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Chapter 1

The Solution of Ordinary Differential Equations

1.1 INTRODUCTION

The first order linear differential equation

$$\frac{dw}{dz} + f(z)w = g(z),$$

where f and g are given functions of z, always possesses a formal solution in terms of quadratures. Multiplication by the integrating factor $\exp\left[\int f(z)dz\right]$ reduces the equation to

$$\frac{d}{dz}\left[w \exp\left(\int f dz\right)\right] = g(z) \exp\left(\int f dz\right),$$

and integration yields the general solution

$$w = \exp\left(-\int f dz\right) \int g(z) \exp\left(\int f dz\right)dz + A \exp\left(-\int f dz\right).$$

The success of the method depends on whether the two integrals involved can be expressed in terms of the standard elementary or transcendental functions.

But no similar solution in terms of quadratures based on the coefficients is possible for the second order linear homogeneous equation

$$\frac{d^2w}{dz^2} + f(z)\frac{dw}{dz} + g(z)w = 0,$$

and for the second order inhomogeneous linear equation

$$\frac{d^2w}{dz^2} + f(z)\frac{dw}{dz} + g(z)w = h(z),$$

the coefficients being functions of the independent variable z.

Throughout the text, we shall consistently use a prime to denote differentiation with respect to z (or any other independent variable used in the immediate context).

For a homogeneous equation with constant coefficients

$$w'' + aw' + bw = 0,$$

explicit solutions may easily be written down. If α and β denote the distinct roots of the auxiliary quadratic

$$\lambda^2 + a\lambda + b = 0,$$

then

$$w = Ae^{\alpha z} + Be^{\beta z},$$

while if the roots are identical,

$$w = (A + Bz)e^{\alpha z}.$$

The solution is valid even when α and β are complex. When a and b are real, these roots will form a conjugate pair $\mu \pm i\nu$, from which it follows that

$$w = e^{\mu z}(C \cos \nu z + D \sin \nu z)$$

since $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$.

In particular, if

$$w'' + n^2 w = 0 \qquad (n^2 \text{ real and positive})$$

the solution is

$$w = A \cos nz + B \sin nz \qquad (n \neq 0)$$

representing an oscillatory solution, or in complex exponential form,

$$w = Ce^{inz} + De^{-inz}.$$

Again, if

$$w'' - m^2 w = 0 \qquad (m^2 \text{ real and positive})$$

we have

$$w = A \cosh mz + B \sinh mz = Ce^{mz} + De^{-mz},$$

a non-oscillatory solution. The complete difference in the nature of these two types of solutions, one oscillatory and one non-oscillatory, will pervade the theory of the propagation of waves in homogeneous media of differing properties.

The inhomogeneous equation with constant coefficients

$$w'' + aw' + bw = h(z)$$

may be solved (i) by the D operator method when $h(z)$ is a polynomial, an exponential function, a sine or cosine, or sums or products of such functions, and (ii) by the method of variation of parameters when $h(z)$ is a general function. We assume that the reader is familiar with the D operator method, but the method of variation of parameters will be dealt with in Section 1.6.

Equations with variable coefficients will form a large portion of the work of this book. Under certain circumstances, solutions exist expressible in terms of the standard functions, though this fact may not be obvious upon a first inspection of the equation, since changes of the independent and dependent variables may be necessary to transform the equation into standard form with recognisable solutions. More generally, formal texts on the analytical theory of differential equations provide what are known as existence theorems that show when equations possess solutions (even though not expressible in explicit analytical form). But the knowledge that solutions exist permits the investigator to seek approximate solutions or numerical solutions. There would be no point in seeking an approximate solution of an equation if an existence theorem is not satisfied. Throughout the text, which will largely be occupied with (i) approximate solutions and their physical meaning, and (ii) general properties of exact solutions which cannot necessarily be obtained explicitly, the existence theorems will be satisfied. Although these theorems cannot be dealt with in the text, readers should be familiar with such analytical approaches, since they form an essential basis of the applied mathematician's toolkit, and experience must decide when such tools are essential in any particular investigation.

