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List of Contributors

Numbers in parentheses indicate the pages on which the authors' contributions begin.

L. C. BIEDENHARN (1), Physics Department, Duke University, Durham, North Carolina

ESKO BLOKKER (199), Institute of Theoretical Physics, University of Stockholm, Stockholm, Sweden

STIG FLODMARK (199), Institute of Theoretical Physics, University of Stockholm, Stockholm, Sweden

W. J. HOLMAN, III (1), Physics Department, Duke University, Durham, North Carolina*

JEAN-MARC LÉVY-LEBLOND (221), Laboratoire de Physique Théorique, Université de Paris VII, France

HAROLD V. McINTOSH (75), Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, Mexico

CARL E. WULFMAN (145), University of the Pacific, Stockton, California

*Present address: Department of Physics, University of Colorado, Boulder, Colorado.

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The Representations and Tensor Operators of the Unitary Groups U(n)

W. J. HOLMAN, III, and L. C. BIEDENHARN*

PHYSICS DEPARTMENT, DUKE UNIVERSITY
DURHAM, NORTH CAROLINA

I. Introduction: The Connection between the Representation Theory of $S(n)$ and That of $U(n)$, and Other Preliminaries

In the first chapter of Volume I (*1*) the Killing–Cartan program for the classification of all compact simple Lie groups and the elementary theory of their representations was reviewed. In this chapter we consider in some detail the representation theory of the unitary groups, $U(n)$, deriving a system of basis vectors (the Gel'fand basis) and the matrix elements for the infinitesimal operators which span the Lie algebras of these groups. We shall give a brief introduction to the theory of tensor operators defined on the unitary groups (*2a–f*). Our presentation will be based largely on the results contained in the literature (3*a*, *b*, *c*).

The unitary groups, most particularly $SU(3)$ and $SU(6)$, have recently become objects of interest to physicists because of their usefulness in the study of elementary particle symmetries. (It needs no emphasis that $SU(2)$, the quantum angular momentum group, is of fundamental importance.) The $U(n)$ groups, as a family, have a further importance in that all of the classical groups can be embedded as subgroups; this property is very much more useful for Lie groups than the corresponding embedding of all finite groups in the symmetric group, $S(n)$.

In our exposition we shall adopt the following procedure: We shall first give an informal proof (due to Wigner and Stone) of the Peter–Weyl theorem,

*Present address: Department of Physics, University of Colorado, Boulder, Colorado.

which has the corollary that all the irreducible unitary representations of a compact matrix group are generated by Kronecker products of a single faithful representation of the group, which, for convenience, we shall take to be the defining representation, that is, the set of all $n \times n$ unitary matrices in the case of $U(n)$, ... The problem of the reduction of the Kronecker products of the defining representation is then solved with the assistance of the representation theory of the finite symmetric groups. Hence, we follow the Wigner–Stone proof with a brief review of this theory using the treatment of Hamermesh (4). We then proceed with a derivation of the characters of the irreducible unitary representations of $U(n)$ and a proof of the Weyl branching law and the Weyl dimension formula. We thus follow the classical, or global method of Hermann Weyl in dealing only with finite elements of the group, rather than a purely algebraic, or infinitesimal approach, which starts from the main theorems of Cartan, enunciated at the end of the work of Kleima *et al.* (1), and uses invariant operators to classify irreducible representations and to label the states of a given representation by means of the Weyl branching law. In the literature of group theory there is often no sharp distinction made between the infinitesimal and the global approaches; the classic researches of Schur, Cartan, and Weyl employed both techniques where expedient. The algebraic method is quite well adapted to the discussion of invariants and similar aspects of group-theoretic analysis for which an explicit basis is not required, but it becomes quite cumbersome in further research. Hence a study of the representation theory of the $U(n)$ groups, the principal object of the present work, would normally employ either method wherever advantageous and so make our distinction arbitrary. We have made it in view of its applicability to the noncompact groups $U(p, q)$, for which the algebraic method—in contrast to the global techniques—may still be applied successfully in many problems (5). Perhaps the sharpest distinction between the purely algebraic-constructive approach and the global methods lies in the application of the symmetric group to the articulation of the structure of the unitary groups.

