LINEAR AND NONLINEAR PROGRAMMING:

An Introduction to Linear Methods in Mathemathical Programming

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Preface

This book deals with linear programming and a selection of other topics which can be handled by extending linear programming methods. It arose out of a course given to undergraduate and postgraduate students from a wide range of numerate disciplines. The minimal common mathematical background of these students imposed severe restrictions on the prior knowledge I could assume. In striving to avoid either excessive preliminary material or the trap of the 'cookbook', I have adopted an approach which is rigorous and complete but informal in presentation. The only mathematical prerequisites are an ability to handle equations and inequalities (knowledge of the theory of equations is not required) and familiarity with summation notation. However, the reader will be expected to follow arguments running, in some cases, over several chapters.

The text is also written to reflect some of the massive research effort that has been directed towards linear programming and related areas. In particular, aspects of linear programming computation, integer programming, network problems and multiple objective methods have undergone considerable development in the last decade. This work has influenced the choice of topics.

Informality of presentation means that, typically examples are solved *before* generalities are discussed. The student is strongly advised to study the worked examples carefully, even better, to try solving them himself – and then to attempt some or all of the exercises. These are not optional. The only way to really understand the material is by plenty of practice, both on routine and on more demanding exercises. Answers or hints are provided for all exercises.

The book emphasises theory and computational methods which are widely applied in all areas of industry and planning, and it is written with the idea of computer implementation in mind (the ambitious reader might even try writing his own code).

Chapters 1, 2, 3, 5 and 6 constitute the basic material on linear programming most of which is used in most of the subsequent chapters. Chapter 4 is more demanding, but not referred to again (except in exercises). Chapter 7 is used in Section 8.5 and Chapter 10 and the introductory section of Chapter 9 is referred to in Chapters 10 and 12. Apart from this, Chapters 8, 9, 10, 11 and 12 and even certain sections from within them, are independent and can be read or skipped as desired.

It gives me great pleasure to acknowledge the stimulation derived from Doug White, Lyn Thomas, and Simon French of the Department of Decision Theory at the University of Manchester and to thank Regina Benveniste for discussions on quadratic programming. My students have offered many helpful and valuable suggestions over the years. All these have shaped my thoughts on mathematical programming and how it should be presented. Some of the exercises use parts (sometimes modified) of exam questions set for students at the University of Manchester and permission to use them is gratefully acknowledge. Finally, a *sine qua non*, the excellent typing of Jill Weatherall is much appreciated.

April 1984 Roger Hartley

Selective Index of Notation

This index does not include 'local notation' used temporarily within a short section of the text, or the different characters used to represent 'variables' in particular contexts.

Symbol	Explanation	First introduced on page
s_i	slack variable	14
x_{B_i}	ith basic variable	22
x_{N_j}	jth non-basic variable	22
α_{ij}	coefficient of x_{N_j} in ith equation	22
z	objective function	36
a_i	artificial variable	41
$c_{B_i}(c_{N_j})$	objective function coefficient of $x_{B_i}(x_{N_j})$	46
β_{ij}	coefficient of X_j in ith equation	55
η_i	ith element in η-list	56
π_i	output of backwards transformation	57
δ_k	reciprocal of column weighting factor	62
w	minus dual objective function	68
t_j	dual slack variable	70
$b_{B_i}(b_{N_j})$	RHS of constraint in which $x_{B_i}(x_{N_j})$ is slack	88
U_J	upper bound on x_J	100
σ_J	slack variable associated with x_J	100
L_J	lower bound on x_J	104
$P(j)$	parent (vertex) of j	107
w_k	kth objective function weight	129
γ_{kj}	canonical coefficient of x_{N_j} in kth objective	138
$d_k(e_k)$	deviation variables	148
$v_{\mathrm{I}}(v_{\mathrm{II}})$	optimal value of a game to player I(II)	155
z_k	optimal objective value of kth subproblem	167
f_{i0}	fractional part of α_{i0}	168
$D_k(.)$	down-penalty	168

