CONTENTS

CHAPTER 1

Trigonometric Series and Fourier Series

1.1 The Genesis of Trigonometric Series and Fourier Series

1.1.1. **The Beginnings.** D. Bernoulli, D'Alembert, Lagrange, and Euler, from about 1740 onward, were led by problems in mathematical physics to consider and discuss heatedly the possibility of representing a more or less arbitrary function f with period 2π as the sum of a *trigonometric series* of the form

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1.1.1}$$

or of the formally equivalent series in its so-called "complex" form

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{1.1.1*}$$

in which, on writing $b_0 = 0$, the coefficients c_n are given by the formulae

$$c_n = \tfrac{1}{2}(a_n - ib_n), \quad c_{-n} = \tfrac{1}{2}(a_n + ib_n) \qquad (n = 0, 1, 2, \cdots).$$

This discussion sparked off one of the crises in the development of analysis.

Fourier announced his belief in the possibility of such a representation in 1811. His book *Théorie Analytique de la Chaleur*, which was published in 1822, contains many particular instances of such representations and makes widespread heuristic use of trigonometric expansions. As a result, Fourier's name is customarily attached to the following prescription for the coefficients a_n, b_n, and c_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx, \tag{1.1.2}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} \, dx, \tag{1.1.2*}$$

the a_n and b_n being now universally known as the "real," and the c_n as the "complex," *Fourier coefficients* of the function f (which is tacitly assumed to be integrable over $(-\pi, \pi)$). The formulae (1.1.2) were, however, known earlier to Euler and Lagrange.

The grounds for adopting Fourier's prescription, which assigns a definite trigonometric series to each function f that is integrable over $(-\pi, \pi)$, will be scrutinized more closely in 1.2.3. The series (1.1.1) and (1.1.1*), with the coefficients prescribed by (1.1.2) and (1.1.2*), respectively, thereby assigned to f are termed the "real" and "complex" *Fourier series* of f, respectively.

During the period 1823–1827, both Poisson and Cauchy constructed proofs of the representation of restricted types of functions f by their Fourier series, but they imposed conditions which were soon shown to be unnecessarily stringent.

It seems fair to credit Dirichlet with the beginning of the rigorous study of Fourier series in 1829, and with the closely related concept of function in 1837. Both topics have been pursued with great vigor ever since, in spite of more than one crisis no less serious than that which engaged the attentions of Bernoulli, Euler, d'Alembert, and others and which related to the prevailing concept of functions and their representation by trigonometric series. (Cantor's work in set theory, which led ultimately to another major crisis, had its origins in the study of trigonometric series.)

1.1.2. The rigorous developments just mentioned showed in due course that there are subtle differences between trigonometric series which converge at all points and Fourier series of functions which are integrable over $(-\pi, \pi)$, even though there may be no obvious clue to this difference. For example, the trigonometric series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\log n}$$

converges everywhere; but, as will be seen in Exercise 7.7 and again in 10.1.6, it is not the Fourier series of any function that is (Lebesgue-)integrable over $(-\pi, \pi)$.

The theory of trigonometric series in general has come to involve itself with many questions that simply do not arise for Fourier series. For the express purpose of attacking such questions, many techniques have been evolved which are largely irrelevant to the study of Fourier series. It thus comes about that Fourier series may in fact be studied quite effectively without reference to general trigonometric series, and this is the course to be adopted in this book.

The remaining sections of this chapter are devoted to showing that, while Fourier series have their limitations, general trigonometric series have others no less serious; and that there are well-defined senses and contexts in which Fourier series are the natural and distinguished tools for representing functions in useful ways. Any reader who is prepared to accept without question the restriction of attention to Fourier series can pass from 1.1.3 to the exercises at the end of this chapter.

1.1.3. The Orthogonality Relations. Before embarking upon the discussion promised in the last paragraph, it is necessary to record some facts that provide the heuristic basis for the Fourier formulae (1.1.2) and (1.1.2*) and for whatever grounds there are for according a special role to Fourier series.