We shall develop this method, then proceed to a determination of the characters of the irreducible unitary representations of $U(n)$ and a proof of the Weyl branching law, which will provide us with sufficient invariants to label the $U(n)$ irreducible representations and also with a system of labels for the states of these representations. We shall then provide a realization of the representations by means of a boson calculus, discussing the $U(2)$ case at length, then proceeding to the general case of $U(n)$. We shall add Racah's determination of invariant operators and state labels, a determination which is trivial in view of the results of Weyl's branching law, but which is of interest in that it takes place from the standpoint of the algebraic-constructive method rather than from global considerations. Finally, we shall treat the determination of the matrix elements of the operators of the $SU(n)$ Lie algebra, a determination which we shall complete only in the context of a general treatment of tensor operators on the $U(n)$ groups. Since the $SU(n)$ groups are all simply connected, the representation theory of the groups is completely determined by that of their Lie algebras.

First of all we shall establish the *Wigner–Stone version of the Peter–Weyl theorem* (6): *For a compact group G which is isomorphic to a set of finite-dimensional matrices there exists a set of finite-dimensional matrix representations whose elements form a complete set of functions defined over the group manifold.*

The well-known orthogonality theorem for the irreducible unitary representations of compact Lie groups assigns a property to any two *given* irreducible representations, whereas the completeness theorem requires the *construction* of a set of inequivalent irreducible representations, then proves their completeness. The procedure of Wigner and Stone is slightly different: Given one faithful matrix representation (which may be that realization which furnishes the abstract definition of the group, but which does not have to be irreducible) we may build up all the desired representations by forming Kronecker products of this one and then reducing the products. Taking D to be the original faithful representation, we construct the Kronecker products $D \boxtimes D$, $D \boxtimes D \boxtimes D, \ldots$. At each stage only finite-dimensional representations occur, and hence we need deal only with the problems of the reduction of finite matrices. The functions on the group manifold defined by these products will not all be linearly independent, but by the usual Schmidt process, beginning with the elements D_{ij}, we may define an orthonormal set of vectors $v_i(g)$, where g represents an element of the group, in other words, a point on the group manifold; that is,

$$\int d\mu(g)\, v_i(g)\, v_j(g^{-1}) = \delta_{ij}, \tag{1.1}$$

where $d\mu(g)$ represents the invariant Haar measure on the manifold.

The property of completeness is the assertion that a function $f(g)$ orthogonal to all $v_i(g)$ is necessarily equal to zero for all g. We obtain a proof by assuming the contrary, then establishing a contradiction.

For simplicity we assume that D is real. This assumption leads to no loss of generality because if D is complex we may always construct an isomorphic real representation of twice the original dimension.

Let us assume that $f(g)$ is not equal to zero when $g = g_0$, that is, $f(g_0) \neq 0$, and, moreover, that $f(g)$ is orthogonal to all $v_i(g)$:

$$\int d\mu(g) f(g) v_i(g^{-1}) = 0, \qquad \text{all} \quad v_i(g). \tag{1.2}$$

Consider now the function

$$u(g, \lambda) \equiv N(\lambda) \exp(-\lambda \sum_{ij} [D_{ij}(g) - D_{ij}(g_0)]^2), \tag{1.3}$$

where the function $N(\lambda)$ is normalized by the requirement that

$$\int d\mu(g) u(g, \lambda) = 1. \tag{1.4}$$

The function $u(g, \lambda)$ has the property that in the limit $\lambda \to \infty$ it becomes proportional to the Dirac delta function $\delta(g - g_0)$. Since, however, the function $\exp(-x^2)$ has a uniformly convergent power series in any closed domain of x, it follows that $u(g, \lambda)$ may be expanded in terms of matrix elements $D_{ij}(g)$ and all their powers. This is equivalent to the statement that $u(g, \lambda)$ may be expanded linearly in terms of the $v_i(g)$. Thus, if $f(g)$ is orthogonal to all $v_i(g)$, then it is orthogonal to $u(g, \lambda)$ for all λ, that is,

$$\int d\mu(g) u(g, \lambda) f(g) = 0, \qquad \text{all} \quad \lambda. \tag{1.5}$$

By taking the limit $\lambda \to \infty$, however, we find

$$\lim_{\lambda \to \infty} \int d\mu(g) u(g, \lambda) f(g) = \int d\mu(g) f(g) \delta(g - g_0) = f(g_0), \tag{1.6}$$

and by hypothesis $f(g_0) \neq 0$. This is the desired contradiction, and it shows that the only vector orthogonal to all $v_i(g)$ is the null vector. Since the matrix elements of the irreducible unitary representations form a complete set of functions on the group manifold, the corollary follows immediately that we may obtain every irreducible representation for a compact group from a faithful representation (and its complex conjugate) by taking Kronecker products and reducing these to irreducible constituents. This is the result which we shall need in the following discussion of the representations of the unitary groups.

