FLUID MECHANICS

A CONCISE INTRODUCTION
TO THE THEORY

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CHAPTER ONE

FUNDAMENTALS

1. DEFINITION OF A FLUID

The defining property of a fluid is that it cannot withstand shearing forces, however small, without sustained motion. Since both gases and liquids have this property, they both are fluids and subject to a unified treatment as far as their macroscopic motion is concerned. Fluids may, of course, be in a state of equilibrium under the action of surface forces applied normal to their boundary. Indeed, the entire subject of hydrostatics deals with fluids in such a state.

In this book, only fluids which have no privileged directions are considered. Such fluids are called *isotropic*.

2. VELOCITIES

If matter were infinitely divisible, it would be meaningful to define the velocity of a material point as its time rate of displacement; but matter in general and fluids in particular are not infinitely divisible. Strictly

speaking, we can comprehend the velocity only of a molecule, an atom, a nucleus, or an electron; the "velocity" of a geometrical point in the empty space between the electrons and the nucleus in an atom, between atom and atom in a molecule, or between the molecules themselves is physically meaningless.

It would be a hopeless situation indeed if in order to study fluid motion we had to deal with the molecules directly. Fortunately, although there is much empty space between molecules, the number of molecules per unit volume of a liquid or of a gas under ordinary conditions is extremely great. A mole of gas has approximately 6.024×10^{23} molecules (Loschmidt number) and under normal conditions occupies a volume of 22.4 liters, so that 1 cm^3 contains 2.687×10^{19} molecules. The number of molecules in 1 μ^3 (1 $\mu = 1/1,000$ mm) is about 2.687×10^7. For such a small volume this is an exceedingly great number. Thus gases under ordinary conditions—and, a fortiori, liquids—can be considered for all practical purposes to be continuous. The velocity of a fluid particle of a very small volume (1 μ^3, say) can be defined to be the average of the momenta of the vast number of molecules contained therein divided by the total mass of the particle. Since the volume of this particle is very small, the velocity so defined can be considered to be the velocity of the material point situated at the center of mass of the fluid particle, as if the fluid were indefinitely divisible. This approach is valid unless a highly rarefied gas is being studied.

According to this definition, the velocity v of a fluid particle consisting of N_i molecules $(i = 1, 2, \ldots, n)$ of the ith substance present in the particle is

$$\mathbf{v} = \frac{\sum_{i=1}^{n}\left[m_i \sum_{j=1}^{N_i} \mathbf{v}(i,j) \right]}{\sum_{i=1}^{n} N_i m_i}, \tag{1}$$

in which m_i is the molecular weight of the ith substance and $\mathbf{v}(i,j)$ is the velocity of the jth of the N_i molecules of molecular weight m_i.

The velocity of a fluid particle can also be defined as

$$\mathbf{v} = \frac{\sum_{j=1}^{N} \mathbf{v}(j)}{N}, \tag{2}$$

in which $$N = \sum_{i=1}^{n} N_i$$

and $\mathbf{v}(j)$ is the velocity of the jth of the N molecules of the n substances taken together without regard to the *kind* of molecules. The particle velocity defined by (1) is a mean velocity of the molecules weighted by the molecular weights, whereas that defined by (2) is an unweighted mean

velocity. In Sec. 5, when the equation of continuity is discussed, it will be evident that the two definitions do not correspond to the same form of the equation of continuity if the fluid is a mixture. In Chap. 8, the matter will be further discussed in connection with the diffusion equation. In order to facilitate later discussions, Eq. (1) will be put into a slightly more convenient form. If for each constituent of a fluid mixture the mean velocity is defined to be

$$\mathbf{v}(i) = \sum_{j=1}^{N_i} \frac{\mathbf{v}(i,j)}{N_i}$$

Eq. (1) can be written as

$$\mathbf{v} = \frac{\sum\limits_{i=1}^{n} N_i m_i \mathbf{v}(i)}{\sum\limits_{i=1}^{n} N_i m_i} \qquad (1a)$$

Whatever the volume of the fluid particle, the density ρ_i of the ith constituent must be proportional to $N_i m_i$. Thus $(1a)$ can be written as

$$\mathbf{v} = \frac{\sum\limits_{i=1}^{n} \rho_i \mathbf{v}(i)}{\rho} , \qquad (1b)$$

in which ρ is the total density, or

$$\rho = \sum_{i=1}^{n} \rho_i. \qquad (3)$$

The full meaning of the definition of the velocity of a particle must be understood. The average of the momenta of the molecules contained in a fluid particle may be zero, even though these molecules are moving with great speeds individually. In fact, it is zero if the molecules are in completely random motion. The average of the molecular momenta is therefore a measure of the ordered part of molecular motion. The intensity of random molecular motion is manifested in a property called *temperature*.

