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CHAPTER I

THE AXIOMATIC METHOD

§ 1. Geometry and Axiomatic Systems

Each scientific theory involves a body of concepts and a collection of assertions. When questioned of the meaning of a concept, we often explain it or define it in terms of other concepts. Similarly, when questioned of the truth or the reason for believing the truth of an assertion, we usually justify our belief by indicating that it follows from or can be deduced from certain other assertions which we accept. If somebody, as many children do, continues indefinitely to ask for definitions or deductions, it is obvious that sooner or later one of two things will happen. Either we find ourselves travelling in a circle, making use, in our answers, of concepts and assertions whose meaning and justification we originally set out to explain; or, at some stage, we refuse to supply any more definitions and deductions, and reply bluntly that the concepts and assertions we employ in our answer are already the most basic which we take for granted. When the problem is to understand the meaning of a concept or to see that a proposition is true, there is no basic objection to circular procedures, and, indeed, mutual support may in many cases prove to be the best sort of evidence we can ever obtain. But when we are able to start merely with a small number of primitive ideas and propositions, the linear mode of approach does have a special appeal and fascination in that questions of meaning and truth become concentrated in these few initial primitives plus certain typical ways of definition and deduction.

Usually, the primitive propositions are called axioms or postulates. When the concepts and propositions of a theory are thus arranged according to the connections of definability and deducibility, we have an axiomatic system for the theory.

The best known axiom system is undoubtedly Euclid's for geometry. His *Elements* is said to have had a wide circulation next only to the Bible. Admiration for its rigour and thoroughness has been expressed frequently. Spinoza, for example, attempted to attain the same formal perfection in his *Ethics* (Ethica more geometrico demonstrata).

There are in the *Elements* ten primitive propositions (axioms) of which five are called common notions, five are called postulates. From these and a number of definitions, 465 propositions (theorems) are deduced with considerable logical rigour. The same deductive method was used by Newton in mechanics, by Lagrange in analytic mechanics, by Clausius in thermodynamics. In recent years, axiom systems for many branches of mathematics and natural sciences have appeared.

While Euclid's unification of masses of more or less isolated discoveries was undoubtedly an impressive success in the program of systematizing mathematics, his actual axiom system is, according to the standard generally accepted now, far from formally perfect. For example, instead of taking point and line as primitive concepts, Euclid defines them respectively as "something which has no parts" and "length without breadth". Moreover, investigations in the latter part of last century have revealed many axioms which are implicitly assumed or inadequately formulated by Euclid.

The development of views on axiomatic systems was closely connected with the discovery of non-euclidean geometries. On the one hand, Euclid's axioms as a whole seemed so natural and obvious that they were regarded as logically necessary or, according to Kant, synthetic a priori. On the other hand, since ancient times, many mathematicians have found Euclid's fifth postulate (the "parallel axiom") not sufficiently self-evident and tried to derive it from the other axioms. This disputed axiom states in effect that through a given point one and only one straight line can be drawn which is parallel to a given straight line. Unlike the other axioms, it involves a reference to infinity through the concept of a parallel. During the Renaissance, controversy over the axiom renewed. In the eighteenth century, Lambert and Saccheri tried, with no success, to derive contradictions from the negation of the axiom.

During the first third of the nineteenth century, Lobachevski, Bolyai, and Gauss independently discovered a consistent geometry in which the parallel axiom was replaced by the assumption that there exist more than one parallels through a given point. In 1854, Riemann discussed the possibility of a finite but unbounded space and invented geometries in which there exist no parallel lines at all. All these geometries in which Euclid's fifth postulate is false are called non-euclidean geometries.

The realization of different possible geometries led to a desire to separate abstract mathematics from spatial intuition. For example, Grassmann stressed in his *Ausdehnungslehre* (1844) the distinction between a purely mathematical discipline and its application to nature. Since the axioms are no longer necessarily true in the physical world, deductions must be made independently of spatial intuition. Reliance on diagrams and meaning of geometrical concepts must, therefore, be avoided.

In his book on geometry (1882), Pasch adhered to this changed viewpoint and found out many shortcomings in Euclid's axiomatization. He disclosed the most hidden axioms, those of order. For example, he noted the need of the following axiom: a straight line that intersects one side of a triangle in any point other than a vertex must also intersect another side of the triangle.

