# THE UNIVERSAL ENCYCLOPEDIA OF MATHEMATICS

WITH A FOREWORD BY

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*Editor, The World of Mathematics*

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## PUBLISHER'S NOTE

We believe this may be the first popular encyclopedia or reference book of mathematics of its kind, arranged in alphabetical order of subjects. It is based on the *Rechenduden* of the Bibliographisches Institut in Mannheim. Well over 200,000 copies of this were sold within two years. The present version has been thoroughly adapted to Anglo-American needs and has been considerably altered in the process. We are grateful to the mathematicians on both sides of the Atlantic who have undertaken this task or helped us with their advice.

The book starts from the beginning of secondary school mathematics and includes many of the topics studied for a university degree. It is not addressed to the professional mathematician, but it will help the student to become one. It is intended for the Man in the Street, the harassed parent, and the technical or engineering student; or for the scientist, engineer and accountant for whom mathematics have not lost their fascination.

The encyclopedia is reliable and the explanations clear. It contains a large collection of formulae (arithmetical, algebraic, geometric, trigonometric, special functions, series, differential and integral calculus). There are also tables of mathematical functions (powers, square and cube roots, logarithms, trigonometrical functions, exponential functions, length of arcs and angles in degrees and radians, tables of differences)—every value electronically calculated separately, with no interpolated values.

# FOREWORD

I AM GLAD of the opportunity to say a few words about this clearly written, sensibly arranged and reliable reference work.

It is a translation of a widely used German compendium, designed to serve the needs of high school and college students, which encompasses many branches of mathematics from arithmetic through the calculus and includes a collection of essential formulae and tables. While the higher branches such as group theory or algebraic topology are not treated, the coverage within the limits indicated is succinct and to the point, and it is safe to predict that the book will be a much consulted companion, a comfort for the student, engineer or teacher to have within arm's reach.

The thought may readily cross one's mind that there must be several, if not indeed many, references of similar scope already available. Curiously enough, despite the immense growth in popularity of mathematical studies since the last world war, and the expanding mathematical curricula in the schools, this is not the case. Popularizations of mathematics abound, and the textbook field has been almost literally swamped with new approaches and innumerable revisions of standard works; also, there has been an outpouring of self-help and self-teaching guides. But if you are looking for an explicit, simply-written, alphabetically-arranged lexicon of mathematics, geared to average requirements, which explains such matters as angles, the design and operation of calculating machines, conic sections, continued fractions, mathematical induction, regular polygons, the Pythagorean theorem, the real number system, infinite series, nomography, logarithms, linear transformations, the solution of equations, the binomial distribution, complex numbers, Archimedes's spiral, the computation of interest, the indefinite integral, Pascal's triangle, the Roman number system, rigid motions, Platonic solids, the use of determinants, the nature of vectors, trigonometric relations, and many other concepts, methods and applications of mathematics, you will find this book unique.

The treatment of most topics seems to me just right. A straightforward, unencumbered definition is usually followed by a more detailed discussion, which often includes a number of carefully selected and worked out examples. Where diagrams are used they are helpful and utilitarian, not introduced for decoration. It is not assumed that you already know about a subject before looking it up, but it is assumed, and quite properly, that you are equipped to make your own way at that level. For instance, if you look up affine transformations, you are assumed to be reasonably conversant with the fundamentals of analytical geometry, and if logarithms are your quarry you must at least have some notion of the behaviour of exponents. It would, of course, be a mistake to suppose that you can teach yourself all about a topic by reading the relevant article in the encyclopedia. That is not its purpose. What it sets out to do and what I believe it is successful in doing in a remarkably high proportion of its entries is to explain the elements of a concept or method, to indicate its connections and relations with other parts of mathematics and to give the reader the proper bearings and a good start in following up, if he is so inclined, by consulting appropriate texts and monographs.

Teachers will, I believe, be grateful for this book, for themselves and for their students. Parents will like it, as will the adult reader who still retains a flickering interest in mathematics which his earlier schooling has not extinguished. The book is a prize I wish I had had when I was a student, and even now I look forward to having it on my shelf.