The second prolegomenon which we shall need is a brief review of the representation theory of the symmetric group. The group of permutations of n objects is of order $n!$, and any given element may be expressed in the following notation. Let us label the n objects in their initial ordering as $(1, 2, \ldots, n)$, that is, with the positive integers in consecutive order; then a permutation of these objects will take them into a new order $(i_1, i_2, \ldots, i_n)$, in which the i_j are composed of the first n positive integers in the sequence to which they have been taken by our chosen element of the permutation group. This element of the permutation group is uniquely specified by the initial and final orderings of the n objects on which the group acts. So, we write the element in the form

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{pmatrix},$$

where we have written the initial ordering above and the final below. The law of composition for the group then becomes

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}\begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}. \quad (1.7)$$

Every element of the permutation group is composed of one or more (up to n) *cycles*. Consider the permutation of degree eight,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 1 & 5 & 4 & 7 & 6 & 8 \end{pmatrix};$$

we see that the numbers 1, 2, 3 are permuted among themselves, as are the numbers 4, 5 and the numbers 6, 7. Each subset of numbers which are permuted cyclically among themselves is called a cycle. In this example there is one three-cycle, composed of the numbers 1, 2, 3, two two-cycles, composed of 4, 5 and 6, 7, respectively, and one one-cycle, the number 8. We can then abbreviate this permutation by writing it in terms of its cycles in the following manner: (123)(45)(67)(8). Having no elements in common, of course, the cycles are commutative with one another.

Let us suppose that we have resolved a given member of the permutation group on n objects, S_n, into its cycles, and let the number of one-cycles be ν_1, of two-cycles be ν_2, of j-cycles be ν_j, and so forth. Since the total number of objects to be permuted is n, we must have

$$\nu_1 + 2\nu_2 + \cdots + n\nu_n = n. \quad (1.8)$$

A permutation which when resolved into independent cycles has ν_1 one-cycles, ν_2 two-cycles, $\ldots$, ν_n n-cycles is said to have the *cycle structure* $(1^{\nu_1}, 2^{\nu_2}, \ldots, n^{\nu_n})$. All the permutations which have the same cycle structure form an equivalence class within the group S_n. Likewise, each solution of (1.8) for positive integers $\nu_1, \nu_2, \ldots, \nu_n$ defines a class in S_n, and hence the number of classes is just the number of such solutions. If we let

$$\begin{aligned} \nu_1 + \nu_2 + \cdots + \nu_n &= \lambda_1 \\ \nu_2 + \nu_3 + \cdots + \nu_n &= \lambda_2 \\ &\vdots \\ \nu_n &= \lambda_n, \end{aligned} \tag{1.9}$$

then

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = n \tag{1.10}$$

and

$$\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n \geqq 0. \tag{1.11}$$

This decomposition of n into a sum of n integers is called a *partition of n*. Each solution of (1.8), then, uniquely specifies an equivalence class and a partition of n into $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Conversely, given a partition (1.10), we also have a cycle structure and an equivalence class of permutations.

At this point we shall merely state the theorem that for a finite group the number of nonequivalent irreducible representations is equal to the number of equivalence classes in the group; for proof, we refer to Hamermesh (*4*, p. 110). Hence, it is possible to construct a one-to-one correspondence between the partitions of n and the irreducible representations of the permutation group on n objects.