1.2 FORMATION OF DIFFERENTIAL EQUATIONS FROM GIVEN SOLUTIONS

A linear homogeneous differential equation of the second order has two independent solutions. Conversely, if two independent functions are given, a second order equation may be formed possessing these two functions as two independent solutions.

If the two given functions are $u(z)$ and $v(z)$, the proposed general solution w will be

$$w = Au(z) + Bv(z),$$

A and B being arbitrary constants. We form w' and w'':

$$w' = Au'(z) + Bv'(z),$$
$$w'' = Au''(z) + Bv''(z),$$

and eliminate A and B by the determinantal equation

$$\begin{vmatrix} w & u & v \\ w' & u' & v' \\ w'' & u'' & v'' \end{vmatrix} = 0.$$

Clearly the coefficient of w'' must not vanish if a second order equation is to be produced. This means that $uv' - u'v$, known as the Wronskian of the two given functions, must not vanish, this being the condition for u and v to be independent functions.

Example If the two given independent functions are e^{z^2} and e^{-z^2}, we have

$$w = Ae^{z^2} + Be^{-z^2},$$
$$w' = 2Aze^{z^2} - 2bze^{-z^2},$$
$$w'' = (2 + 4z^2)Ae^{z^2} + (-2 + 4z^2)Be^{-z^2}.$$

Eliminating A and B, we have

$$\begin{vmatrix} w & 1 & 1 \\ w' & 2z & -2z \\ w'' & 2 + 4z^2 & -2 + 4z^2 \end{vmatrix} = 0,$$

reducing to

$$w'' - \frac{w'}{z} - 4z^2 w = 0.$$

Conversely, this equation may be solved by transforming the independent variable to $t = z^2$.

When the coefficients of w' and/or w have a singularity at a point $z = a$, the solution usually has a singularity at $z = a$. In the present equation, the coefficient of w' has a singularity at $z = 0$, but this is an apparent singularity since the given form of w has no singularity at $z = 0$.

Example If $g(z)$ is a thrice-differentiable function of z, find the differential equation satisfied by the two independent functions $g'^{-\frac{1}{2}}e^{ikg}$ and $g'^{-\frac{1}{2}}e^{-ikg}$ valid at points where g' does not vanish.

In this case, differentiation is easier if we first multiply the general solution by $g'^{\frac{1}{2}}$, giving

$$g'^{\frac{1}{2}}w = Ae^{ikg} + Be^{-ikg},$$
$$g'^{\frac{1}{2}}w' + \tfrac{1}{2}g'^{-\frac{1}{2}}g''w = ikAg'e^{ikg} - ikBg'e^{-ikg}.$$

Before differentiating a second time, divide by the coefficient g' that appears on the right-hand side:

$$g'^{-\frac{1}{2}}w' + \tfrac{1}{2}g'^{-3/2}g''w = ikAe^{ikg} - ikBe^{-ikg}.$$

Differentiation now produces five terms on the left-hand side, two of which cancel:

$$g'^{-\frac{1}{2}}w'' - \tfrac{1}{2}g'^{-3/2}g''w' + \tfrac{1}{2}g'^{-3/2}g''w' + \tfrac{1}{2}g'^{-3/2}g'''w - \tfrac{3}{4}g'^{-5/2}g''^{2}w$$
$$= -k^2Ag'e^{ikg} - k^2Bg'e^{-ikg}.$$

In this case, there is no need to form a determinant in order to eliminate A and B, since the right-hand side of this equation is merely $-k^2g'^{3/2}w$. Multiplying by $g'^{\frac{1}{2}}$, and rearranging, we obtain

$$w'' + k^2\left[g'^2 + \frac{1}{k^2}\left(\frac{g'''}{2g'} - \frac{3g''^2}{4g'^2}\right)\right]w = 0. \qquad (1)$$

This equation is stated to be in <u>normal form</u>, since the term in w' is absent.

For example, if $g(z) = z^4$, the equation

$$w'' + k^2\left(16z^6 - \frac{15}{4k^2z^2}\right)w = 0$$

has the general solution $w = Az^{-3/2}e^{ikz^4} + Bz^{-3/2}e^{-ikz^4}$.