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CHEVY CHASE, MD.
SEPTEMBER 1963

# ALPHABETICAL ENCYCLOPEDIA UNDER SUBJECTS

**Absolute value**

The absolute value of a real number $a$, written $|a|$ (read: mod $a$), equals $a$ if $a$ is positive, and $-a$ if $a$ is negative, *e.g.* $|-2| = 2$, $|+2| = 2$. The absolute value of a complex number $z = a + ib$ is reckoned as $z = \sqrt{a^2 + b^2}$. In the *Argand diagram* (see below), the absolute value (modulus) of a complex number is the distance from the origin of the point representing the number.

**Acute triangle**

A triangle is said to be acute if each of the three interior angles is less than 90° (*i.e.* if all interior angles are acute).

**Addition**

Addition is one of the four fundamental operations of arithmetic. The addition sign is '+' (read: plus). The numbers which are added are called summands.

$$4 \quad + \quad 3 \quad = \quad 7$$

Summand plus Summand equals Sum

*Addition of fractions*

Fractions can be added directly only if they have equal denominators, *e.g.* $\frac{3}{17} + \frac{4}{17} = \frac{7}{17}$.

If fractions with unequal denominators are to be added they must first be brought to their *least common denominator* (see below).

*Addition of literal numbers*

Identically-named numbers can be added (and subtracted) by adding the coefficients:

$$2a + 3b + a + 4b = 3a + 7b.$$

Summands may be interchanged (*commutative law*):

$$a + b = b + a.$$

With more than two summands, brackets may be inserted (*associative law*):

$$a + b + c = (a + b) + c = a + (b + c).$$

*Addition of directed numbers*

No further rule is needed for the addition of positive numbers:

$$a + (+b) = a + b, \quad 5 + (+3) = 5 + 3 = 8.$$

To add one negative number to another, its absolute value is subtracted:

$$a + (-b) = a - b, \qquad (-a) + (-b) = -(a + b);$$
$$5 + (-3) = 5 - 3 = 2, \quad (-5) + (-3) = -(5 + 3) = -8.$$

*Addition of powers and surds*

Powers and surds must be treated like literal numbers for the purposes of addition, *e.g.*

$$a^2 + 4 \,.\, b^3 + 4 \,.\, c^3 + 4 \,.\, a^2 = 5 \,.\, a^2 + 4(b^3 + c^3);$$
$$3\sqrt{2} + 5\sqrt{3} + \sqrt{2} = 4\sqrt{2} + 5\sqrt{3}.$$

*Addition of complex numbers*

In adding complex numbers the real and imaginary parts must be added separately, *e.g.*

$$(5 + 3i) + (17 - i) = 22 + 2i.$$

## Addition theorems for trigonometric functions

The addition theorems enable us to calculate the functions $\sin(\alpha + \beta)$, $\cos(\alpha + \beta)$, $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$, given $\sin\alpha$, $\sin\beta$, $\cos\alpha$ and $\cos\beta$, and to calculate $\tan(\alpha + \beta)$, $\tan(\alpha - \beta)$, $\cot(\alpha + \beta)$, $\cot(\alpha - \beta)$, given $\tan\alpha$, $\tan\beta$, $\cot\alpha$, $\cot\beta$.

The addition theorems are as follows:

$$\text{I.} \quad \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\text{II.} \quad \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

III. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

IV. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

V. $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

VI. $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

VII. $\cot(\alpha + \beta) = \dfrac{\cot\alpha\cot\beta - 1}{\cot\beta + \cot\alpha}$

VIII. $\cot(\alpha - \beta) = \dfrac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$

The proof of formulae I–IV can be derived with the help of the following figures.

