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Shock Waves and Reaction–Diffusion Equations

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Acknowledgment

I get by with a little help from my friends.

LENNON AND MCCARTNEY

Throughout my career, I have had the good fortune to be closely associated with many very gifted mathematicians. I learned a great deal from each of them; this book is really not much more than a compilation of what they taught me.

First, I want to thank Louis Nirenberg for initially inspiring me and for giving me an early opportunity to visit the Courant Institute. It was there that I met Edward Conway, my friend and collaborator, who showed me the mathematics of shock waves, and convinced me (with difficulty!) of the value of difference schemes. Years later, he introduced me to the equations of mathematical ecology, the study of which led to much fruitful joint research.

I must also thank Peter Lax, whose work has always interested me, for inviting me to visit the Courant Institute once again. The second time at the "Courant" I met my other good friend and collaborator, Charles Conley. It was he who enthusiastically taught me so much deep and exciting mathematics. Among N ($\gg 1$) other things, I learned from him the power and beauty of topological methods in analysis. He has also generously shared with me many of his brilliant insights. His ideas continue to influence my research to the present day.

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Preface

> . . . the progress of physics will to a large extent depend on the progress of nonlinear mathematics, of methods to solve nonlinear equations . . . and therefore we can learn by comparing different nonlinear problems.
>
> WERNER HEISENBERG

I undertook to write this book for two reasons. First, I wanted to make easily available the basics of both the theory of hyperbolic conservation laws and the theory of systems of reaction–diffusion equations, including the generalized Morse theory as developed by C. Conley. These important subjects seem difficult to learn since the results are scattered throughout the research journals.[1] Second, I feel that there is a need to present the modern methods and ideas in these fields to a wider audience than just mathematicians. Thus, the book has some rather sophisticated aspects to it, as well as certain textbook aspects. The latter serve to explain, somewhat, the reason that a book with the title *Shock Waves and Reaction–Diffusion Equations* has the first nine chapters devoted to linear partial differential equations. More precisely, I have found from my classroom experience that it is far easier to grasp the subtleties of nonlinear partial differential equations *after* one has an understanding of the basic notions in the linear theory.

This book is divided into four main parts: linear theory, reaction–diffusion equations, shock wave theory, and the Conley index, in that order. Thus, the text begins with a discussion of ill-posed problems. The aim here was to show that partial differential equations are not divorced from side conditions; indeed specific side conditions are required for specific equations. And in view of Lewy's example, which is presented in its entirety, no side conditions can force solutions on some equations. We discuss an example of a nonlinear scalar conservation law which has no global classical solution, thereby foreshadowing the notion of "weak" solution. In Chapter 2 we consider characteristics, an important notion which comes up widely in nonlinear contexts. Chapter 3 deals with the simple one-dimensional wave equation. Here is where we introduce the reader to the important ideas of

[1] This is not quite true; there are some good survey articles on shock waves (e.g., [Lx 5]) but these do not contain many proofs. Also in the theory of reaction–diffusion equations, there are the books [Fi] and [Mu], but they both seem to me to be research monographs.

domains of dependence, energy integrals, and finite differences. The purpose of the next chapter is to demonstrate the power, generality, and elegance of energy integral methods. In the course of the development we present several basic techniques for obtaining inequalities.

The next chapter is devoted to Holmgren's uniqueness theorem. We view it in a modern context, where we can use it later to motivate Oleinik's uniqueness theorems for conservation laws. In Chapter 6 we consider general hyperbolic operators and show how energy integrals, together with Fourier transform methods, are used to prove global existence theorems. The uniqueness of these solutions is obtained via Holmgren's theorem. Chapter 7 is devoted to the theory of distributions. The importance of this subject for linear operators is, of course, well known. This author firmly believes that the great advances in *nonlinear* partial differential equations over the last twenty years could not have been made were it not for distribution theory. The ideas of this discipline provided the conceptual framework for studying partial differential equations in the context of weak solutions. This "philosophy" carried over, rather easily, to many important nonlinear equations. In Chapters 8 and 9 we study linear elliptic and parabolic equations, respectively, and we prove the basic maximum principles. We also describe the estimates of Schauder, as well as those of Agmon, Douglis, and Nirenberg, which we need in later chapters. The proofs of these important estimates are (happily) omitted since it is difficult to improve upon the exposition given in Gilbarg–Trudinger [GT]. (We point out here that the material in Chapters 1–9 can serve as an introductory course in partial differential equations.)