Hilbert's famous work on the foundations of geometry appeared in 1899. It further emphasized the point that strict axiomatization involves total abstraction from the meaning of the concepts. Apart from Hilbert's axioms for geometry, alternative systems have been proposed by Peano, Veblen, Huntington, and others. Hilbert arranged the axioms in five groups: The axioms of incidence, of order (betweenness), of congruence, of parallels, and of continuity.

In these systems it is customary to take for granted a basic logic of inference (the theory of quantification or the predicate calculus) which deals with the logical constants "if-then", "not", "all", "some", "or", "and", "if and only if". There are, as we know, standard axiom systems for quantification theory. If we adjoin one such system to an axiom system for geometry, we get a more thoroughly formalized system.

In general, there are different degrees of formalization. If Euclid thought wrongly that his axiom system was completely formal, how do we know that a system considered formal now will not turn out to be imperfectly formalized?

In the evolution of axiom systems, there has emerged a sharp criterion of formalization in terms, not of meaning and concepts, but of notational features of terms and formulae.

Before stating the criterion, let us recapitulate the process of formalization. In a given mathematical discipline, there is a body of asserted and unasserted propositions. Out of the asserted propositions, choose some as axioms from which others can be deduced. In order that the axioms be adequate, they must express all the relevant properties of the undefined technical terms so that it should be possible to perform the deductions even if we treat the technical terms as meaningless words or symbols. Then we turn our attention to the logical particles or nontechnical words and make explicit the principles which determine their meaning or, in other words, govern their use. As a result, we should be able to recognize, merely by looking at the notational pattern, axioms and proofs.

From now on we shall speak of formal or axiomatic systems only when the systems satisfy the following criterion: there is a mechanical procedure to determine whether a given notational pattern is a symbol occurring in the system, whether a combination of these symbols is a well-formed formula (meaningful sentence) or an axiom or a proof of the system. Thus the formation rules, i.e. rules for specifying well-formed formulae, are entirely explicit in the sense that theoretically a machine can be constructed to pick out all well-formed formulae of the system if we use suitable physical representation of the basic symbols. The axioms and rules of inference are also entirely explicit. Every proof in each of these systems, when written out completely, consists of a finite sequence of lines such that each line is either an axiom or follows from some previous lines in the sequence by a definite rule of inference. Therefore, given any proposed proof, presented in conformity with the formal requirements for proofs in these systems, we can check its correctness mechanically. Theoretically, for each such formal system, we can also construct a machine which continues to print all the different proofs of the system from the simpler ones to the more complex, until the machine finally breaks down through wear and tear. If we suppose that the machine will never break down, then every proof of the system can be printed by the machine. Moreover, since a formula is a theorem if and only if it is the last line of a proof, the machine will also, sooner or later, print every theorem of the system. (Following a nearly established usage, we shall always count the axioms of a system among its theorems.)

Mathematical objects such as numbers and functions are studied in ordinary mathematical disciplines. Metamathematics, which constitutes nowadays an important part of mathematical logic, takes, on the other hand, mathematical theories as its objects of study. This is made possible by formalizing mathematical theories into axiomatic systems, which, unlike, for instance, the psychology of invention, are suitable objects of exact mathematical study. Indeed, if the powerful method of representing symbols by positive integers is employed, many problems in metamathematics become problems about positive integers, and the difference in subject matter between metamathematics and mathematics becomes even less conspicuous.

Apart from formal systems we shall also have occasion to study quasiformal systems in the following sense. A quasiformal system is obtained from a formal system by adding "nonconstructive rules of proof", which superficially provide definite methods of proof but really leave open the methods of proof. The best known is the rule of infinite induction according to which if $F(n)$ is a theorem for every positive integer n, then "$(n)F(n)$" is also a theorem. This leaves open the methods by which it is established that $F(n)$ is a theorem for every n.

Concerning each formal system, we can ask a number of different questions which are usually divided into two categories: the syntactical questions which deal with the system taken as a pure formalism or a machine for manufacturing formulae, and the semantical questions which are concerned with interpreting the system.

For example, with regard to a formal system, it is natural to ask whether it might not happen that not only proofs but also provability can be mechanically checked. If this is true for a system, then there is a definite method such that given any sentence of the system, the method enables us to decide whether it is a theorem. Such systems are called decidable systems. Other questions are: whether a given formal system is satisfiable, i.e. admits an interpretation; whether a system is consistent, i.e. contains no contradictory theorems.