We shall now introduce the concept of the regular representation of a finite group. If we label the elements of S_n as s_i, $1 \leqq i \leqq n!$, then multiplication of the elements $s_1, \ldots, s_{n!}$ on the left by s_k merely permutes the $s_1, \ldots, s_{n!}$ among themselves. Considering $s_1, \ldots, s_{n!}$ as coordinates in an $(n!)$-dimensional space, we can represent the element s_k by a permutation of the $n!$ coordinates. Thus if $s_k s_i = s_{j_i}$ $(i = 1, \ldots, n!)$, we define the *regular representation* as the correspondence of s_k with the $n! \times n!$ matrix $D_{il}(s_k) = \delta_{lj_i}$. In this representation the diagonal elements of all matrices are zero except for the s_k which has the property that $s_k s_i = s_i$, that is, for the identity element. Each irreducible representation of the group is contained in the regular repre-

If a subalgebra B has the property that for any $u \in B$, su is also in B for any $s \in A$, where A is the whole algebra, then B is called a *left ideal.*

In the regular representation a left ideal I_1 is an invariant subspace since $sI_1 = I_1$ for any element s of the algebra A. Since the regular representation is fully reducible, the space A must be a direct sum of left ideals, that is, $A = \sum_{\oplus i} I_i$, where $sI_i = I_i$ for all i. Every element of the algebra A can be uniquely expressed as the sum of one element in each of the left ideals I_i; only the element zero is common to all I_i. The matrices of the regular representation $D(s)$ are reducible to $\sum_{\oplus i} D^{(i)}(s)$, where $D^{(i)}(s)$ is the matrix of the linear transformation induced in I_i by left multiplication with s.

The unit element e of the group S_n is contained in the group algebra A and has the property that $es = se = s$ for all $s \in A$. If A is the direct sum of two left ideals, $A = I_1 + I_2$, then the unit element e can be uniquely expressed as a sum $e = e_1 + e_2$, $e_1 \in I_1$, $e_2 \in I_2$. Similarly, any element s of A can be uniquely expressed as $s = s_1 + s_2 = se = s(e_1 + e_2) = se_1 + se_2$. Since I_1 and I_2 are left ideals, $s_1 = se_1$, $s_2 = se_2$. If s is in I_1, then $s = s_1$, $s_2 = 0$, and as a result $s = se_1$, $se_2 = 0$.

The element e_1 is *idempotent*, that is $e_1^2 = e_1$, and it is a generator of the ideal I_1 since se_1 is in I_1 for all s in A. If s is in I_1 then $se_1 = s$. The same remarks apply for e_2 in I_2. Also $e_1 e_2 = e_2 e_1 = 0$.

The left ideals I_1 and I_2 may in turn contain subalgebras which are left ideals. If an ideal I contains no proper subideal, it provides us with an irreducible representation of the algebra A. Such an ideal is said to be *minimal*. Continuing this process, we may express the algebra A as a direct sum of minimal left ideals,

$$A = I_1 \oplus \cdots \oplus I_k .$$

The left ideal I_i is generated by the idempotent e_i, and $e_i^2 = e_i$, $e_i e_j = 0$ for $i \neq j$. From the previous argument it is clear that we find the generators e_i by resolving the unit element e into components in the spaces $I_1, \ldots, I_k$. An idempotent which cannot be resolved into a sum of idempotents satisfying $e_i^2 = e_i$ and $e_i e_j = 0$, $i \neq j$, is called *primitive*. An idempotent e is primitive if and only if the ideal $I = Ae$ is minimal.

Any idempotent at all in the group algebra of S_n will generate a left ideal which gives a representation contained in the regular representation. In particular, let us consider the element $p = \sum_i s_i$, where the sum runs over all the elements of the group S_n. Now for any permutation s_k, $s_k p = \sum_i s_k s_i = p$. Hence, $p^2 = \sum_{k,i} s_k s_i = n! p$, and p is a generator. The quantity $(1/n!)p$ is idempotent. Multiplying p on the left by the quantity $R = \sum_i \alpha_i s_i$, we get $Rp = (\sum_i \alpha_i)p$; thus the left ideal Ap generated by p consists of the multiples of p. This is a one-dimensional vector space. Left multiplication by a permutation s does not change ap, so our representation assigns the number 1 to every group element and is the identity representation.

Similarly $q = \sum_i \delta_i s_i$, where δ_i is the signature of the permutation s_i ($\delta_i = 1$ for s_i an even permutation and $\delta_i = -1$ for s_i odd), is *idempotent up to a factor*, since $s_k q = \sum_i \delta_i s_k s_i = \delta_k q$, so that $q^2 = n!q$. Likewise $(1/n!)q$ is idempotent. The generator q then generates a one-dimensional ideal which consists of multiples of q. Left multiplication by an element s_k then provides us with the alternating representation which associates with each element of the group its signature. This representation is irreducible.