$U_k(.)$	up-penalty	169
ϕ	PWL approximation	176
$\lambda_i(a_k)$	variables (breakpoints) in SOS	177
$\lambda_i(\mu_j)$	Kuhn–Tucker multiplier associated with $s_i(x_j)$	186

CHAPTER 1

Introduction

1.1 AN EXAMPLE – RECYCLING WASTE PAPER

The primary impetus for the study of **Linear Programming Problems** (LPs) is the wide range of practical problems which can be, or have been, modelled as LPs. LP modelling constitutes a study in itself (see Further Reading) and, as our intention is to concentrate on solution procedures, we will limit the discussion to a single example. This is a simplified version of a model designed to explore some of the possible benefits from recycling waste paper.

Let us assume that the paper-making industry in a certain country makes, say, twelve different types of paper and that current annual production of type i is p_i tons ($i = 1, \ldots, 12$). Some of the paper produced is lost permanently from the system – exported, burnt, stored in libraries in the form of books, etc. – and the remaining waste paper, v_i tons of type i, has to be disposed of. One possible means of mitigating the disposal problem is to recycle some of this waste into secondary pulp, which can then be used as part of the input to the production process, the other ingredient being virgin pulp. The technology of paper-making imposes a lower limit, α_i for paper of type i, on the proportion of input which must be virgin pulp. The costs of collecting and processing waste paper into secondary pulp should also be taken into account, but such costs can be very difficult to measure and so we will seek to determine how much paper should be recycled in order to minimise the total amount of virgin pulp used, when a proportion λ of the total waste paper is available for recycling. Fulfilling this objective will also minimise the residual amount of waste paper to be disposed of.

Let us define $y_i(z_i)$ to be the amount of virgin (secondary) pulp used as input to the production of paper of type i ($i = 1, \ldots, 12$) and w_{ij} to be the amount of waste paper of type i used in the production of paper of type j in a year (measured in tons). If a 5 per cent weight loss is involved in the production process,

$$0.95y_i + 0.95z_i = p_i, \quad i = 1, \ldots, 12. \tag{1.1}$$

The minimum virgin pulp requirement can be restated as

$$0.95y_i \geqslant \alpha_i p_i, \qquad i = 1, \ldots, 12. \tag{1.2}$$

The production process dictates that only certain waste papers can be used in the production of other papers. So let us put $a_{ij} = 1$, if waste paper of type i can be used in secondary pulp for the production of paper of type j, and $a_{ij} = 0$, otherwise. Then we must have

$$z_j = \sum_{i=1}^{12} a_{ij} w_{ij}, \qquad j = 1, \ldots, 12, \tag{1.3}$$

and, since λv_i tons of paper of type i are available for recycling,

$$\sum_{j=1}^{12} a_{ij} w_{ij} \leqslant \lambda v_i, \qquad i = 1, \ldots, 12, \tag{1.4}$$

for the left-hand side of (1.4) is the total amount of waste paper of type i consumed. When $a_{ij} = 0$, w_{ij} is absent from (1.3)/(1.4).

Our objective is to minimise total virgin pulp used, i.e.

$$\sum_{i=1}^{12} y_i \tag{1.5}$$

whilst also satisfying y_i, z_i, $w_{ij} \geqslant 0$ for $i, j = 1, \ldots, 12$ and (1.1. . .(1.4). This example exhibits the typical features of an LP: a linear **objective function** (1.5), to be maximised or **minimised subject** to linear restrictions, or **constraints** (1.1) –(1.4) on the non-negative variables. Any non-negative solution of the constraints is called a **feasible solution**. Any feasible solution maximising or minimising the objective function is called **optimal**.

