Differential Equation Models

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Preface

The purpose of this four volume series is to make available for college teachers and students samples of important and realistic applications of mathematics which can be covered in undergraduate programs. The goal is to provide illustrations of how modern mathematics is actually employed to solve relevant contemporary problems. Although these independent chapters were prepared primarily for teachers in the general mathematical sciences, they should prove valuable to students, teachers, and research scientists in many of the fields of application as well. Prerequisites for each chapter and suggestions for the teacher are provided. Several of these chapters have been tested in a variety of classroom settings, and all have undergone extensive peer review and revision. Illustrations and exercises are included in most chapters. Some units can be covered in one class, whereas others provide sufficient material for a few weeks of class time.

Volume 1 contains 23 chapters and deals with differential equations and, in the last four chapters, problems leading to partial differential equations. Applications are taken from medicine, biology, traffic systems and several other fields. The 14 chapters in Volume 2 are devoted mostly to problems arising in political science, but they also address questions appearing in sociology and ecology. Topics covered include voting systems, weighted voting, proportional representation, coalitional values, and committees. The 14 chapters in Volume 3 emphasize discrete mathematical methods such as those which arise in graph theory, combinatorics, and networks. These techniques are used to study problems in economics, traffic theory, operations research, decision theory, and other fields. Volume 4 has 12 chapters concerned with mathematical models in the life sciences. These include aspects of population growth and behavior, biomedicine (epidemics, genetics and bio-engineering), and ecology.

These four volumes are the result of two educational projects sponsored by The Mathematical Association of America (MAA) and supported in part by the National Science Foundation (NSF). The objective was to produce needed material for the undergraduate curriculum. The first project was undertaken by the MAA's Committee on the Undergraduate Program in Mathematics (CUPM). It was entitled Case Studies and Resource Materials for the Teaching of Applied Mathematics at the Advanced Undergraduate Level, and it received financial support from NSF grant SED72-07370 between September 1, 1972 and May 31, 1977. This project was completed under the direction of Donald Bushaw. Bushaw and William Lucas served as chairmen of CUPM during this effort, and George Pedrick was involved as the executive director of CUPM. The resulting report, which appeared in late 1976, was entitled *Case Studies in Applied Mathematics*, and it was edited by Maynard Thompson. It contained nine chapters by eleven authors, plus an introductory chapter and a report on classroom trials of the material.

The second project was initiated by the MAA's Committee on Institutes and Workshops (CIW). It was a summer workshop of four weeks duration entitled Modules in Applied Mathematics which was held at Cornell University in 1976. It was funded in part by NSF grant SED75-00713 and a small supplemental grant SED77-07482 between May 1, 1975 and September 30, 1978. William F. Lucas served as chairman of CIW at the time of the workshop and as director of this project. This activity lead to the production of 60 educational modules by 37 authors.

These four volumes contain revised versions of 9 of the 11 chapters from the report *Case Studies in Applied Mathematics*, 52 of the 60 modules from the workshop Modules in Applied Mathematics, plus two contributions which were added later (Volume 2, Chapters 7 and 14), for a total of 63 chapters. A preliminary version of the chapter by Steven Brams (Volume 2, Chapter 3), entitled "One Man, *N* Votes," was written in connection with the 1976 MAA Workshop. The expanded version presented here was prepared in conjunction with the American Political Science Association's project Innovation in Instructional Materials which was supported by NSF grant SED77-18486 under the direction of Sheilah K. Mann. The unit was published originally as a monograph entitled *Comparison Voting*, and was distributed to teachers and students for classroom field tests. This chapter was copyrighted by the APSA in 1978 and has been reproduced here with its permission.

An ad hoc committee of the MAA consisting of Edwin Beckenbach, Leonard Gillman, William Lucas, David Roselle, and Alfred Willcox was responsible for supervising the arrangements for publication and some of the extensive efforts that were necessary to obtain NSF approval of publication in this format. The significant contribution of Dr. Willcox throughout should be noted. George Springer also intervened in a crucial way at one point. It should be stressed, however, that any opinions or recommendations are those of the particular authors, and do not necessarily reflect the views of NSF, MAA, the editors, or any others involved in these project activities.

There are many other individuals who contributed in some way to the realization of these four volumes, and it is impossible to acknowledge all of them here. However, there are two individuals in addition to the authors, editors and people named above who should receive substantial credit for the ultimate appearance of this publication. Katherine B. Magann, who had provided many years of dedicated service to CUPM prior to the closing of the CUPM office, accomplished the production of the report *Case Studies in Applied Mathematics*. Carolyn D. Lucas assisted in the running of the 1976 MAA Workshop, supervised the production of the resulting sixty modules, and served as managing editor for the publication of these four volumes. Without her efforts and perseverance the final product of this major project might not have been realized.

July 1982 W. F. LUCAS

Preface for Volume 1

Volume 1 consists of twenty-three chapters concerned with mathematical modeling and problem solving using differential equations. The chapters in Part I deal with the very beginning, and often the most important part of the modeling process: how to translate the given problem into a mathematical problem. The first chapter by Henderson West shows how to translate various word problems into differential equations, while Chapter 3 by Frauenthal deals with the special case of population growth models, a subject of much current interest. The second chapter, also by Henderson West, describes how to analyze a differential equation, and how to draw qualitative conclusions from it. These three chapters were written with a clarity and painstaking attention to detail that is not often found in textbooks, and thus are "must reading" for the beginning student of modeling.