1.3 CHANGE OF THE DEPENDENT VARIABLE

There is given the inhomogeneous equation

$$w'' + f(z)w' + g(z)w = h(z).$$

If $\phi(z)$ is a given function, we introduce the new dependent variable u, defined by $w = \phi u$. The first and second derivatives,

$$w' = \phi'u + \phi u',$$

and

$$w'' = \phi''u + 2\phi'u' + \phi u'',$$

are substituted into the equation, yielding

$$\phi''u + 2\phi'u' + \phi u'' + f(\phi'u + \phi u') + g\phi u = h,$$

or

$$u'' + \left(\frac{2\phi'}{\phi} + f\right) u' + \left(\frac{\phi''}{\phi} + \frac{f\phi'}{\phi} + g\right) u = \frac{h}{\phi}.$$

If ϕ is suitably chosen, this may reduce to a standard equation with recognisable solutions.

Example Solve the equation

$$z^2w'' + (4z + 3z^2)w' + (2 + 6z + 2z^2)w = 2e^{-3z},$$

using the suggested transformation $w = z^n u$ (where n is to be found) to reduce the equation to one with constant coefficients.

We substitute into the equation the derivatives

$$w' = z^n u' + nz^{n-1}u, \quad w'' = z^n u'' + 2nz^{n-1}u' + n(n-1)z^{n-2}u,$$

obtaining upon simplification,

$$u'' + \left(3 + \frac{2n + 4}{z}\right) u' + \left(2 + \frac{3n + 6}{z} + \frac{n^2 + 3n + 2}{z^2}\right)u = 2z^{-n-2}e^{-3z}.$$

The coefficients reduce to constants when

$$2n + 4 = 0, \quad 3n + 6 = 0, \quad n^2 + 3n + 2 = 0,$$

possessing a common root $n = -2$. Clearly the numbers in the coefficients of the original equation had to be of a special form in order to achieve this result. The equation now becomes

$$u'' + 3u' + 2u = 2e^{-3z}.$$

The complementary function is $Ae^{-z} + Be^{-2z}$.

The particular integral found by the D operator method is

$$u = \frac{2e^{-3z}}{D^2 + 3D + 2} = \frac{2e^{-3z}}{3^2 - 3.3 + 2} = e^{-3z}.$$

Hence

$$w = \frac{Ae^{-z} + Be^{-2z} + e^{-3z}}{z^2}.$$

The transformation $w = \phi u$ reduces the equation

$$w'' + fw' + gw = 0$$

to normal form when ϕ is chosen to ensure that the coefficient of u' vanishes. Hence $f + 2\phi'/\phi = 0$, so the required transformation is

$$w = \exp\,(-\tfrac{1}{2}\textstyle\int f dz)\, u,$$

a result that should be committed to memory. Differentiating $\phi'/\phi = -\frac{1}{2}f$, we obtain

$$\frac{\phi''}{\phi} - \frac{\phi'^2}{\phi^2} = -\tfrac{1}{2}f' \quad \text{and} \quad \frac{\phi''}{\phi} = \tfrac{1}{4}f^2 - \tfrac{1}{2}f'.$$

The normal form of the equation then becomes

$$u'' + (g - \tfrac{1}{4}f^2 - \tfrac{1}{2}f')u = 0. \qquad (2)$$

<u>Example</u> Reduce to normal form the equation

$$w'' - 4zw' + (4z^2 - z - 2)w = 0.$$

The required transformation is $w = \phi u$, where

$$\phi = \exp\left[-\tfrac{1}{2}\textstyle\int(-4z)dz\right] = e^{z^2}.$$

Equation (2) then becomes

$$u'' + \left[4z^2 - z - 2 - \tfrac{1}{4}(4z)^2 - \tfrac{1}{2}(-4)\right]u = 0,$$

reducing to $u'' = zu$, a transcendental differential equation often referred to as <u>the Stokes equation</u>. Its properties will be studied in Chapter 7.