Quantification theory occupies a special place with regard to formal systems. It can be viewed either as a subsidiary part of each axiom system which has its own special subject matter, or as a basic common framework such that each special system is but one of its applications. The latter viewpoint leads to an inclination to treat all important problems about formal systems as problems about quantificational formulae. For instance, both the consistency and the decidability of formal systems are reduced to the "decision problem", i.e., the problem of deciding the satisfiability or validity of quantificational formulae.

§ 2. The Problem of Adequacy

One is led to an interest in mathematical logic through diverse paths. The approach determines the posing of questions which in turn determines the replies. It is not so much that different approaches yield different answers to the same problems. Rather logicians of different backgrounds tend to ask different questions. They hold, therefore, different views on what the business of logic is. To settle these differences impartially is beyond the capability of an individual logician qua logician. For a settlement inevitably depends on judgments as to whether one type of question is more interesting than another: a highly partial and subjective matter.

What can be profitably done is to make a declaration of interest in certain problems, coupled with an enunciation of reasons for considering such questions interesting. If there is an attractive unifying principle among these problems, the chances of the declaration being accepted as a definition of logic will increase.

Consider the body of all mathematical disciplines, with their concepts and theorems, as it is formulated in a crystallized form (say, as in textbooks). One main problem of mathematical logic is to organize and systematize each discipline separately and the whole body altogether more or less in the manner of Euclid or, if possible, even more thoroughly. The scale of the program of systematizing the whole mathematics ought to satisfy everyone whose instinctive urge toward system building is reasonably modest. Yet there is no program of comparable scale, with the possible exception of that of unifying physical theory, which enjoys with it the unusual advantages of objectivity of results and proximity to vital human endeavours.

The organization of each discipline separately is not only a necessary preliminary step to the construction of a "grand logic", but important on its own account. In general, if two disciplines are equivalent, we need only study one of them. On the other hand, if an important discipline A is reducible to but substantially weaker than a discipline B, we cannot legitimately conclude that since *A* is a part of *B*, *A* need not be studied separately. For example, number theory and analysis are both reducible to set theory, analysis includes number theory as a proper part; we nonetheless make separate investigations of the three disciplines because each of them presents its own peculiar problems. Indeed, the business of mathematical logic is primarily the systematic study of these three regions and the underlying more elementary discipline of quantification theory. More explicitly, number theory deals with non-negative integers, analysis deals with non-negative integers and real numbers (or sets of integers), set theory deals with arbitrary sets or classes or functions. It is natural to choose to treat these branches because their concepts and methods are familiar and basic. On the one hand, masses of rather objective facts in them have to be accounted for, so that there is little room for capricious preferences. On the other hand, because of the central position of these branches, a clarification of their principles will bring the foundations of other branches of mathematics under control.

For each of the three fields, the first problem is to find a formal system which formalizes the intuitive theory. Given such a formal system, since it is intended as the formalization of an intuitive theory, the natural question is its adequacy. This question may take different forms. One form is, can all known proofs in the intuitive theory be formalized in the system? In many cases, the answer is usually yes: we believe that all known theorems and their proofs in the intuitive theory get close replicas in those formal systems of it which are now generally accepted. The actual derivation of mathematics from any such system is, however, long and tedious; it is practically impossible to verify conclusively that the intuitive theory, with all its details, is derivable in the system. On the other hand, it is also hard to refute such a claim for that would require the discovery of some premise or principle of inference, which has so far been tacitly assumed but unrecognized.

Even where a formal system is said to be adequate to an intuitive theory, it is often possible to construct, in terms of the whole formal system, arguments which can no longer be formalized in the system but are nonetheless of the same general type as arguments formalizable in the system. Next, in the intuitive theory, we are often less hesitant to use methods from other disciplines; for example, the use of analytic methods in number theory. The boundaries between formal systems are generally sharper so that, for example, usual formal systems of number theory do not include the analytic methods. Last but not least, there is always the question of faithful representation: although all known theorems and proofs of the intuitive theory have images in a formal system, we can still query how close these images resemble the originals, whether they are natural or rather distorted pictures. Indeed, if we merely attempt to embed actually recorded arguments but disregard intentions to admit general patterns, diverse artificial formal systems can be constructed for a given theory.