If the fluid is considered as a continuum, the velocity at any point in the fluid is postulated as a function of time t and the coordinates x_i ($i = 1, 2, 3$) of the point. It is evident that without some modification the continuum concept is incompatible with the concept of diffusion; this basic incompatibility, however, presents no insurmountable difficulties.

Other properties, kinematic or dynamic, can also be defined either from the molecular or the continuum point of view. In this book a fluid will be considered as a continuum, although wherever mass diffusion is involved, the appropriateness of the continuum approach must be judged in the light of molecular considerations.

3. ACCELERATION

To find the acceleration of a particle, we must see how its velocity changes with time and therefore must keep track of its identity—at least for a short interval of time. There are two different descriptions of fluid motion. In the one, the coordinates of fluid particles are considered to be a function of time and of their permanent identifications, such as their initial coordinates. In the other, the velocities and other properties of fluid particles are considered to be functions of time and fixed spatial coordinates independent of time. The former is called the *material* or *Lagrangian* description, after Joseph Louis Lagrange (1736–1813), and the latter the *spatial* or *Eulerian* description, after Leonard Euler (1707–1783), although historians argue that both descriptions should be attributed to Euler.

With the Lagrangian description, the Cartesian coordinates (c_1,c_2,c_3) of the position of a fluid particle at the initial time serve to identify the particle. The subsequent (Cartesian) coordinates of the position of the same particle will be denoted by (X_1,X_2,X_3). These coordinates are functions of c_i and time t. For a definite particle the identifying coordinates are fixed, and the coordinates X_i are functions of time alone. If the coordinates are Cartesian, the velocity components are

$$u_i = \frac{\partial X_i}{\partial t}, \qquad i = 1, 2, 3,$$

and the acceleration components are simply

$$a_i = \frac{\partial^2 X_i}{\partial t^2}, \qquad i = 1, 2, 3.$$

With the Eulerian description, the velocity and the acceleration of a fluid in motion are considered to be functions of time and position. The Cartesian coordinates (x_1,x_2,x_3) describing the position are now independent of time. However, to find the velocity or the acceleration we still have to follow the particle for a short interval of time dt. During this interval, the coordinates of the particle followed have changed by the amounts dX_i. The corresponding change of the velocity consists of two parts: the first part is a local change with time, and the second part is a change due to the change of position of the particle. If the velocity component in the direction of x_i is again denoted by u_i, then since u_i is a function of t and x_i,

$$du_i = \frac{\partial u_i}{\partial t} \, dt + \frac{\partial u_i}{\partial x_j} \, dx_j$$

for all increments dt and dx_j. The physical process of following the particles corresponds to the identification of dx_j with the particle

displacements dX_j. Thus, when a particle is followed,

$$du_i = \frac{\partial u_i}{\partial t}\, dt + \frac{\partial u_i}{\partial x_j}\, dX_j, \qquad i = 1, 2, 3,$$

in which, as in the preceding equation, the last term stands for the sum of three terms, with j ranging over 1, 2, and 3. This summation convention will always be used unless otherwise stated. If the coordinates are Cartesian, the displacement components are

$$dX_i = u_i\, dt, \qquad i = 1, 2, 3.$$

Dividing (3) by dt and taking the limit, we have

$$a_i = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}, \qquad i = 1, 2, 3. \tag{4}$$

The operator

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \tag{5}$$

stands for what is commonly called the *substantial differentiation*, which means differentiation with respect to time by following the substance. From (5) it follows that

$$a_i = \frac{Du_i}{Dt}, \qquad i = 1, 2, 3. \tag{4a}$$

A flow is *steady* if all dependent variables, such as velocity, acceleration, density, temperature, and pressure, are independent of time at any fixed point.

3.1. The Proper Definition of Acceleration When Diffusion Is Present

In employing the continuum approach special care must be taken when diffusion is present, as can be illustrated by considering the force acting on a fluid particle of volume V. Suppose that there are two gases, one of molecular weight m_1 and the other of molecular weight m_2, and that there are N_1 molecules of the first gas and N_2 of the second gas in ΔV. The density in the cgs system is

$$\rho = \frac{N_1 M_1 + N_2 M_2}{\Delta V} \qquad \text{in g/cm}^3, \tag{6}$$

in which M_1 and M_2 are the molecular masses of the gases, which are related to m_1 and m_2 by

$$M_1 = \frac{m_1}{L}, \qquad M_2 = \frac{m_2}{L},$$

L being the Loschmidt number, i.e., the number of molecules in m g of a gas of molecular weight m. Now, because of diffusion, the two gases in the particle do not have the same mean velocity or acceleration.