<u>Example</u> Reduce to normal form <u>Bessel's equation</u> of order ν:

$$w'' + z^{-1}w' + (1 - \nu^2 z^{-2})w = 0.$$

The required transformation is

$$w = \exp\,(-\tfrac{1}{2}\textstyle\int z^{-1}dz)u = \exp\,(-\tfrac{1}{2}\log z)u = z^{-\frac{1}{2}}u,$$

where u satisfies the equation

$$u'' + (1 - \nu^2 z^{-2} - \tfrac{1}{4}z^{-2} + \tfrac{1}{2}z^{-2})u = 0,$$

or

$$u'' + \left[1 - (\nu^2 - \tfrac{1}{4})z^{-2}\right]u = 0.$$

In particular, when $\nu = \frac{1}{2}$, the equation $u'' + u = 0$ has solutions

$e^{\pm iz}$, showing that Bessel functions of order $\frac{1}{2}$ can be expressed linearly in terms of the simpler functions $z^{-\frac{1}{2}}e^{\pm iz}$, or $z^{-\frac{1}{2}}\cos z$ and $z^{-\frac{1}{2}}\sin z$.

For general values of ν, when $|z|$ is large the equation for u reduces approximately to $u'' + u = 0$, showing that approximate solutions of Bessel's equation of order ν for large $|z|$ are $z^{-\frac{1}{2}}e^{\pm iz}$ But the reader should be warned that such solutions are valid only in restricted domains in the complex z-plane; Chapter 7 examines this subject in more detail.

A second kind of transformation of the dependent variable, non-linear in character, may be made yielding a non-linear differential equation. Let $w = F(u)$, where F denotes a suitable function of u. Then the chain rule gives

$$\frac{dw}{dz} = \frac{dF}{du} \cdot \frac{du}{dz}, \qquad \frac{d^2w}{dz^2} = \frac{d^2F}{du^2}\left(\frac{du}{dz}\right)^2 + \frac{dF}{du} \cdot \frac{d^2u}{dz^2} .$$

The equation $w'' + fw' + gw = 0$ becomes

$$\frac{dF}{du}u'' + f\frac{dF}{du}u' + \frac{d^2F}{du^2}u'^2 + gF = 0,$$

a non-linear equation of the second order in the dependent variable u.

Example . Reduce an equation in normal form to a Riccati equation.

In the given equation $w'' + gw = 0$, substitute

$$w = \exp\left(\int u\,dz\right).$$

Then

$$w' = u \exp\left(\int u\,dz\right), \quad w'' = (u' + u^2)\exp\left(\int u\,dz\right),$$

yielding

$$u' + u^2 + g = 0 . \tag{3}$$

The second order linear equation is thus reduced to a first order equation, but at the expense of being rendered non-linear. Riccati equations of this type will form part of certain approximation procedures discussed in Chapter 7.

1.4 CHANGE OF THE INDEPENDENT VARIABLE

In the given equation

$$w'' + fw' + gw = h,$$

where z is the independent variable, we introduce the given substitution $t = t(z)$ and its inverse $z = z(t)$. The derivatives with respect to z become:

$$\frac{dw}{dz} = \frac{dw}{dt} \cdot \frac{dt}{dz}, \qquad \frac{d^2w}{dz^2} = \frac{d^2w}{dt^2}\left(\frac{dt}{dz}\right)^2 + \frac{dw}{dt} \cdot \frac{d^2t}{dz^2},$$

yielding

$$\left(\frac{dt}{dz}\right)^2 \frac{d^2w}{dt^2} + \left(\frac{d^2t}{dz^2} + f(z)\frac{dt}{dz}\right) \frac{dw}{dt} + g(z)w = h(z),$$

where dt/dz, d^2t/dz^2, $f(z)$, $g(z)$, $h(z)$ must be expressed in terms of the new variable t. In suitable cases, the resulting equation may be more susceptible to solution than the former equation in the independent variable z.