The domain of an intuitive theory is quite indeterminate, and the domain of its known theorems is more so. It is, therefore, desirable to introduce, besides the demonstrability of known theorems, other criteria for the adequacy of a formalization. Since the formal system is constructed to approximate the intuitive theory, the theory is the intended interpretation of the system. One criterion of adequacy is that the system do admit its intended interpretation. A separate and independent criterion is that the system be categorical, i.e., admit essentially only one interpretation, any two interpretations of it are isomorphic or, in other words, essentially equivalent. The most ideal formalization should be adequate by both criteria. Actually, most interesting formal systems are not categorical. It is not even clear that they always admit the intended interpretations: sometimes the intuitive theory to be formalized is so complex that we are unable to specify unambiguously the intended interpretation.

The reference to intuitive interpretations of a formal system inevitably brings in elements which are less precise and exact than the formal system. To avoid this, one is led to the introduction of a criterion of adequacy which refers only to the formal system: a formal system is complete if and only if for every unambiguous sentence in the system, either it or its negation is a theorem. This makes no reference to the intuitive theory and can therefore serve as a criterion of adequacy only after we have been independently convinced that all thoughts in the intuitive theory can be expressed in the formal system and that ordinary theorems can be proved. Once we agree on these matters the criterion of completeness is sufficient since we obviously do not wish to have both a sentence and its negation demonstrable in any theory. On the other hand, it is not clear that a formalization, to be adequate, must be complete in this sense. It is quite possible that an incomplete system reproduces faithfully our incomplete intuitive theory because it may happen that our intuitive procedure of proof is not capable of settling certain questions in the theory.

However that may be, the sharp question of completeness has led to sharp answers. Gödel's famous theorem establishes the conclusion that the usual formal systems for number theory, analysis and set theory are incomplete unless they contain contradictions. Moreover, given any consistent formal system for one of these theories, a sentence of the system can be constructed which is demonstrably indemonstrable. This result and its proof have also as a corollary the impossibility of finding a categorical formal system for any of the disciplines. In other words, in the process of answering the sharper problem of completeness one is led to significant conclusions on the original less precise question of adequacy. This illustrates a rather general phenomenon of studying a vaguer original problem through a related, though apparently different, one which is capable of exact treatment. Another example is the study of the problem of evidence by way of an attack on questions of consistency. The determination of completeness or consistency of a formal system required so much information: in the process of answering such questions one cannot help getting significant conclusions on the more basic problems of adequacy and comprehensibility of the formal system.

Tabelle 3
IR-Spektren der Komplexe $^{3}LFe(CO)(NO)^{2}D$ (ν in cm^{-1}).

Nr.	^{3}L-Ligand	^{2}D-Ligand	$\nu(CO)$	$\nu(NO)$ und $\nu(COR)$	Lösungsmittel	Lit.
1	C_3H_5	$P(C_4H_9\text{-n})_3$	1926	1692	Toluol	[3]
2	C_3H_5	$P(C_6H_5)_3$	1935	1700	Toluol	[3]
3	C_3H_5	$P(C_6H_5)_2Cl$	1960	1725	Toluol	[3]
5	C_3H_5	$P(OCH_3)_3$	1944	1710	Toluol	[3]
6	C_3H_5	$P(OC_2H_5)_3$	1939	1703	Toluol	[3]
7	C_3H_5	$P(OC_4H_9)_3$	1939	1703	Toluol	[3]
8	C_3H_5	$P(OC_6H_5)_3$	1958	1723	Toluol	[3]
9	C_3H_5	$P(OCH(CH_3)_2)_3$	1936	1702	Toluol	[3]
10	C_3H_5	$P(OCH_2)_3CCH_3$	1960	1722	Toluol	[3]
11	C_3H_5	$As(C_6H_5)_3$	1929	1704	Toluol	[3]
12	$1\text{-}CH_3\text{-}C_3H_4$	$P(C_6H_5)_3$	1930, 1926	1685, 1690	CCl_4 Toluol	[2] [5]
13	$1\text{-}CH_3COCH_2\text{-}C_3H_4$	$P(C_6H_5)_3$	1925	1690, 1680	Film	[2]
14	$1\text{-}CH_3COCH_2\text{-}C_3H_4$	$P(OC_6H_5)_3$	1957	1689	–	[2]
15	$1\text{-}C(CH_3)_2OHCH_2\text{-}C_3H_4$	$P(C_6H_5)_3$	1915	1685	KCl-Preßling	[2]
16	$1\text{-}C(C_2H_5)_2OHCH_2\text{-}C_3H_4$	$P(C_6H_5)_3$	1927	1681	–	[2]
17	$2\text{-}CH_3\text{-}C_3H_4$	$P(C_6H_5)_3$	1924	1688	Toluol	[5]
18	$2\text{-}CH_3\text{-}C_3H_4$	$P(OC_2H_5)_3$	1930	1691	Toluol	[5]
19	$1\text{-}CH_3(1\text{-}CH_3COCH_2)C_3H_3$	$P(C_6H_5)_3$	1923	1681	Film	[2]
20	$1\text{-}CH_3(1\text{-}CH_3COCH_2)C_3H_3$	$P(OC_6H_5)_3$	1908	1669	Film	[2]
21	$1\text{-}CH_3(1\text{-}C(CH_3)_2OHCH_2)C_3H_3$	$P(C_6H_5)_3$	1923	1678	Film	[2]
23	$1\text{-}CH_3COCH_2(1,2\text{-}(CH_3)_2)C_3H_2$	$P(C_6H_5)_3$	1920, 1910	1710, 1688, 1662	–	[2]
24		$P(C_6H_5)_3$	1942	1669	KCl-Preßling	[2]
25		$P(C_6H_5)_3$	1920	1663	KCl-Preßling	[2]