We obtain the remaining irreducible representations in the following manner: For any partition of n, $\lambda_1 + \cdots + \lambda_r = n$, $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_r$, $r \leqq n$, we draw a diagram of the following kind which will be called a "tableau." We draw a row of λ_1 boxes, (or nodes) then under it a row of λ_2 boxes, and under this row another row of λ_3 boxes, and so on, until we draw the final row of λ_r boxes. We arrange these rows with their left-hand ends directly under one another. The right end of each row then either lies directly above the right end of the row beneath it or extends beyond it. Each such Young *frame* then corresponds to a particular cycle structure, that is, a particular equivalence class. We then get the *Young tableau* from this frame by filling the squares with the numbers $1, \ldots, n$ in any order. For such a tableau we now consider two special kinds of permutations, P and Q. P will denote any permutation which interchanges only the numbers of each row among themselves, then the rows can be said to be invariant under the P, which are called "horizontal permutations." The Q are defined similarly as permutations which interchange only the numbers of each column among of the Kronecker product of p defining representations is then given by the direct product of p such vector spaces $\mathfrak{A}^{(1)} \times \mathfrak{A}^{(2)} \times \cdots \times \mathfrak{A}^{(p)}$:

$$A_{i_1}'^{(1)} A_{i_2}'^{(2)} \cdots A_{i_p}'^{(p)} = \sum_{j_1=1}^{n} \cdots \sum_{j_p=1}^{n} u_{i_1 j_1} \cdots u_{i_p j_p} A_{j_1}^{(1)} A_{j_2}^{(2)} \cdots A_{j_p}^{(p)}. \quad (1.13)$$

We now recall from our discussion of the Wigner–Stone theorem that all the irreducible unitary representations of the group can be generated from the reduction of Kronecker products of the defining representation. Hence, all the irreducible representations of $U(n)$ can be extracted from expressions of the form (1.13). The direct product of the p vector spaces $\mathfrak{A}^{(1)}, \ldots, \mathfrak{A}^{(p)}$ forms the set of tensors of rank p. These tensors are defined with respect to the group $U(n)$. Weyl noted that the transformations induced by the operations of $U(n)$ commute with the transformations which permute the p vector spaces among themselves. The transformations of this latter group can be represented completely in terms of the Young symmetry patterns defined by the partitions of p. Each pattern uniquely denotes an irreducible representation of S_p and hence also an invariant subspace of the tensor $\mathfrak{A}^{(1)} \times \cdots \times \mathfrak{A}^{(p)}$. We shall illustrate this decomposition of the carrier space for the case $p = 2$. For S_2 there are only two elements, e (the identity) and s (the permutation of two objects). We denote the second-rank tensor by F_{ij}. Then $eF_{ij} \equiv F_{ij}$ and $sF_{ij} \equiv F_{ji}$. The operator s commutes with the transformations (1.13) in tensor space:

$$(sF')_{j_2 j_1} = F'_{j_1 j_2} = u_{j_1 i_2} u_{j_2 i_1} F_{i_2 i_1} = u_{j_1 i_1} u_{j_2 i_2} F_{i_1 i_2} = u_{j_2 i_2} u_{j_1 i_1} (sF)_{i_2 i_1}. \quad (1.14)$$

The Young operators of the S_2 group are just $(e + s)$ and $(e - s)$; applied to F_{ij} they project out the symmetric and antisymmetric components of F_{ij}. Since both e and s commute with the transformations of the direct product $U(2) \times U(2)$, these components are invariant; that is, because the Young operator commutes with the transformation of the product $U(2) \times U(2)$ it projects out invariant subspaces both of the carrier space and of the matrices of the transformation. Hence the Young operator projects the product space into the invariant subspace defined by the Young tableau; it can be shown that the reduction of the Kronecker product into invariant subspaces provided by these operators corresponds exactly to its reduction into irreducible representations of $U(n)$. Then, from Schur's Lemma we have it that an irreducible representation of $U(n)$ which is labeled by the partition $[\lambda]$ appears in the reduction of the direct product $\mathfrak{A}^{(1)} \times \cdots \times \mathfrak{A}^{(p)}$ with a multiplicity equal to the degree of the irreducible representation $[\lambda]$ of S_n. The degree of the irreducible representation $[\lambda]$ of $U(n)$ is equal to the multiplicity with which the irreducible representation $[\lambda]$ of S_n occurs in the regular representation.