These facts, which result from straightforward and elementary calculations, are expressed in the following so-called *orthogonality relations* satisfied by the circular and complex exponential functions:

$$\left.\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & (m \neq n, m \geqslant 0, n \geqslant 0) \\ \tfrac{1}{2} & (m = n > 0), \\ 1 & (m = n = 0) \end{cases} \\
&\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & (m \neq n, m \geqslant 0, n \geqslant 0) \\ \tfrac{1}{2} & (m = n > 0), \\ 0 & (m = n = 0) \end{cases} \\
&\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0, \\
&\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx} e^{-inx} \, dx = \begin{cases} 0 & (m \neq n) \\ 1 & (m = n); \end{cases}
\end{aligned}\right\} \quad (1.1.3)$$

in these relations m and n denote integers, and the interval $[-\pi, \pi]$ may be replaced by any other interval of length 2π.

1.2 Pointwise Representation of Functions by Trigonometric Series

1.2.1. Pointwise Representation. The general theory of trigonometric series was inaugurated by Riemann in 1854, since when it has been pursued with vigor and to the great enrichment of analysis as a whole. For modern accounts of the general theory, see [Z_1], Chapter IX and [$Ba_{1,2}$], Chapters XII–XV.

From the beginning a basic problem was that of representing a more or less arbitrary given function f defined on a period-interval I (say the interval $[-\pi, \pi]$) as the sum of at least one trigonometric series (1.1.1), together with a discussion of the uniqueness of this representation.

A moment's thought will make it clear that the content of this problem depends on the interpretation assigned to the verb "to represent" or, what comes to much the same thing, to the term "sum" as applied to an infinite series. Initially, the verb was taken to mean the pointwise convergence of the series at all points of the period interval to the given function f. With the passage of time this interpretation underwent modification in at least two ways. In the first place, the demand for convergence of the series to f at *all* points of the period-interval I was relaxed to convergence at *almost all* points of that interval. In the second place, convergence of the series to f at all or almost all points was weakened to the demand that the series be *summable* to f by one of several possible methods, again at all or almost all points. For the purposes of the present discussion it will suffice to speak of just one such summability method, that known after Cesàro, which consists of replacing the partial sums

$$s_0(x) = \tfrac{1}{2}a_0,$$
$$s_N(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \qquad (N = 1, 2, \cdots) \quad (1.2.1)$$

of the series (1.1.1) by their arithmetic means

$$\sigma_N = \frac{s_0 + \cdots + s_N}{N + 1} \qquad (N = 0, 1, 2, \cdots). \quad (1.2.2)$$

Thus we shall say that the series (1.1.1) is summable at a point x to the function f if and only if

$$\lim_{N \to \infty} \sigma_N(x) = f(x).$$

It will be convenient to group all these interpretations of the verb "to represent" under the heading of *pointwise representation* (everywhere or almost everywhere, by convergence or by summability, as the case may be) of the function f by the series (1.1.1).

In terms of these admittedly rather crude definitions we can essay a bird's-eye view of the state of affairs in the realm of pointwise representation, and in particular we can attempt to describe the place occupied by Fourier series in the general picture.

1.2.2. Limitations of Pointwise Representation. Although it is undeniably of great intrinsic interest to know that a certain function, or each member of a given class of functions, admits a pointwise representation by some trigonometric series, it must be pointed out without delay that this type of representation leaves much to be desired on the grounds of *utility*. A mode of representation can be judged to be successful or otherwise useful as a tool in subsequent investigations by estimating what standard analytical operations applied to the represented function can, via the representation, be expressed with reasonable simplicity in terms of the expansion coefficients a_n and b_n. This is, after all, one of the main reasons for seeking a representation in series form. Now it is a sad fact that pointwise representations are in themselves not very useful in this sense; they are simply too weak to justify the termwise application of standard analytical procedures.

Another inherent defect is that a pointwise representation at almost all points of I is never unique. This is so because, as was established by Men'shov in 1916, there exist trigonometric series which converge to zero almost everywhere and which nevertheless have at least one nonvanishing coefficient; see 12.12.8. (That this can happen came as a considerable surprise to the mathematical community.)

1.2.3. The Role of the Orthogonality Relations. The a priori grounds for expecting the Fourier series of an integrable function f to effect a pointwise representation of f (or, indeed, to effect a representation in any reasonable sense) rest on the orthogonality relations (1.1.3). It is indeed a simple consequence of these relations that, if there exists *any* trigonometric series (1.1.1) which represents f in the pointwise sense, and if furthermore the s_N (or the σ_N) converge dominatedly (see [W], p. 60) to f, then the series (1.1.1) must be the Fourier series of f. However, the second conditional clause prevents any very wide-sweeping conclusions being drawn at the outset.