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are different ways of doing this. Hence, we get different axiomatic systems. Each system is constructed with a view to yield, short of contradictions, as much set theory as possible, and as naturally as possible. The systems are only such that we do not know how to get contradictions in them. We do not know that contradictions will never arise in them.

If it is agreed that reality cannot be contradictory, then this ignorance indicates either that sets are not real or at least that we do not have a clear concept of set. This leads to the broad questions of mathematical reality and mathematical evidence. If our intuition is not able to assure us that the theorems in an axiomatic set theory are true, what mathematical propositions can be seen to be true by our intuition? What kind of evidence distinguishes these intuitive mathematical truths from other mathematical propositions?

There seems to be a relative character in the nature of evidence. What is viewed as evident at one stage of the intellectual process may lose its intuitive evidence at a more advanced stage. For example, the information about physical things which sense experiences supply is no longer regarded as evident when a distinction between real and apparent qualities is introduced. Or, Euclidean axioms are no longer regarded as evident after the discovery of non-Euclidean geometries. Similarly, the discovery of paradoxes seems to deprive axioms of set theory of evidence. One is led to the search for some primitive or absolute evidence which will not be discredited at a higher stage of intellectual development.

Poincaré and Russell blamed the use of impredicative definitions. Russell introduced his vicious circle principle to rationalize the exclusion of these definitions, and constructed a formal system, commonly known as the ramified theory of types, according to the principle. There are certain shortcomings in the formal system which led Russell to introduce an axiom of reducibility which violates the vicious circle principle and nullifies completely the initial efforts to exclude impredicative definitions. It now seems possible to construct formal and quasi formal systems which yield most of the fruits of the axiom of reducibility but which still conform to the vicious circle principle. All these will be referred to as systems of predicative set theory.

There is much in common between Brouwer's intuitionism and what Hilbert considered to be finitist methods. A most striking coincidence in the two somewhat different approaches is the denial of the general validity of the law of excluded middle. In other words, even though tertium non datur holds for many sentences, there are others such that neither they nor their negations are true according to the intuitionist or the finitist interpretation.

This position can be made to appear less strange if we accept the following plausible propositions. In the first place, there are more arbitrary functions of integers than computable functions; or, in other words, there are certain functions of integers which are not computable. This can be established rigorously if we get a sharp and reasonable concept of computable functions.