This subject is completely standard and has received a classic and beautifully lucid treatment in the Princeton lectures of Weyl (7). To fix the notation we consider the irreducible representation of the group $U(n)$ defined by the Young pattern $[\lambda] = (\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_p \geqq 0)$, $\sum_{i=1}^{p} \lambda_i = p$. The bases for this representation are the p-th rank tensors whose components are $i_1, \ldots, i_p$, where $1 \leqq i_j \leqq n$. The Young symmetrizer $Y_{[\lambda]}$ is the operator associated with the Young tableau (see Fig. 1), and is the sum of products of the opera-

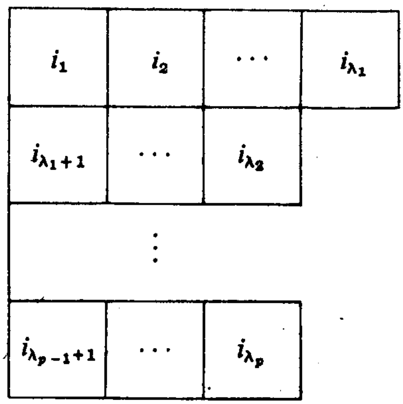

FIG. 1

tors denoting permutations of the rows (denoted by P_i) multiplied by the operators of the permutations of the columns (denoted by Q_j) with the sign (+) for even and (−) for odd permutations. Thus, $Y_{[\lambda]} = \sum_{ij} \delta_j Q_j P_i$. The order $\mathscr{Q}\mathscr{P}$ is fixed by definition; we could define $Y_{[\lambda]}$ as $\sum_{ij} P_i \, \delta_j Q_j$, but this convention would be distinct. Each of these definitions provides us with a distinct set of basis elements which span the group algebra of the permutaprimarily concerned here with the theory of the symmetric group, we shall not derive its primitive characters, but assume them known. They can be obtained by a variety of methods, notably Frobenius' determinantal method and the graphical method, on which a wide literature exists. We refer the reader to the works of Hamermesh (*4*), Weyl (*8*), Boerner (*9*), and Robinson (*10*), to mention only a few. We shall denote the character of the representation which belongs to a frame with rows of lengths $[\lambda] = [\lambda_1, \ldots, \lambda_n]$ as $\chi^{[\lambda]}_{(\nu)}$, where (ν) denotes the equivalence class of the permutations which have ν_1 one-cycles, ν_2 two-cycles, and so forth. The character of the unitary group $U(n)$ which belongs to the irreducible unitary representation $[\lambda]$ we denote $\varphi^{[\lambda]}(u)$, where $u \in U(n)$. We now refer back to Eq. (1.13) and restrict our attention to those transformations for which $i_1 \leqq i_2 \leqq i_3 \leqq \cdots \leqq i_p$, and these we denote (i) [following Boerner (*9*)]. We combine into a single summand all terms for which the systems $(j) = (j_1, \ldots, j_p)$ differ only in a permutation of indices, that is, all those terms for which $A^{(1)}_{j_1} A^{(2)}_{j_2} \cdots A^{(p)}_{j_p}$ has the same value, recalling that $\mathfrak{A}^{(1)} \times \mathfrak{A}^{(2)} \times \cdots \times \mathfrak{A}^{(p)}$ is totally symmetric. We can now abbreviate (1.13) as

$$A'_{(i)} = \sum_{(j)} \sum_{s}{}' u^{s(j)}_{(i)} A_{(j)}, \tag{1.13a}$$

where $A'_{(i)} = A'_{i_1} A'_{i_2} \cdots A'_{i_p}$, $i_1 \leqq \cdots \leqq i_p$, and $u^{s(j)}_{(i)} = S u_{i_1 j_1} \cdots u_{i_p j_p}$, where both the i_k and the j_k are subject to the restriction $i_1 \leqq \cdots \leqq i_p$ and S is a permutation which operates on the indices j_k. The summation $\sum'$ is carried out over a set of permutations $s(j)$ which transform (j) into the same numbers in a different order, each order occurring exactly once. If (j) contains 1 μ_1 times, 2 μ_2 times, ..., n μ_n times, then there are $p!/1!2!\cdots n!$ different orderings, with $\sum \mu_k = p$, $0 \leq \mu_k \leq p$. The character of the reducible representation which is formed by the Kronecker product $u^{s(j)}_{(i)}$ is the sum of diagonal elements