As will be seen in due course, the requirements expressed by the second conditional clause are fulfilled by the Fourier series of sufficiently smooth functions f (for instance, for those functions f that are continuous and of bounded variation). But, alas, the desired extra condition simply does not obtain for more general functions of types we wish to consider in this book. True, a greater degree of success results if convergence is replaced by summability (see 1.2.4). But in either case the investigation of this extra condition itself carries one well into Fourier-series lore. This means that this would-be simple and satisfying explanation for according a dominating role to Fourier series can scarcely be maintained at the *outset* for functions of the type we aim to study.

1.2.4. Fourier Series and Pointwise Representations. What has been said in 1.2.3 indicates that Fourier series can be expected to have but limited success in the pointwise representation problem. Let us tabulate a little specific evidence.

The Fourier series of a periodic function f which is continuous and of bounded variation converges boundedly at all points to that function. The Fourier series of a periodic continuous function may, on the contrary, diverge at infinitely many points; even the pointwise convergence almost everywhere of the Fourier series of a general continuous function remained in doubt until 1966 (see 10.4.5), although it had been established much earlier and much more simply that certain fixed subsequences of the sequence of partial sums of the Fourier series of any such function is almost everywhere convergent to that function (the details will appear in Section 8.6). The Fourier series of an integrable function may diverge at all points.

If ordinary convergence be replaced by summability, the situation improves. The Fourier series of a periodic continuous function is uniformly summable to that function. The Fourier series of any periodic integrable function is summable at almost all points to that function, but in this case neither the s_N nor the σ_N need be dominated.

1.2.5. **Trigonometric Series and Pointwise Representations.** Having reviewed a few of the limitations of Fourier series vis-à-vis the problem of pointwise representation, we should indicate what success is attainable by using trigonometric series in general.

In 1915 both Lusin and Privalov established the existence of a pointwise representation almost everywhere by summability methods of any function f which is measurable and finite almost everywhere. For 25 years doubts lingered as to whether summability could here be replaced by ordinary convergence, the question being resolved affirmatively by Men'shov in 1940. This result was sharpened in 1952 by Bary, who showed that, if the function f is measurable and finite almost everywhere on the interval I, there exists a continuous function F such that $F'(x) = f(x)$ at almost all points of I, and such that the series obtained by termwise differentiation of the Fourier series of F converges at almost all points x of I to $f(x)$. Meanwhile Men'shov had in 1950 shown also that to any measurable f (which may be infinite on a set of positive measure) corresponds at least one trigonometric series (1.1.1) whose partial sums s_N have the property that $\lim_{N\to\infty} s_N = f$ in *measure* on I. This means that one can write $s_N = u_N + v_N$, where u_N and v_N are finite-valued almost everywhere, $\lim_{N\to\infty} u_N(x) = f(x)$ at almost all points x of I, and where, for any fixed $\varepsilon > 0$, the set of points x of I for which $|v_N(x)| > \varepsilon$ has a measure which tends to zero as $N \to \infty$. (The stated condition on the v_N is equivalent to the demand that

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \frac{|v_N|\,dx}{1 + |v_N|} = 0;$$

and the circuitous phrasing is necessary because f may take infinite values on a set of positive measure.) This sense of representation is weaker than pointwise representation. For more details see [Ba_2], Chapter XV.

These theorems of Men'shov and Bary lie very deep and represent enormous achievements. However, as has been indicated at the end of 1.2.2, the representations whose existence they postulate are by no means unique.

Cantor succeeded in showing that a representation at *all* points by a convergent trigonometric series is necessarily unique, if it exists at all. Unfortunately, only relatively few functions f admit such a representation: for instance, there are continuous periodic functions f that admit no such representation. (This follows on combining a theorem due to du Bois-Reymond and Lebesgue, which appears on p. 202 of [Ba_1], with results about Fourier series dealt with in Chapter 10 of this book.) It is indeed the case that, in a sense, "most" continuous functions admit no representation of this sort.