Example Find the general solution of the differential equation

$$4z\frac{d^2w}{dz^2} + 2(1 - z^{\frac{1}{2}}) \frac{dw}{dz} - 6w = \exp\,(3z^{\frac{1}{2}}) \qquad z \neq 0$$

by using the substitution $t = z^{\frac{1}{2}}$.

The required derivatives of the dependent variable are

$$\frac{dw}{dz} = \frac{dw}{dt} \cdot \tfrac{1}{2}z^{-\frac{1}{2}}, \qquad \frac{d^2w}{dz^2} = \frac{d^2w}{dt^2} \cdot \tfrac{1}{4}z^{-1} - \frac{dw}{dt} \cdot \tfrac{1}{4}z^{-3/2},$$

reducing the given equation directly to

$$\frac{d^2w}{dt^2} - \frac{dw}{dt} - 6w = e^{3t}$$

The complementary function is $w = Ae^{3t} + Be^{-2t}$.

The particular integral must be obtained by the rule that extracts e^{3t} from the operand by replacing the operator $D \equiv d/dt$ by $D + 3$:

$$w = \frac{1}{D^2 - D - 6} e^{3t} = e^{3t} \frac{1}{(D+3)^2 - (D+3) - 6}$$

$$= e^{3t} \frac{1}{D^2 + 5D} = \frac{te^{3t}}{5},$$

since the only relevant term in the denominator of the operator is $5D$, meaning integrate once with respect to t. Hence the general solution is

$$w = (A + \tfrac{1}{5}t)\, e^{3t} + Be^{-2t}$$

$$= (A + \tfrac{1}{5}z^{\frac{1}{2}}) \exp (3z^{\frac{1}{2}}) + B \exp (-2z^{\frac{1}{2}}).$$

<u>Example</u> Excluding the origin, change the dependent variable w to zu and the independent variable z to $1/t$ in the given equation

$$z^5 \frac{d^2w}{dz^2} - zw = 1,$$

and hence obtain its general solution.

<u>*(1) The dependent variable*</u>: The derivatives of w with respect to z become

$$\frac{dw}{dz} = z\frac{du}{dz} + u, \qquad \frac{d^2w}{dz^2} = z\frac{d^2u}{dz^2} + 2\frac{du}{dz},$$

yielding upon substitution,

$$z^6 \frac{d^2u}{dz^2} + 2z^5 \frac{du}{dz} - z^2 u = 1.$$

<u>*(2) The independent variable*</u>: If $t = 1/z$, we have

$$\frac{du}{dz} = -z^{-2}\frac{du}{dt}, \qquad \frac{d^2u}{dz^2} = z^{-4}\frac{d^2u}{dt^2} + 2z^{-3}\frac{du}{dt},$$

yielding upon substitution and simplification,

$$\frac{d^2u}{dt^2} - u = z^{-2} = t^2.$$

The complementary function is $u = Ae^{t} + Be^{-t}$.
The particular integral is given by

$$u = \frac{t^2}{D^2 - 1} = -(1 + D^2 + D^4 + \ldots)t^2 = -t^2 - 2.$$

Hence the general solution is given by

$$u = Ae^{1/z} + Be^{-1/z} - z^{-2} - 2,$$

and

$$w = zu = z(Ae^{1/z} + Be^{-1/z}) - z^{-1} - 2z.$$

1.5 GENERAL SOLUTION DERIVED FROM A KNOWLEDGE OF ONE SOLUTION

We shall suppose that one solution of the differential equation

$$w'' + fw' + gw = 0$$

is known, where f and g are functions of z. Such a solution may be spotted by inspection, or some special property of the variable coefficients f and g may suggest a simple solution. Again, a simple solution may be thrown up by the method consider in Section 1.8, whereby power series solutions of the equation are obtained.