Some of the constraints are inequalities, such as (1.2) and (1.4); the remainder are equalities. Inequality constraints can always be converted to equalities by adding or subtracting a non-negative variable. Thus (1.2) can be rewritten

$$0.95y_i - s_i = \alpha_i p_i, \qquad i = 1, \ldots, 12,$$

where $s_i \geqslant 0$, and (1.4) can be rewritten

$$\sum_{j=1}^{12} a_{ij} w_{ij} + t_i = \lambda v_i, \qquad i = 1, \ldots, 12,$$

where $t_i \geqslant 0$. The variables introduced into the constraints are called **slack** variables and often have a natural interpretation in the model. For example, t_i above is the amount of waste paper of type i, potentially available for recycling, that is not actually used. Another useful trick for standardising problems is based on the observation that minimising a function gives the same optimal solution(s) as maximising its negative, so that, in our waste paper problem, we could have chosen to maximise

$$-\sum_{i=1}^{12} y_i.$$

We can now write the LP as

$$\begin{array}{llll}
\text{P1:} & \text{maximise} & \sum_{i=1}^{12} -y_i & \\
& \text{subject to} & 0.95y_i + 0.95z_i = p_i, & i = 1, \ldots, (12) \\
& & 0.95y_i - s_i = \alpha_i p_i, & i = 1, \ldots, (12) \\
& & \sum_{i=1}^{12} a_{ij} w_{ij} - z_j = 0, & j = 1, \ldots, (12) \\
& & \sum_{j=1}^{12} a_{ij} w_{ij} + t_i = \lambda v_i, & i = 1, \ldots, (12) \\
& & y_i, z_i, s_i, w_{ij}, z_j, t_i \geqslant 0, & i, j = 1, \ldots, (12).
\end{array}$$

Any LP, such as P1, which is written so that its objective function must be maximised and with only equality constraints (in non-negative variables) is said to be in **standard equality form** and this form will prove particularly valuable when we come to the development of computational procedures.

Occasionally, problems arise in which not all the variables are required to be non-negative. Such unrestricted variables are called **free** variables and it is straightforward to modify computational procedures to accommodate them (see Exercise 2 of Chapter 3). Alternatively, if x_j is a free variable we can write $x_j = y_j - z_j$ (i.e. substitute $y_j - z_j$ throughout the problem for x_j) where y_j, $z_j \geqslant 0$. In this way, at the expense of introducing extra variables, we can convert the problem to standard equality form. In some problems there may be both free variables and equality constraints and one can adopt the strategy of using the equality constraint to express a free variable in terms of other variables and thereby eliminating it from the other constraints and the objective function (see Exercise 3). This has the advantage that the numbers of contraints and variables are both reduced by one.

1.2 A GRAPHICAL METHOD

For the rest of this chapter we will concentrate on LPs with two variables and describe a graphical procedure for solving them. Since such problems are unlikely to arise in realistic models, the procedure is offered, not as a serious competitor to more sophisticated methods but, rather, to illuminate some essential features of linear optimisation. This geometrical approach can be developed into a systematic methodology of linear programming, but the mathematical level involved would exceed the limits set for this book. In any case, such a development is probably better employed in the elucidation of non-linear (especially convex) programming. Instead, we will adopt an algebraic and computational approach, but geometric ideas will sometimes be used to provide an alternative viewpoint on important concepts.

To start, we shall examine the problem P2.

P2: maximise $-2x_1 + 3x_2$ $(=z)$

subject to

$$2x_1 - x_2 \leqslant 4 \qquad \text{(I)}$$

$$x_1 - 2x_2 \geqslant -2 \qquad \text{(II)}$$

$$2x_1 + x_2 \geqslant 2 \qquad \text{(III)}$$

$$x_1, x_2 \geqslant 0$$

In which we have labelled the constraints for future use. We will ignore the objective function for the moment. The constraints are represented in Fig. 1.1

