), the flow of heat in a body of thermal conductivity k, specific heat c and density ρ is governed by *Fourier's equation*

$$\Delta T = \frac{c\rho}{k}\frac{\partial T}{\partial t} - \frac{Q}{k},$$

where $T(M, t)$ is the temperature at the point M, and Q is the density of heat sources within the body.[3] The boundary conditions to be satisfied on the surface of the body (or its parts) depend on the particular problem under consideration. Most often it is assumed that the surface of the body has a given temperature $T|_S = f(P, t)$, where P is a point of the surface S, or that the body radiates heat into the surrounding medium according to *Newton's law*, which states that the amount of heat radiated by a unit area of the surface per unit time is proportional to the difference between the temperature of the surface and that of the surrounding medium. In the latter case, the boundary condition takes the form

$$\left(\frac{\partial T}{\partial n} + hT\right)\bigg|_S = hT_{\text{med}},$$

where $\partial/\partial n$ indicates differentiation with respect to the exterior normal to S, T_{med} is the temperature of the surrounding medium, and h is the heat exchange coefficient or emissivity. Without loss of generality, we can assume that $T_{\text{med}} = 0$; this assumption is made in all the problems involving heat conduction except Prob. 155.[4]

We now give a few problems on the formulation of initial and boundary conditions for the equation of heat conduction (and for the related diffusion equation).

12. Let the temperature of a conductor in the form of an infinite cylinder of radius a be initially the same as that of the surrounding medium. Suppose that starting from the time $t = 0$, the conductor is heated by a constant

[3] The density of heat current (i.e., the heat flux) is described by the vector

$$\mathbf{q} = -k \operatorname{grad} T.$$

[4] Examples of other boundary conditions encountered in the applications are given in Probs. 365, 367 and 370.

electric current releasing an amount of heat Q per unit volume of the conductor. Give a mathematical formulation of the corresponding problem of heat conduction, assuming that the heat exchange at the surface of the conductor obeys Newton's law.[5]

Ans. The temperature $T(r, t)$ satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{\partial T}{\partial \tau} - \frac{Q}{k}, \qquad \tau = \frac{kt}{c\rho},$$

with initial condition

$$T\big|_{\tau=0} = 0$$

and boundary condition

$$\left(\frac{\partial T}{\partial r} + hT\right)\bigg|_{r=a} = 0.$$

13. A homogeneous sphere of radius a is heated for a long time by heat sources uniformly distributed throughout its volume with density Q. Write the equations which describe the cooling of the sphere after the sources are turned off, assuming that the heat exchange between the surface of the sphere and the surrounding medium, during both the heating and cooling, obeys Newton's law.

Ans.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{\partial T}{\partial \tau}, \qquad \left(\frac{\partial T}{\partial r} + hT\right)\bigg|_{r=a} = 0,$$

$$T\big|_{\tau=0} = \frac{Q}{6k}(a^2 - r^2) + \frac{Qa}{3kh}.$$

14. Two slabs of thicknesses a_1 and a_2, made from different materials and heated to temperatures T_1^0 and T_2^0, are put into contact with each other at the time $t = 0$. Write the equations governing the resulting process of temperature equalization, assuming that the free surfaces are thermally insulated from the surrounding medium.

Ans.

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{c_1\rho_1}{k_1}\frac{\partial T_1}{\partial t} \quad (0 < x < a_1), \qquad \frac{\partial^2 T_2}{\partial x^2} = \frac{c_2\rho_2}{k_2}\frac{\partial T_2}{\partial t} \quad (a_1 < x < a_1 + a_2),$$

with initial conditions

$$T_1\big|_{t=0} = T_1^0, \qquad T_2\big|_{t=0} = T_2^0,$$

[5] It is recommended that the problem be solved directly from underlying physical principles, without regarding Fourier's equation as known.

and boundary conditions

$$\frac{\partial T_1}{\partial x}\Big|_{x=0} = 0, \quad T_1\big|_{x=a_1} = T_2\big|_{x=a_1}, \quad k_1\frac{\partial T_1}{\partial x}\Big|_{x=a_1} = k_2\frac{\partial T_2}{\partial x}\Big|_{x=a_1} \quad \frac{\partial T_2}{\partial x}\Big|_{x=a_1+a_2} = 0.$$

15. A nonuniformly heated body in the form of a circular ring of radius a with a small cross section cools by giving off heat from its lateral surface. Write the equations describing the corresponding process of temperature equalization, assuming that the temperature drop inside the ring can be neglected and that the surface cooling obeys Newton's law.