Let $p(z)$ be the known solution, satisfying

$$p'' + fp' + gp = 0.$$

If possible, let a second independent solution be given by the product $w = pu$, where u is to be found. Differentiating twice and substituting, we obtain

$$pu'' + 2p'u' + p''u + f(pu' + p'u) + gpu = 0.$$

The three terms $(p'' + fp' + gp)u$ vanish, since p satisfies the given equation. The remaining three terms may be separated as follows:

$$\frac{u''}{u'} = -\frac{2p' + fp}{p},$$

since the underived variable u has disappeared from the equation. Logarithmic integration yields

$$\log u' = -2 \log p - \int f dz,$$

where no constant of integration need be added to the indefinite integral. Hence

$$u' = p^{-2} \exp\left(-\int f dz\right),$$

and

$$w = pu = p(z)\int \frac{\exp\,(-\int f dz)}{p^2}\,dz. \qquad (4)$$

The general solution of the equation is therefore

$$w = Ap(z) + Bp(z)\int \frac{\exp\,(-\int f dz)}{p^2}\,dz.$$

Example The given differential equation

$$2z(1 - z)w'' + (1 - z)w' + 3w = 0$$

has one simple solution $p = z^{\frac{1}{2}}(1 - z)$. Find the other solution.

It may easily be checked that $z^{\frac{1}{2}}(1 - z)$ satisfies the equation. Then u is given by

$$u = \int \frac{\exp\,(-\frac{1}{2}\int z^{-1}dz)}{z(1 - z)^2}\,dz$$

where the function f in solution (4) has been obtained by dividing by the coefficient of w'', the coefficient of w'' being assumed equal to unity in the general theory. Since

$$\exp\,(-\tfrac{1}{2}\int z^{-1}dz) = \exp\,(-\tfrac{1}{2}\log z) = z^{-\frac{1}{2}},$$

we have

$$u = \int \frac{dz}{z^{3/2}(1 - z)^2}$$

$$= \int \frac{2dt}{t^2(1 - t^2)} \qquad \text{(place } z = t^2\text{)}$$

$$= \int \left(\frac{2}{t^2} + \frac{\frac{1}{2}}{(1 - t)^2} + \frac{\frac{1}{2}}{(1 + t)^2} + \frac{\frac{3}{2}}{1 + t} + \frac{\frac{3}{2}}{1 - t}\right) dt$$

$$= -\frac{2}{t} + \frac{\frac{1}{2}}{1 - t} - \frac{\frac{1}{2}}{1 + t} + \frac{3}{2}\log\,(1 + t) - \frac{3}{2}\log\,(1 - t)$$

$$= \frac{3t^2 - 2}{t(1 - t^2)} + \frac{3}{2}\log\frac{1 + t}{1 - t}\,.$$

Hence

$$w = z^{\frac{1}{2}}(1 - z)u$$

$$= 3z - 2 + \frac{3}{2}z^{\frac{1}{2}}(1 - z)\log\frac{1 + z^{\frac{1}{2}}}{1 - z^{\frac{1}{2}}}.$$

The form of the this answer must of course be modified along various domains of the real z-axis.

1.6 VARIATION OF PARAMETERS

This is a general method in terms of quadratures for solving the inhomogeneous equation

$$w'' + fw' + gw = h, \qquad (5)$$

where f, g and h are functions of z, when the general solution of the homogeneous equation (with h replaced by zero) is known. The method is applicable to linear equations of order n.

Let $p(z)$ and $q(z)$ be two independent solutions of the equation

$$w'' + fw' + gw = 0.$$

We seek suitable functions $a(z)$ and $b(z)$, so that the linear combination

$$w = a(z)p(z) + b(z)q(z)$$

is a solution of equation (5) satisfying the initial conditions $w(z_0) = w'(z_0) = 0$. Such functions $a(z)$ and $b(z)$ are not, of course, unique, though the resulting expression for w will be unique. We choose $a(z)$ and $b(z)$ to be the most useful and symmetrical in form. In fact, since the two functions will satisfy only one equation (5) upon substitution, we are at liberty to choose a second equation to be satisfied by $a(z)$ and $b(z)$ more or less arbitrarily.

Differentiation of w yields

$$w' = ap' + bq' + a'p + b'q.$$

At this stage, we deliberately choose $a'p + b'q$ to vanish. No further choice may be made for a second order equation. Further differentiation yields

$$w'' = ap'' + bq'' + a'p' + b'q'.$$