A quick glance at the contents serves to explain the flavor of those topics which form the major portion of the book. I have made a deliberate effort to explain the main ideas in a coherent, readable manner, and in particular I have avoided excess generality. To be specific, Chapter 10 contains a discussion of how far one can go with the maximum principle for a scalar nonlinear parabolic (or elliptic) equation. It is used to prove the basic comparison and existence theorems; the latter done via the method of upper and lower solutions. The text contains several carefully chosen examples which are used both to illustrate the theorems and to prepare the way for some later topics; e.g., bifurcation theory. The next chapter begins with a development of the variational properties of the eigenvalues for a linear second-order elliptic operator on a bounded domain in $\mathbf{R}^n$. There follows a careful discussion of linearized stability for a class of evolution equations broad enough to include systems of reaction–diffusion equations. In Chapter 12, we give a complete development of degree theory for operators in Banach spaces of the form (Id. + compact). The discussion begins with the finite-dimensional case, culminating with Brouwer's fixed point theorem. This is applied to flows on Euclidean spaces; specifically, we give two applications, one to flows on spheres and one to flows on tori. The Leray–Schauder degree is then developed, and we illustrate its use in nonlinear elliptic equations. The second half of this chapter is devoted to Morse theory. Our goal is to re-interpret the Morse index in an intrinsic topological way (using the stable manifold theorem), as the homotopy type of a quotient space. This is done in preparation for Chapters 22 and 23, where we consider Conley's extension of the Morse index. We give a proof of Reeb's theorem on the characterization of spheres in terms of Morse functions. The chapter ends with an appendix on algebraic topology where homotopy theory, homology theory, and cohomology theory are discussed. The goal was to make these important ideas accessible to analysts.

In Chapter 13, some of the standard bifurcation theorems are proved; namely, those which come under the heading "bifurcation from a simple eigenvalue." We then use degree theory to prove the bifurcation theorems of both Krasnoselski and Rabinowitz. Again, these theorems are illustrated by applications to specific differential equations. In the final section we discuss, with an example, another more global type of bifurcation which we term "spontaneous" bifurcation. This is related back to earlier examples, and it is also made use of in Chapter 24.

Chapter 14 may be considered the "high point" in this group. It is here where the notion of an invariant region is defined, and all of the basic theorems concerning it are proved. As a first application, we prove a comparison theorem which allows us to obtain rather precise (but somewhat coarse) qualitative statements on solutions. We then give a general theorem on the asymptotic behavior of solutions. Thus, we isolate a parameter which, when positive, implies that for large time, every solution gets close to a spatially independent one; in particular, no bifurcation of nonconstant steady-state solutions can occur. There follows a section which makes quantitative the notion of an invariant region; the statement is that the flow is gradient-like near the boundary of this region. This means that attracting regions for the kinetic equations are also attracting regions for the full system of reaction–diffusion equations, provided that the geometry of the region under consideration is compatible with the diffusion matrix. In the final section, these results are applied to the general Kolmogorov form of the equations which describe the classical two-species ecological interactions, where now diffusion and spatial dependence are taken into account. One sees here how the standard ecological assumptions lead in a fairly direct way to the mathematical conditions which we have considered.

In Chapter 15, we begin to discuss the theory of shock waves. This is a notoriously difficult subject due to the many subtleties not usually encountered in other areas of mathematics. The very fact that the entire subject is concerned with *discontinuous* functions, means that many of the modern mathematical techniques are virtually inapplicable. I have given much effort in order to overcome these obstacles, by leading the reader gently along, step by step. It is here where I have leaned most upon my classroom experience. Thus, the development begins with a chapter describing the basic phenomena: the formation of shock waves, the notion of a weak solution and its consequences, the loss of uniqueness, the entropy conditions, etc. These things are all explained with the aid of examples. There follows next a chapter which gives a rather complete description of the theory of a single conservation law: existence, uniqueness, and asymptotic behavior of solutions. The existence proof follows Oleinik and is done via the Lax–Friedrichs difference scheme. The reasons why I have chosen this method over the several other ones available are discussed at the beginning of the chapter; suffice it to say that it requires no sophisticated background, and that the method of finite differences is, in principle, capable of generalization to systems. The entrance into systems of conservation laws, is made via a discussion of the Riemann problem for the "p-system." Here it is possible to explain things geometrically, by actually drawing the shock- and rarefaction-wave curves. We then develop the basic properties of these waves, and following Lax, we solve the Riemann problem for general systems. These ideas are applied in the next chapter to the equations of gas dynamics, where we solve the Riemann problem for arbitrary data, both analytically and geometrically. We prove Weyl's entropy theorem, as well as von Neumann's shock-interaction theorem. The next chapter, the Glimm Difference Scheme, is one of the most difficult ones in the book (the others being Chapters 22 and 23 on the Conley index). Glimm's theorem continues to be the most important result in conservation laws, and it must be mastered by anyone seriously interested in this field. I feel that the proof is not nearly as difficult as is commonly believed, and I have tried hard to make it readable for the beginner.