In Fig. 1, clearly,

$$\overline{OD} = \cos\beta,\ \overline{DE} = \overline{OD}\sin\alpha = \sin\alpha\cos\beta,$$

$$\overline{OE} = \overline{OD}\cos\alpha,\ \overline{AD} = \sin\beta,\ \overline{CD} = \overline{AD}\sin\alpha.$$

Further, $\overline{AC} = \overline{AD}\cos\alpha = \sin\beta\cos\alpha.$

Then $\sin(\alpha + \beta) = \dfrac{\overline{AB}}{\overline{AO}} = \overline{AB} = \overline{AC} + \overline{CB} = \overline{AC} + \overline{DE},$

that is, $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha.$

Similarly,

$$\cos(\alpha + \beta) = \frac{\overline{OB}}{1} = \overline{OB} = \overline{OE} - \overline{BE} = \overline{OE} - \overline{CD},$$

that is, $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

In Fig. 2, similarly,

$$\overline{AD} = \sin\beta,\ \overline{DE} = \overline{AD}\cos\alpha = \cos\alpha\sin\beta,$$

$$\overline{OD} = \cos\beta,\ \overline{DC} = \overline{OD}\sin\alpha = \cos\beta\sin\alpha,$$

$$\overline{OC} = \overline{OD}\cos\alpha = \cos\beta\cos\alpha,$$

$$\overline{AE} = \overline{AD}\sin\alpha = \sin\beta\sin\alpha.$$

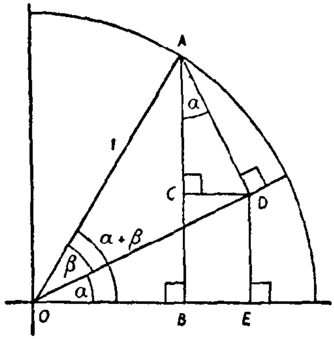


FIG. 1

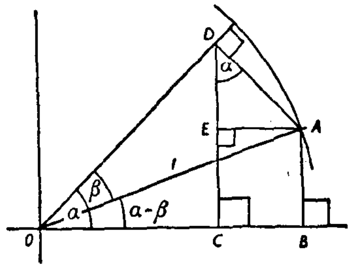


FIG. 2

Then $\sin(\alpha - \beta) = \overline{AB} = \overline{EC} = \overline{DC} - \overline{DE} = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$

and $\cos(\alpha - \beta) = \overline{OB} = \overline{OC} + \overline{CB} = \overline{OC} + \overline{EA} = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

Formulae V–VIII are obtained immediately from the relations

$$\tan\phi = \frac{\sin\phi}{\cos\phi} \quad \text{and} \quad \cot\phi = \frac{\cos\phi}{\sin\phi}.$$

*Trigonometric functions of double angles*

If the values of $\sin\alpha$, $\cos\alpha$, $\tan\alpha$ and $\cot\alpha$ are known, $\sin 2\alpha$, $\cos 2\alpha$, $\tan 2\alpha$ and $\cot 2\alpha$ can be calculated by putting $\beta = \alpha$ in the addition theorems:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha;$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha;$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}; \quad \cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha}.$$

More generally, $\sin(n\alpha)$ and $\cos(n\alpha)$ can be calculated, given $\sin\alpha$ and $\cos\alpha$ ($n$ a positive integer):

$$\sin(n\alpha) = \binom{n}{1}\cos^{n-1}\alpha\sin\alpha - \binom{n}{3}\cos^{n-3}\alpha\sin^3\alpha + \binom{n}{5}\cos^{n-5}\alpha\sin^5\alpha - \ldots,$$

$$\cos(n\alpha) = \cos^n \alpha - \binom{n}{2} \cos^{n-2} \alpha \sin^2 \alpha + \binom{n}{4} \cos^{n-4} \alpha \sin^4 \alpha - \dots .$$

*Trigonometric functions of half angles*

Given $\sin \alpha$, $\cos \alpha$, $\tan \alpha$, and $\cot \alpha$, then $\sin \frac{\alpha}{2}$, $\cos \frac{\alpha}{2}$, $\tan \frac{\alpha}{2}$ and $\cot \frac{\alpha}{2}$ can be calculated as follows.

Since we know $\cos 2\alpha = 2\cos^2 \alpha - 1 = 1 - 2\sin^2 \alpha$,

we have
$$\cos \alpha = 2\cos^2 \frac{\alpha}{2} - 1 = 1 - 2\sin^2 \frac{\alpha}{2},$$

that is, $\sin \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{2}}$; $\cos \frac{\alpha}{2} = \sqrt{\frac{1 + \cos \alpha}{2}}$.