The three chapters by Braun in Part II deal with three diverse and important problems that can be modeled, and completely solved, by first order differential equations. It is interesting to note that the work described in these units (and indeed, many of the modules in this volume) was originally done not by mathematicians, but by chemists, biologists and sociologists.

Part III is essentially a continuation of Part II, the difference being that the problems in this section are modeled by higher order linear equations and by solvable systems of first order equations. Systems of differential equations can be used to model very complex and even esoteric problems, and the results obtained are often very exciting, as seen in the three modules by Braun, Coleman, and Powers.

The five chapters by Baker and Drew in Part IV describe some applications of mathematics to problems in traffic theory, another popular source of interesting modeling problems. The results obtained are not as powerful and spectacular as the results obtained in the previous chapters. Nevertheless, they are still extremely important as they illustrate how mathematical modeling can often aid in our understanding of complex and even uncontrollable and unsolvable problems.

The five chapters by Braun and Coleman in Part V deal with systems of non-linear equations and their application to important problems in biology and ecology. Powerful results are obtained via the methods of the qualitative theory of differential equations. It is interesting to note that the qualitative theory of differential equations evolved, originally, from problems in physics and astronomy. These same techniques have had important applications to problems in the biological sciences. Indeed, the existence of such a powerful and polished theory has motivated many mathematicians to undertake the study of several outstanding problems in biology and ecology.

The chapters in Part VI deal with systems that can be modelled by partial differential equations, one of the more difficult areas of mathematical analysis. The unit by Borrelli is basic in that it carefully describes the theory and the actual modeling. The remaining three chapters by Drew, Meyer and Porsching deal with concrete and important real life applications.

July 1982

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Part VI. Models Leading to Partial Differential Equations

PART I

DIFFERENTIAL EQUATIONS, MODELS, AND WHAT TO DO WITH THEM

CHAPTER 1

Setting Up First-Order Differential Equations from Word Problems

Beverly Henderson West*

1. Introduction

"Word problems" are sometimes troublesome; but you have learned that most noncalculus applied problems can be conquered with careful translating and attention to the kinds of units involved. A trivial illustration of this type is as follows.

EXAMPLE 1. One Sunday a man in a car leaves A at noon and arrives at B at 3:20 p.m. If he drove steadily at 55 mi/h, how far is B from A?

Solution: • distance = rate × time

$$= (55 \text{ mi/h})(3\tfrac{1}{3} \text{ h})$$

$$= (55)(\tfrac{10}{3}) \text{ mi}$$

$$\bullet = 183\tfrac{1}{3} \text{ mi}.$$

Note: In this and the other examples of this chapter, the *key* mathematical statements (equations, solutions, initial conditions, answers, etc.) are preceded by bullets (•) to stand out among the calculations. An even better way to emphasize the key statements, especially in handwritten work, would be to draw boxes around them or to use color highlighting.

Word problems involving differential equations may be more difficult than the applied problems you have dealt with heretofore. Contrast Example 1 with Example 2.

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EXAMPLE 2. One Sunday a man in a car leaves A at noon and arrives at B at 3:20 p.m. He started from rest and steadily increased his speed, as indicated on his speedometer, to the extent that when he reached B he was driving at 60 mi/h. How far is B from A?

Solution: An inexperienced student might suspect that not enough information is provided. However, the steadily increasing speedometer reading means that the man's speed or velocity is a *linear function* of time, and velocity is the derivative of distance S as a function of time. So,

$$\bullet \frac{dS}{dt} = at + b \quad \text{(mi/h)}$$

and, by integration,

$$\bullet\ S = \tfrac{1}{2}at^2 + bt + c \quad \text{(mi)}.$$

If t is measured in hours, the remaining information in the problem tells us that

$$\bullet\ ① \ S(0) = 0; \quad ② \ \frac{dS}{dt}(0) = 0; \quad ③ \ \frac{dS}{dt}(3\tfrac{1}{3}) = 60; \quad ④ \ S(3\tfrac{1}{3}) = ?.$$

The first three conditions are enough to evaluate the three constants a, b, c, and the fourth will then give us the answer to the question in the problem:

$$\begin{aligned} c &= 0 \quad &(\text{from } ①) \\ b &= 0 \quad &(\text{from } ②) \\ a &= 18 \quad &(\text{from } ③) \end{aligned}$$

so

$$S(3\tfrac{1}{3}) = 9t^2 = 9(\tfrac{10}{3})^2 = \bullet\ 100 \text{ mi, the distance from } A \text{ to } B.$$

Now, you may well have solved this problem differently, but all these ingredients must have been implicit in your solution. For instance, you might have realized that starting from rest would immediately give $dS/dt = at$, but you would still have been using information from the problem (condition (2)) to evaluate a constant which would otherwise have been there.

Consider another simple differential equation word problem which is commonly encountered.

EXAMPLE 3. The growth rate of a population of bacteria is in direct proportion to the population. If the number of bacteria in a culture grew from 100 to 400 in 24 h, what was the population after the first 12 h?

Solution: The first sentence tells what is true at *any* instant; the second gives information on specific instants. If we denote the population by $y(t)$, the first tells us that

$$\bullet \frac{dy}{dt} = ky,$$