The final chapter in this group is designed to give the reader a flavor of some of the general results that are known for systems, the emphasis being on systems of two equations. I have also given a proof of Oleinik's uniqueness theorem for the p-system; her paper is available only in the original Russian. Having been sufficiently "turned on" by the superb lectures of T. Nishida at Michigan (in academic year 1981/82), I was unable to resist including a chapter on quasilinear parabolic systems. The main result here is Kanel's existence proof for the isentropic gas dynamics equations with viscosity.

With Chapter 22, I begin Part Four of the book. These last three chapters deal mainly with the Conley index, together with its applications. Thus, the first chapter opens with a long descriptive discussion in which the basic ideas of the theory are explained; namely the concept of an isolated invariant set and its index, together with their main properties. These are illustrated by an easily understood example, in which things are worked out in detail and the connections with the classical Morse index are noted. I have also included a discussion of the so-called "Hopf bifurcation," from this point of view. Although the sections which follow are independent of this one, I strongly recommend that the reader not skim over it, but rather that he give it serious thought. The remaining sections in this chapter contain all of the basic definitions, together with proofs of the existence of an isolating block, and the theorem that the index is independent of the block which contains it. This is all done for flows, where the reader can "see" the geometrical and topological constructions. I have also given some applications to differential equations in $\mathbf{R}^n$, as well as a proof of the "connecting orbit" theorem. In Chapter 23, the theory is developed from a more general, more abstract point of view, in a form suitable for applications to partial differential equations. We define the notions of index pairs, and Morse decompositions of an isolated invariant set. The concept of local flow is also introduced, again with an eye towards the applications. We prove both the existence of index pairs for Morse decompositions, as well as the well-definedness of the Conley index. That is, we show that the index $h(S)$ of an isolated invariant set S, depends only on the homotopy class of the space N_1/N_0, where (N_1, N_0) is any index pair for S. This result immediately puts at our disposal the algebraic invariants associated with the cohomology groups which form exact sequences on the Morse decomposition of S. These are powerful tools for computing indices, in addition to being of theoretical use. They lead, for example, to an easy proof of the "generalized" Morse inequalities. We then prove the continuation property of the Conley index, in a rather general context. The final section serves both to illustrate some of the theorems, as well as to derive additional results which will be used in the applications. We point out that these two chapters monotonically increase in difficulty as one proceeds. This is done by design in order to meet the needs of readers having assorted degrees of mathematical maturity—one can proceed along as far as his background will take him (and further, if he is willing to work hard!).

The last chapter contains a sample of the applications to travelling waves. We first study the shock structure problem of the existence of an orbit connecting two rest points, and in particular, we solve the shock structure problem for magnetohydrodynamic shock waves having arbitrary strength. We then prove the existence of a periodic travelling wave solution for the Nagumo equations. An isolating neighborhood is constructed, and the Conley index is explicitly computed, in order to demonstrate the different topological techniques which are involved. We also show how to obtain the desired information a different way by using an exact sequence of cohomology groups in order to determine the nontriviality of the index. Next follows a long section, where we apply the theory to reaction-diffusion equations, and we use the Conley index together with some previously obtained (global) bifurcation diagrams, to study the stability of steady-state solutions, and to determine in some cases, the entire global picture of the solution set. The chapter closes with a section in which we give some instability theorems for nonconstant stationary solutions of the Neumann problem.

Each of the four sections in this book (in any order) is suitable for a one-semester graduate course. In particular, as we have remarked earlier, the first section can be used for an introductory graduate-level course in partial differential equations. The prerequisite for this is one of graduate-level mathematics as given in the average American university.