In any case it seems reasonable to believe this to be the case. In the second place, as is commonly asserted, existence means constructibility according to the intuitionistic and the finitist reading of "there exists". It follows in particular that a sentence "for every m, there exists an n, such that $R(m,n)$" is true only if there is a constructive procedure which yields, for each m, its corresponding n, or, in other words, there is a computable function of m such that "for every m, $R(m,f(m))$" is true. In the third place, negating an assertion about an infinite collection does not merely mean that the assertion is false, because to know that would require going through infinitely many cases to decide whether there is a counter example, and it is not effectively possible to do so. Hence, from the effective approach, the negation of an assertion about infinitely many things can only be taken as an assertion of impossibility or absurdity: the assumption that the original assertion is true leads to a contradiction. According to this interpretation, the negation of "for all m, there exists an n, such that $R(m,n)$" means not the nonexistence of a computable function but rather that of an arbitrary function f such that "for every m, $R(m,f(m))$" is true. Hence, there is an asymmetry between a sentence and its negation. Thus, if a general assertion is satisfied only by a noncomputable function, then it is neither absurd nor effectively true. Hence, the law of excluded middle no longer holds for such an assertion.

Some examples may serve to clarify the matter a little more. According to widely accepted rigorous concepts of computable functions, it is possible to find a definite formula $T(m,n)$ such that there is no computable function but one noncomputable function f such that "for every m, $T(m,f(m))$" is true.

Consider now the following three sentences:

(1) For every positive integer m, there exists a positive integer n, n is greater than m and n is a prime (viz., one of the numbers such as 2, 3, 5, 7, etc., which are greater than 1 and cannot be resolved into smaller factors).

(2) For every positive integer m, there exists a positive integer n such that $T(m,n)$.

(3) For every positive integer m, there exists a positive integer n, n is greater than m, n is a prime, and $n+2$ is also a prime.

They are all of the form

(4) $(m)(En)R(m,n)$,

where $R(m,n)$ is, for (1) and (3), a formula such that, for any given values of m and n, the truth or falsity of $R(m,n)$ can be checked by elementary calculations.

Let us now consider, with regard to the three examples, the law of excluded middle:

(5) $(m)(En)R(m,n) \vee \sim(m)(En)R(m,n)$.

In the first place, if (2) is taken as the sentence "$(m)(En)R(m,n)$", (5) is false according to the effective interpretation of (m), (En) (quantifiers) and $\sim$ (negation). This is clear from the discussions given above.

The sentence (1) is known to be true and the familiar proof goes as follows. Given any positive integer m. Consider the number $m!+1$ ($m!$ being the factorial of m, i.e. the product of all positive integers from 1 to m). Clearly $m!+1$ is not divisible by any of $2, 3, \cdots, m$. Either $m!+1$ is a prime, or, if not itself a prime, it is divisible by some prime. In either case, there must be some prime greater than m, but not greater than $m!+1$.

From this proof it follows that for every m, there exists an n such that $n \leqslant m!+1$ and $R(m,n)$ (i.e. $m < n$ and n is prime). Hence, given any m, theoretically we can go through the sentences $R(m, m+1), \cdots, R(m, m!+1)$ and find, by elementary calculations, the smallest integer $k, m+1 \leqslant k \leqslant m!+1$, such that $R(m,k)$. In this way, we get a computable function f such that $(m)R(m,f(m))$. In this way, the proof of (1) establishes that (1) is true by the effective interpretation. It follows that the law of excluded middle (5) is true if (1) is taken as "$(m)(En)R(m,n)$".

It is not known whether (3) is classically true or false. It follows that the truth or falsity of (3) is also not known according to the effective interpretation. There is an ambiguity in the notion of effective methods. If we understand this in the classical sense, then (3) can be true only if it is true in the effective interpretation of existence because $\mu_n R(m,n)$ would be general recursive and effective. On the other hand, the intuitionists would require that a constructive proof be given for the effectiveness of a given function before it can be accepted as effective. Hence, the problem is reduced to a more subtle question of determining what a constructive proof is. According to a classical effective interpretation, if (3) is true in the effective sense, it is also true in the classical sense; if (3) is false in the effective sense, it is also false in the classical sense. On the other hand, if we adopt the doubly effective interpretation of the intuitionists, there are altogether three possibilities: (i) the sentence (3) is effectively true; (ii) it is not effectively true but classically true; (iii) it is false. If either (i) or (iii) is the case, then the law of excluded middle (5) is true for (3). If, on the other hand, (ii) is the case, then it no longer holds for (3). Of course, the intuitionists reject the notion of classical truth and say simply we do not know whether (5) is true for (3).

The account thus far attempts to make it reasonably plausible that sometimes tertium non datur may fail, if an effective or constructive interpretation of logical particles in arithmetic formulae is adopted. There is, however, no ambition to give here a full explanation of either intuitionism or finitist methods. For example, the above account does not even touch on the