Ans.

$$\frac{1}{a^2}\frac{\partial^2 T}{\partial \varphi^2} = \frac{\partial T}{\partial \tau} + \frac{hp}{S}T, \qquad \tau = \frac{kt}{c\rho},$$

where p is the perimeter, S the cross-sectional area and h the heat exchange coefficient. The temperature, which must be a periodic function of the angular coordinate φ, satisfies the initial condition

$$T\big|_{\tau=0} = f(\varphi),$$

where f is a given function.

16. Show that the concentration $C(x, y, z, t)$ of a substance diffusing in a gas or liquid obeys the differential equation

$$\Delta C = \frac{1}{D}\frac{\partial C}{\partial t} - \frac{Q}{D},$$

where Q is the source density of the diffusing substance and D is the diffusion coefficient.

Hint. Starting from Nernst's law $\mathbf{q} = -\text{grad}\, C$ (where the vector $\mathbf{q}$ is the density of flow of the diffusing substance), write a conservation equation for an arbitrary volume element.

3. Electricity and Magnetism

An important class of problems of mathematical physics involves integration of the differential equations arising in various branches of electromagnetic theory. Assuming that the reader has previously encountered this subject (see G5, J6, P1), we shall regard the following basic equations as known:

1. The equations of electrostatics

$$\Delta u = -\frac{4\pi\rho}{\varepsilon}, \qquad \mathbf{E} = -\text{grad}\, u,$$

where u is the potential of the electrostatic field $\mathbf{E}$, $\rho = \rho(M)$ is the volume density of charge at the point M, ε is the dielectric constant of the medium, and Δ is the Laplacian operator.

2. The equations

$$\Delta u = -\frac{Q}{\sigma}, \qquad \mathbf{j} = -\sigma \operatorname{grad} u, \tag{1}$$

for the distribution of d-c current density inside a homogeneous conductor, where u is the potential of the current field, $\mathbf{j}$ is the current density vector, $Q = Q(M)$ is the volume density of current sources (in particular, Q may vanish), and σ is the conductivity.

3. The equations

$$\Delta \mathbf{A} = -\frac{4\pi\mu}{c}\mathbf{j}^{(e)}, \qquad \mathbf{H} = \frac{1}{\mu}\operatorname{curl}\mathbf{A}$$

for the magnetic field due to d-c currents, where $\mathbf{A}$ is the vector potential of the magnetic field $\mathbf{H}$, the vector $\mathbf{j}^{(e)}$ is the density of the (external) currents producing the magnetic field, μ is the magnetic permeability of the medium, c is the velocity of light in vacuum, and Δ is the Laplacian operator.[6]

4. *Maxwell's equations*

$$\operatorname{curl}\mathbf{H} = \frac{\varepsilon}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi\sigma}{c}\mathbf{E} + \frac{4\pi}{c}\mathbf{j}^{(e)},$$

$$\operatorname{curl}\mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t},$$

$$\operatorname{div}\mathbf{E} = \frac{4\pi\rho}{\varepsilon},$$

$$\operatorname{div}\mathbf{H} = 0$$

for the electromagnetic field in a homogeneous isotropic medium, where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field vectors, ε, μ and σ are the dielectric constant, the magnetic permeability and the conductivity of the medium, c is the velocity of light, and ρ and $\mathbf{j}^{(e)}$ are the charge and current densities producing the field.[7]

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[6] The components of the vector $\Delta\mathbf{A}$ in a Cartesian coordinate system are ΔA_x, ΔA_y and ΔA_z. To calculate the components of the vector $\Delta\mathbf{A}$ in other coordinate systems, one should use the relation

$$\Delta\mathbf{A} = \operatorname{grad}\operatorname{div}\mathbf{A} - \operatorname{curl}\operatorname{curl}\mathbf{A}.$$

Expressions for the components of $\Delta\mathbf{A}$ in cylindrical and spherical coordinates are given on p. 389–390.

[7] It should be noted that if $\mathbf{j}^{(e)}$ is given, then ρ cannot be chosen arbitrarily, but must satisfy the differential equation

$$\frac{\partial \rho}{\partial t} + \frac{4\pi\sigma}{\varepsilon}\rho = -\operatorname{div}\mathbf{j}^{(e)}$$

implied by the first and third Maxwell equations.