If the acceleration of the particle is defined by a formula similar to (2), i.e., as the unweighted mean of the accelerations of all the molecules in the particle of volume ΔV, again $a_i \rho \, \Delta V$ will not give the components of the net force acting on the particle. Only if $\mathbf{a}$ (or a_i) is defined by a formula like (1*b*), i.e., as a mean acceleration weighted by the density, will it be obvious that $\mathbf{a}\rho \, \Delta V$ is the net force acting on the particle. Fortunately, the differences in accelerations (or velocities) of the various gas components in a mixture are usually small compared with the acceleration of any one gas or with the force acting on the entire fluid particle. Hence the error caused by using an inadequately defined acceleration to compute the net force on a particle of a gas mixture is small. The same is true of liquids. But even the conceptual difficulty disappears if there is only one substance or if the mixture is homogeneous and the flow is steady, so that there is no net diffusion.

4. PATH LINES AND STREAMLINES

The coordinates X_i used in the Lagrangian description are functions of the identifying coordinates c_i and t. For fixed c_i, we have

$$X_i = F_i(t), \qquad i = 1, 2, 3,$$

as the parametric equations describing the locus of the particle under consideration. This locus is called a *path line*.

Although the Eulerian description is less convenient for describing path lines, it is superior to the Lagrangian method for describing streamlines, or lines to which the velocity vectors of the fluid are tangent at a particular *instant*. The differential equations for such lines are, in Cartesian coordinates,

$$\frac{dx_1}{u_1} = \frac{dx_2}{u_2} = \frac{dx_3}{u_3}. \tag{7}$$

As mentioned before, a flow is called steady if at any fixed point in the fluid the velocity does not change with time. For steady flows path lines are coincident with streamlines, for when a particle reaches the position of its predecessor on a streamline, by dint of the steadiness of the motion it has the same velocity as its predecessor and therefore goes the way its predecessor went, and so on. But if the flow is unsteady, path lines and streamlines may (though not necessarily) differ. A simple case of unsteady flow in which path lines and streamlines are coincident is rectilinear parallel flow with a velocity varying with time.

The streamlines passing through a closed curve which does not lie on a surface generated by streamlines form a tubular surface. The fluid contained in such a surface is called a *stream tube*.
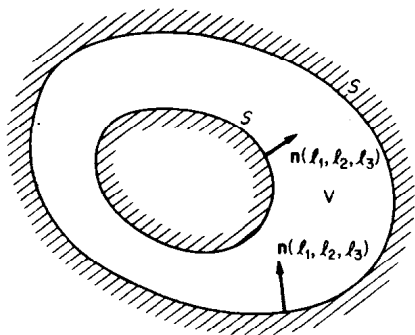
FIGURE 1.　DEFINITION SKETCH FOR GREEN'S THEOREM.

5.　CONTINUITY

Since the principle of mass conservation must not be violated, the velocity distribution in a fluid in motion must satisfy a certain condition. Obviously, since mass is involved, the density (mass per unit volume) of the fluid ρ must be considered. If S is an arbitrary fixed surface enclosing a fluid volume V and u_n is the velocity along the normal to S drawn *into* V, the net amount of mass flowing into V per unit time is

$$\int_S \rho u_n \, dS.$$

This must be equal to the rate at which the mass in V is increasing. Thus

$$\frac{\partial}{\partial t} \int_V \rho \, dV = \int_S \rho u_n \, dS. \tag{8}$$

Equation (8) is an integral equation of continuity and can be reduced to a differential form. Green's theorem states that for any three single-valued and differentiable functions (U_1, U_2, U_3) in Cartesian coordinates,

$$\int_S l_i U_i \, dS = -\int_V \frac{\partial U_i}{\partial x_i} \, dV, \tag{9}$$

in which the l's are the direction cosines of the normal to S drawn into V (Fig. 1). Taking $U_i = \rho u_i$, we have

$$\int_S \rho u_n \, dS = \int_S l_i \rho u_i \, dS = -\int_V \frac{\partial(\rho u_i)}{\partial x_i} \, dV. \tag{10}$$

From (8) and (10) it follows that

$$\int_V \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} \right] dV = 0. \tag{11}$$