Dividing, we have

$$\tan \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha};$$

and similarly

$$\cot \frac{\alpha}{2} = \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}}.$$

**Affine**

Two figures (solid or plane) which are derivable one from the other by an *affine transformation* (see below) are called affine with respect to one another. *E.g.* an ellipse can be regarded as the affine image of the circle on its major axis. Ellipse and circle are affine with respect to one another.

**Affine-symmetrical**

A plane figure is called affine-symmetric if a straight line exists such that the part of the figure lying on one side of the line is transformed into the part lying on the other side by an *affine* *reflection* (see *Reflection, affine*). *E.g.* the oblique kite-like figure of Fig. 3 is affine-symmetrical with respect to the diagonal $\overline{AC}$. The other diagonal $\overline{BD}$ gives the direction of reflection.

An ellipse is affine-symmetrical with respect to a diameter $d_1$. The conjugate diameter $d_2$ gives the direction of reflection (see *Ellipse*).

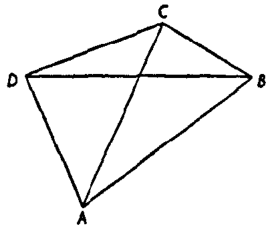


FIG. 3

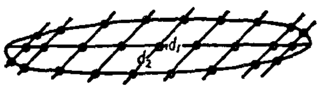


FIG. 4

**Affine transformation**

An affine transformation of the points of space is a reversible *single-valued* (one-to-one) transformation in space, in which the ratios of distances separating three points $P$, $Q$, $R$ which lie on a straight line remain unchanged. *I.e.* if $P'$, $Q'$, $R'$ are the image points of $P$, $Q$, $R$, and if $\frac{\overline{PQ}}{\overline{QR}} = c$, then $\frac{\overline{P'Q'}}{\overline{Q'R'}} = c$.

If a general point $P$ has coordinates $x, y, z$ in a given rectangular coordinate system and the image point $P'$ has coordinates $x', y', z'$ in the same coordinate system, then the sets of coordinates are connected by equations of the following form:

$$x' = a_1x + a_2y + a_3z + a_4,$$
$$y' = b_1x + b_2y + b_3z + b_4,$$
$$z' = c_1x + c_2y + c_3z + c_4.$$

Here the coefficients $a_i$, $b_i$, $c_i$ are real numbers which satisfy the condition that the determinant

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0.$$

(If $D = 0$, all the image points $P'$ lie on a single line.)

Under affine transformation straight lines transform into straight lines. If two straight lines are parallel the image lines are parallel also.

*Types of plane affine transformation*

(In what follows, $P(x, y)$ is to have the image point $P'(x', y')$ in the same rectangular coordinate system.)

1. *Parallel displacement*. Equations:

$$x' = x + a_4$$
$$y' = y + b_4$$

Displacement distance: $d = \sqrt{a_4^2 + b_4^2}$. The direction of the parallel displacement is that of the line joining the origin to the point $Q(a_4, b_4)$.

2. *Reflection in a straight line in the plane*

Reflection in the $x$-axis:

$$x' = x,$$
$$y' = -y;$$

in the $y$-axis:

$$x' = -x,$$
$$y' = y.$$

3. *Rotation about a point in the plane*

(a) Rotation about the origin through angle $\delta$,

$$x' = x \,.\, \cos\delta + y \,.\, \sin\delta,$$
$$y' = (-x) \,.\, \sin\delta + y \,.\, \cos\delta.$$

(b) Rotation about the point $P(x_z, y_z)$ through angle $\delta$,

$$x' - x_z = (x - x_z) \,.\, \cos\delta + (y - y_z) \,.\, \sin\delta,$$
$$y' - y_z = -(x - x_z) \,.\, \sin\delta + (y - y_z) \,.\, \cos\delta.$$

The transformation 1, 3(a) and (b) are so-called *Rigid motions* (see below) or congruent transformations in the plane. In a rigid motion the originating figure and the image figure are always congruent. All rigid motions in the plane can be obtained by compounding transformations 1 and 3(a).

4. *Magnification*

(a) Parallel to the $x$-axis,

$$x' = a_1 x,$$
$$y' = y;$$

(b) Parallel to the $y$-axis,

$$x' = x,$$
$$y' = b_1 y.$$