1.2.6. **Summary.** It can thus be said in summary that pointwise representations are subject to inherent limitations as analytical tools, and that Fourier series can be accorded a distinguished role in respect of this type of representation only for functions of a type more restricted than one might hope to handle.

This being so, it is natural to experiment by varying the meaning assigned to the verb "to represent" in the hope of finding a more operationally effective meaning and of installing Fourier series in a more dominating role.

Before embarking on this program, it is perhaps of interest to add that a similar choice prevails in the interpretation of differentiation (which in fact has connections with the representation problem). The pointwise everywhere or almost everywhere interpretation of the derivative, if deprived of any further qualification, is also not entirely effective operationally. A new interpretation is possible and leads to distributional concepts; Chapter 12 is devoted to this topic.

1.3 New Ideas about Representation

1.3.1. **Plan of Action.** In the preceding section we have recounted some of the difficulties in the way of according a unique position to Fourier series on the grounds of their behavior in relation to the traditionally phrased problem of representing functions by trigonometric series. We have also indicated the shortcomings of this type of representation.

To this it may be added that in cases where the mathematical model of a physical problem suggests the use of expansions in trigonometric series, pointwise representations frequently do not correspond very closely to the physical realities.

Faced with all this, we propose to consider new meanings for the verb "to represent" that are in complete accord with modern trends, and which will in due course be seen to justify fully a concentration on Fourier series as a representational device.

1.3.2. **Different Senses of Convergence and Representation.** In recent times analysts have become accustomed to, and adept at working in diverse fields with, other meanings for the verb "to represent," most of which (and all of which we shall have occasion to consider) are tantamount to novel ways in which a series of functions may be said to converge. Such ideas are indeed the concrete beginnings of general topology and the theory of topological linear spaces.

Thus encouraged, we contemplate some possible relationships between an integrable function f on $(-\pi, \pi)$ and a trigonometric series (1.1.1) or (1.1.1*) expressed by each of equations (A) to (D) below.

For this purpose we write again

$$s_0(x) = \tfrac{1}{2}a_0, \quad s_N(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx),$$

so that

$$s_N(x) = \sum_{|n| \leq N} c_n e^{inx}, \tag{1.3.1}$$

and also

$$\sigma_N(x) = \frac{s_0(x) + \cdots + s_N(x)}{N + 1}.$$

The relationships referred to are (compare 6.1.1, 6.2.6, 12.5.3, and 12.10.1):

(A) $$\lim_{N\to\infty} \int_{-\pi}^{\pi} |f(x) - \sigma_N(x)|\, dx = 0;$$

(B) $$\lim_{N\to\infty} \int_{-\pi}^{\pi} |f(x) - s_N(x)|^p\, dx = 0;$$

(C) $$\lim_{N\to\infty} \sup_x |f(x) - \sigma_N(x)| = 0;$$

(D) $$\lim_{N\to\infty} \int_{-\pi}^{\pi} u(x) s_N(x)\, dx = \int_{-\pi}^{\pi} u(x) f(x)\, dx$$

for each indefinitely differentiable periodic function u.

If any one of these relations holds for a given f and a given trigonometric series, one may say that the trigonometric series represents f in the corresponding sense: in case (A) it would be usual to say that the trigonometric series is *Cesàro-summable in mean with exponent* (or *index*) 1 *to* f; in case (B) that the trigonometric series is *convergent in mean with exponent* (or *index*) p *to* f; in case (C) that the trigonometric series is *uniformly Cesàro-summable to* f; and in case (D) that the trigonometric series is *distributionally convergent to* f.

1.3.3. The Role of Fourier Series. It is genuinely simple to verify that, given f, there is *at most one* trigonometric series for which any one of relations (A) to (D) is true, and that this only contender is the Fourier series of f (see the argument in 6.1.3). Moreover, it is true that the relations *do* hold if the trigonometric series is the Fourier series of f, provided in case (B) that *either* $1 < p < \infty$ and $f \in \mathbf{L}^p$ *or* $p = 1$ and $f \log^+ |f| \in \mathbf{L}^1$ (see 8.2.1, 12.10.1, and 12.10.2); and in case (C) that f is continuous and periodic. (The symbols $\mathbf{L}^1$ and $\mathbf{L}^p$ here denote the sets of measurable functions f on $(-\pi, \pi)$ such that $|f|$ and $|f|^p$, respectively, are Lebesgue-integrable over $(-\pi, \pi)$. A tiny modification to this definition is explained in detail in 2.2.4 and will be adopted thereafter in this book.)