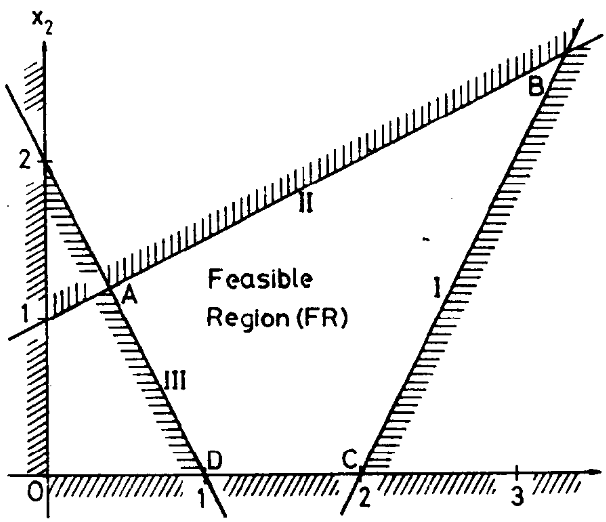

Figure 1.1 Feasible region of P2.

The equality corresponding to constraint I: $2x_1 - x_2 = 4$, gives the line marked I in the figure. The points (x_1, x_2) satisfying the first constraint lie on, or to one side of, this line. To decide which side, we need only substitute a point obviously on one side of the line and see if the constraint is satisfied. The origin (0, 0) is usually the obvious choice and in the present case shows that points to the left of the line I satisfy the constraint. This is indicated in the figure by hatching the side of the line *not* satisfying the constraint, that is those points (x_1, x_2) for which $2x_1 - x_2 > 4$. The same has been done for the second and third constraints. In addition, the x_1-axis has been hatched below to indicate the constraint $x_2 \geqslant 0$ and the left-hand side of the x_2-axis hatched to indicate $x_1 \geqslant 0$. The set of feasible points or solutions, called the **feasible region,** is the quadrilateral ABCD (including its interior).

The feasible region is redrawn in Fig. 1.2, with the hatching omitted. Also included in this figure is a series of lines on which we have set the objective function equal to various constants. Since the coefficients in the objective

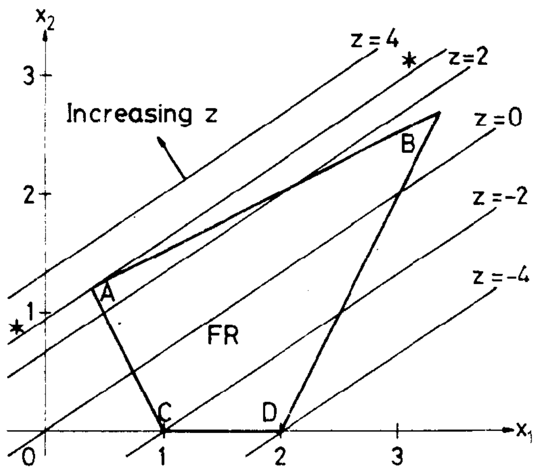

Figure 1.2 Graphical solution of P2.

function do not change, the lines are all parallel. The line $z = 0$ includes points which are feasible and the same is true of $z = 2$, improving the objective function. However, we cannot improve the objective function as far as $z = 4$, since this line includes no feasible points. The best we can achieve is the line indicated with asterisks. This clearly includes the feasible point A, but no feasible points can be found above this line. The optimal solution must be A, which lies on the lines II and III, and must, therefore, satisfy

$$x_1 - 2x_2 = -2$$

$$2x_1 + x_2 = 2,$$

which has the solution $(x_1, x_2) = (\frac{2}{5}, 1\frac{1}{5})$. The maximal objective function value is $z = -2 \times \frac{2}{5} + 3 \times \frac{6}{5} = 2\frac{4}{5}$.

As another example, we shall examine

P3: minimise $2x_1 + 6x_2$

subject to

$$x_1 + 3x_2 \geqslant 3 \qquad \text{(I)}$$

$$2x_1 - x_2 \geqslant 2 \qquad \text{(II)}$$

$$x_1, x_2 \geqslant 0$$