Contents

Part I

Basic Linear Theory

Our present analytical methods seem unsuitable for the solution of the important problems arising in connection with nonlinear partial differential equations and, in fact, with virtually all types of nonlinear problems in pure mathematics. The truth of this statement is particularly striking in the field of fluid dynamics. Only the most elementary problems have been solved analytically in this field . . . .

The advance of analysis is, at this moment, stagnant along the entire front of nonlinear problems. That this phenomenon is not of a transient nature but that we are up against an important conceptual difficulty . . . . yet no decisive progress has been made against them . . . which could be rated as important by the criteria that are applied in other, more successful (linear!) parts of mathematical physics.

It is important to avoid a misunderstanding at this point. One may be tempted to qualify these (shock wave and turbulence) problems as problems in physics, rather than in applied mathematics, or even pure mathematics. We wish to emphasize that it is our conviction that such an interpretation is wholly erroneous.

JOHN VON NEUMANN, 1946

Chapter 1

Ill-Posed Problems

Problems involving differential equations usually come in the following form: we are given an equation for the unknown function u, $P(u) = f$, on a domain Ω together with some "side" conditions on u. For example, we may require that u assumes certain preassigned values on $\partial\Omega$, or that u is in $L^2(\Omega)$, or that u is in class C^k in Ω. At first glance, it would seem that any of these extra conditions are quite reasonable, and that one is as good as the other. However, we shall see that this is far from being true, and that whichever additional supplementary conditions one assigns is intimately connected with the form of equation.

In general, the equations come from the sciences: physics, chemistry, and biology, and the "physical" equations come together with quite specific "side" conditions. At least, this is the way the theory of partial differential equations began. It is the purpose of this chapter to illustrate these ideas by some examples. The chapter ends with the remarkable example of H. Lewy [Le].

§A. Some Examples

1. Let Ω be the region in $\mathbf{R}^2$ defined by

$$\Omega = \{(x, y) : x^2 + y^2 < 1, y > 0\},$$

and consider the "Cauchy problem"[1] in Ω for Laplace's equation:

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (x, y) \in \Omega, \tag{1.1}$$

together with the "initial" conditions

$$u(x, 0) = 0, \qquad u_y(x, 0) = f(x), \qquad -1 < x < 1. \tag{1.2}$$

[1] This is often called an "initial-value" problem, for reasons to be made clear later.

Suppose that $u(x, y)$ is a C^2 solution of (1.1), (1.2) in Ω. We extend u to be a C^2 function in the unit disk by setting $u(x, y) = -u(x, -y)$, in the region $y < 0$. Since the unit disk is simply connected, the function

$$v(x, y) = \int_{(0,0)}^{(x,y)} u_y\, dx - u_x\, dy$$

is a harmonic conjugate of u (because $u + iv$ satisfies the Cauchy–Riemann equations). Thus $u + iv$ is an analytic function, so the same is true of u, and in particular, $u_y(x, 0) = f(x)$ must be a real analytic function. Thus the "data" $f(x)$, assigned along $y = 0$ cannot be arbitrary; it must be a real analytic function.

2. Consider the set of "initial-value" problems in the upper half-plane in $\mathbf{R}^2$, for $n = 1, 2, \ldots,$

$$\begin{aligned} &\Delta u = 0, && y > 0, \\ &u(x, 0) = 0, \quad u_y(x, 0) = \frac{\sin nx}{n}, && x \in \mathbf{R}, \end{aligned} \tag{P_n}$$

and

$$\begin{aligned} &\Delta u = 0, && y > 0, \\ &u(x, 0) = 0, \quad u_y(x, 0) = 0, && x \in \mathbf{R}. \end{aligned} \tag{P_0}$$

The problems (P_n) and (P_0) have the solutions

$$u^n(x, y) = \frac{(\sin ny)(e^{nx} - e^{-nx})}{2n^2},$$

and

$$u^0(x, y) = 0,$$

respectively. Observe that as $n \to \infty$, the data for (P_n) tends uniformly to zero, the data of (P_0). However, we have

$$\overline{\lim_{n \to \infty}}\, |u^n(x, y) - u^0(x, y)| = +\infty,$$

for each point (x, y). In fact, the functions u^n do not converge to u^0 in any reasonable topology. Thus arbitrarily small changes in the data lead to large changes in the solution; the mapping from the "data space" to "solution