Each of the relations (A) to (D) can, therefore, be used to characterize the Fourier series of f under the stated conditions, and each provides some justification for singling out the Fourier series for study. (There are, by the way, numerous other relationships that might be added to the list.)

It turns out that the weakest relationship (D) is suggestive of fruitful generalizations of the concept of Fourier series of such a type that the distinction between Fourier series and trigonometric series largely disappears. It suggests in fact the introduction of so-called *distributions* or *generalized functions* in the manner first done by L. Schwartz [$S_{1,2}$]. It will then appear that any trigonometric series in which $c_n = O(|n|^k)$ for some k may be regarded as the Fourier series of a distribution, to which this series is distributionally convergent. These matters will be dealt with in Chapter 12.

1.3.4. **Summary.** The substance of Section 1.2 and 1.3.3 summarizes the justification for subsequent concentration of attention on Fourier series in particular, at least insofar as reference is restricted to harmonic analysis in its classical setting. We shall soon embark on a program that will include at appropriate points a verification of each of the unproved statements upon which this justification is based. As for trigonometric series in general, we shall do no more than pause occasionally to mention a few of the simpler results that demand no special techniques.

A bird's-eye view of many of the topics to be discussed at some length in this book is provided by the survey article G. Weiss [1].

1.3.5. **Fourier Series and General Groups.** There are still other reasons in favor of the chosen policy which are based upon recent trends in analysis. Harmonic analysis has not remained tied to the study of Fourier series of periodic functions of a real variable; in particular it is now quite clear that Fourier-series theory has its analogue for functions defined on compact Abelian groups (and even, to some extent, on still more general groups); see, for example, [HR], [Re], [E_1]. While the level at which this book is written precludes a detailed treatment of such extensions, we shall make frequent reference to modern developments. However regrettable it may seem, it is a fact that these developments cluster around the extension of precisely those portions of the classical theory which do not depend upon the deeper properties of pointwise convergence and summability, and that a detailed treatment of the analogue for compact groups of the theory of general trigonometric series appears to lie in the future. Moreover, the portions of the classical theory that have so far been extended appear to be those most natural for handling those problems which are currently the center of attention in general harmonic analysis. Of course, these prevailing features may well change with the passage of time. While they prevail, however, they add support to the view that it is reasonable to accord some autonomy to a theory in which the modes of representation mentioned in 1.3.2 take precedence over that of pointwise representation.

EXERCISES

1.1. Establish the formulae

$$D_N(x) = \sum_{|n| \leqslant N} e^{inx} = \frac{\sin (N + \frac{1}{2})x}{\sin \frac{1}{2}x},$$
$$F_N(x) = (N + 1)^{-1} [D_0(x) + \cdots + D_N(x)]$$
$$= (N + 1)^{-1} \left[\frac{\sin \frac{1}{2}(N + 1)x}{\sin \frac{1}{2}x}\right]^2$$

for $N \geqslant 0$ an integer and $x \not\equiv 0$ modulo 2π, where the equality signs immediately following $D_N(x)$ and $F_N(x)$ are intended as definitions for all real x.

1.2. Prove that if p and q are integers and $p < q$, and if $x \not\equiv 0$ modulo 2π, then

$$\left|\sum_{p \leqslant n \leqslant q} e^{inx}\right| \leqslant |\operatorname{cosec} \tfrac{1}{2}x|.$$

By using partial summation (see 7.1.2 and [H], p. 97 ff.) deduce that if $c_p \geqslant c_{p+1} \geqslant \cdots \geqslant c_q \geqslant 0$, then, for $x \not\equiv 0$ modulo 2π,

$$\left|\sum_{p \leqslant n \leqslant q} c_n e^{inx}\right| \leqslant c_p \, |\operatorname{cosec} \tfrac{1}{2}x|.$$

1.3. Assume that $c_n \geqslant c_{n+1}$ and $\lim_{n \to \infty} c_n = 0$. Show that the series

$$\sum_{n=0}^{\infty} c_n e^{inx}$$

is convergent for $x \not\equiv 0$ modulo 2π, and that the convergence is uniform on any compact set of real numbers x which contains no number $\equiv 0$ modulo 2π.

1.4. Assume that $c_n \geqslant c_{n+1} \geqslant 0$ and $nc_n \leqslant A$. Show that

$$\left|\sum_{n=1}^{N} c_n \sin nx\right| \leqslant A(\pi + 1).$$

Hints: One may assume $0 < x < \pi$. Put $m = \min (N, [\pi/x])$ and split the sum into $\sum_1^m + \sum_{m+1}^N$, an empty sum being counted zero. Estimate the partial sums separately, using Exercise 1.2 for $\sum_{m+1}^N$.

1.5. Assume that the c_n are as in Exercise 1.4. Show that the series $\sum_{n=1}^{\infty} c_n \sin nx$ is boundedly convergent, and that the sum function is continuous, except perhaps at the points $x \equiv 0$ modulo 2π. (More general results will appear in Chapter 7.)

1.6. Compute the complex Fourier coefficients of the following functions, each defined by the prescribed formula over $[-\pi, \pi)$ and defined elsewhere so as to have period 2π:

(1) $f(x) = x$;

(2) $f(x) = |\sin x|$;

(3) $f(x) = x$ for $-\pi \leqslant x \leqslant 0$, $f(x) = 0$ for $0 < x < \pi$.

1.7. By a *trigonometric polynomial* is meant a function f admitting at least one expression of the form

$$f(x) = \sum_{|n| \leqslant N} c_n e^{inx},$$

where the c_n are f-dependent complex numbers.

(1) Use the orthogonality relations to show that, if f is a trigonometric polynomial, then

$$\hat{f}(n) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

vanishes for all but a finite number of integers n and that $f(x) = \sum_{n \in Z} \hat{f}(n) e^{inx}$. Show also that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \sum_{n \in Z} |\hat{f}(n)|^2$$

whenever f is a trigonometric polynomial. (This is a special case of Parseval's formula, to which we shall return in Chapter 8 and Section 10.5; see also Remark 6.2.7.)

A trigonometric polynomial f such that $\hat{f}(n) = 0$ for $|n| > N$ is said to be *of degree at most* N.

(2) Verify that the set $\mathbf{T}_N$ of trigonometric polynomials of degree at most N forms a complex linear space of dimension $2N + 1$ with respect to pointwise operations, and that if $f \in \mathbf{T}_N$, then also $\operatorname{Re} f \in \mathbf{T}_N$ and $\operatorname{Im} f \in \mathbf{T}_N$.

(3) Show that if $f \in \mathbf{T}_N$, $f \neq 0$, then f admits at most $2N$ zeros (counted according to multiplicity) in the interval $[0, 2\pi)$ (or in any interval congruent modulo 2π to this one).

1.8. (Stečkin's lemma) Suppose $f \in \mathbf{T}_N$ is real-valued, and that

$$\|f\|_\infty \equiv \sup_x |f(x)| = M = f(x_0).$$

Prove that

$$f(x_0 + y) \geqslant M \cos Ny \qquad \text{for } |y| \leqslant \frac{\pi}{N}.$$

Hints: Put $g(y) = f(x_0 + y) - M \cos Ny$. Assuming the assertion false, we choose y_0 so that $|y_0| < \pi/N$ and $g(y_0) < 0$. We assume $0 < y_0 < \pi/N$; otherwise the subsequent argument proceeds with the interval $[-2\pi, 0)$ in place of $[0, 2\pi)$. By examining closely the signs of g at the points $k\pi/N$ $(k = 0, 1, 2, \cdots, 2N)$, show that g admits at least $2N + 1$ zeros in $[0, 2\pi)$. A contradiction results from Exercise 1.7.

1.9. (Bernstein's inequality) Prove that if $f \in \mathbf{T}_N$, then $\|f'\|_\infty \leqslant N \|f\|_\infty$ (the notation being as in the preceding exercise).

Hints: It suffices, by Exercises 1.7 and 1.10, to prove the inequality for real-valued $f \in \mathbf{T}_N$. If $f'(x_0) = m \equiv \|f'\|_\infty$ (which can be arranged by changing f into $-f$ if necessary) and $M = \|f\|_\infty$. Exercise 1.8 gives $f'(x_0 + y) \geqslant m \cos Ny$ for $|y| \leqslant \pi/N$. Integrate this inequality.

Notes: Many other proofs are known; the above, due to Stečkin, is perhaps the simplest. For a proof based upon interpolation methods, see $[Z_2]$, p. 11. More general results, also due to Bernstein, apply to entire functions of order one and exponential type; see $[Z_2]$, p. 277.

See also the approach in [Kz], p. 17; W. R. Bloom [1], [2]; MR **51** # 1239; **52** ## 6288, 11446; **53** # 11289; **54** # 829.

The inequality has also been extended in an entirely different way by Privalov, who showed that if $I = (a', b')$ and $J = (a, b)$ are any two subintervals of $[-\pi, \pi]$ satisfying $a < a' < b' < b$, then there exists a number $c(I, J)$ such that

$$\sup_{x \in I} |f'(x)| \leqslant c(I, J)N \cdot \sup_{x \in J} |f(x)|$$

for any $f \in \mathbf{T}_N$. It is furthermore established that similarly (but perhaps with a different value for $c(I, J)$) one has

$$\{\int_I |f'(x)|^p \, dx\}^{1/p} \leqslant c(I, J)N \cdot \{\int_J |f(x)|^p \, dx\}^{1/p}$$

for any $f \in \mathbf{T}_N$ and any p satisfying $1 \leqslant p < \infty$. Both inequalities are also valid when $I = J = [-\pi, \pi]$ and $c(I, J) = 1$, the first reducing to that of Bernstein and the second being in this case due to Zygmund. For more details, see $[Ba_2]$, pp. 458–462. See also $[L_2]$, Chapter 3.

1.10. Suppose that $\mathbf{E}$ is a complex linear space of complex-valued functions on a given set (pointwise operations), that $\mathbf{E} = \mathbf{E}_0 + i\mathbf{E}_0$ where $\mathbf{E}_0$ is the set of real-valued functions in $\mathbf{E}$, that l is a complex-linear functional on $\mathbf{E}$ which is real-valued on $\mathbf{E}_0$, and that p is a seminorm on $\mathbf{E}$ (see Appendix B.1.2). Suppose finally that $p(x) \leqslant p(y)$ whenever $x, y \in \mathbf{E}$ and $|x| \leqslant |y|$, and that $|l(x)| \leqslant p(x)$ for $x \in \mathbf{E}_0$. Prove that $|l(x)| \leqslant p(x)$ for $x \in \mathbf{E}$.

Hints: Write $x = a + ib$ with $a, b \in \mathbf{E}_0$ and $l(x) = r(\alpha + i\beta)$ with $r \geqslant 0$, α and β real, and $\alpha^2 + \beta^2 = 1$. Then

$$|l(x)| = r = (\alpha - i\beta)l(x) = l[(\alpha - i\beta)(a + ib)];$$

expanding and taking real parts: $|l(x)| = l(\alpha a + \beta b) \leqslant p(\alpha a + \beta b)$, and so forth.

1.11. Prove that, if a trigonometric polynomial f is real-valued and nonnegative, then $f = |g|^2$ for some trigonometric polynomial g (Fejér and F. Riesz).

Hints: Suppose $f(x) \equiv \sum_{|n| \leq N} c_n e^{inx}$ and consider first the case in which $f(x) > 0$ for all x. Assume (without loss of generality) that $c_{-N} \neq 0$ and examine the polynomial $P(z) = z^N \sum_{|n| \leq N} c_n z^n$. Observe that $P(z) = z^{2N}\overline{P(\bar{z}^{-1})}$ and $f(x) = e^{-iNx}P(e^{ix})$. Verify that the zeros of P are of the form $a_1, a_2, \cdots$, and $\bar{a}_1^{-1}, \bar{a}_2^{-1}, \cdots$, where $0 < |a_r| < 1$, and factorize P accordingly.

In case one knows merely that $f \geqslant 0$, apply the above to the $f_k = f + 1/k$ $(k = 1, 2, \cdots)$ and use a limiting argument.

Remarks. The theorem does not extend in the expected way to other groups; see [R], 8.4.5.