$$\sum_{(j)} \sum_{s}{}' u^{s(j)}_{(j)} = \sum_{(j)} \frac{1}{\mu_1! \mu_2! \cdots \mu_n!} \sum_{s} u^{s(j)}_{(j)}, \tag{1.15}$$

where the summation on the right is taken over all permutations, each ordering being given $(\mu_1! \cdots \mu_n!)$ times. We can also drop the restriction $j_1 \leqq \cdots \leqq j_p$ on the indices (j). No summand of $\sum_{(j)}$ is changed when we replace (j) by the same numbers in any other order. Replacing the summands by the sums over all orders, we get each term $p!/\mu_1! \cdots \mu_n!$ times, so that we must divide by this number. The character, then, is given by

$$\frac{1}{p!} \sum_{s} \sum_{(j)}{}' u^{s(j)}_{(j)} = \sum_{(j)} \sum_{s} u^{s(j)}_{(j)}; \tag{1.16}$$

that is, it is the trace of the transformation with coefficients $u^{s(j)}_{(j)}$ averaged over the symmetric group S_p.

Now let us look more closely at the individual matrix element u_{ij} and its Kronecker products. Because of the unitarity of u, there must exist a unitary matrix a which diagonalizes it, that is, brings it to the form

$$\begin{pmatrix} \epsilon_1 & & & 0 \\ & \epsilon_2 & & \\ & & \ddots & \\ 0 & & & \epsilon_n \end{pmatrix} = a^{-1}ua = a^{\dagger}ua, \tag{1.17}$$

where, again because of the unitarity, the eigenvalues ϵ_k are equal to $\exp(i\omega_k)$, with ω_k real, determined up to a modulus 2π. We denote by σ_1 the trace of u, by σ_2 the trace of $u^2 = \sum_k u_{ik}u_{kj}, \ldots$; thus

$$\sigma_1 = \sum_{k=1}^{n} \epsilon_k, \quad \sigma_2 = \sum_{k=1}^{n} \epsilon_k^2, \ldots, \sigma_r = \sum_{k=1}^{n} \epsilon_k^r. \tag{1.18}$$

It then follows immediately that

$$\sum_{(j)} u^{s(j)}_{(j)} = \sigma_1^{\nu_1}\sigma_2^{\nu_2}\cdots\sigma_p^{\nu_p} \tag{1.19}$$

if the permutation s contains ν_1 one-cycles, ν_2 two-cycles, ... Now, in the class of permutations which have the cycle structure $(\nu) = (1^{\nu_1}, 2^{\nu_2}, \ldots, p^{\nu_p})$ there are $h_\nu = p!/\nu_1!\,1^{\nu_1}\nu_2!\,2^{\nu_2}\cdots\nu_p!\,p^{\nu_p}$ elements. Hence, the average over S_p gives for our compound character the formula

$$\frac{1}{p!}\sum_{(\nu)} h_\nu \sigma_1^{\nu_1}\cdots\sigma_p^{\nu_p} = \sum_{(\nu)} \frac{1}{\nu_1!\,\nu_2!\cdots\nu_p!}\left(\frac{\sigma_1}{1}\right)^{\nu_1}\left(\frac{\sigma_2}{2}\right)^{\nu_2}\cdots\left(\frac{\sigma_p}{p}\right)^{\nu_p}, \tag{1.20}$$

from which we must now project out the primitive characters. Now, by the symmetry properties of the p-fold Kronecker product $u^{(i)}_{(j)}$, we have

$$Su^{(i)}_{(j)} = u^{(i)}_{(j)}S, \tag{1.21}$$

where S is a permutation matrix which operates on the upper (or lower) indices. The matrix S, then, is a representation matrix for the element $s \in S_p$; hence it is equivalent to a direct sum of irreducible representations $C_{[\lambda]}(s)$, each of which occurs with a multiplicity $l([\lambda])$, where we have labeled the irreducible representations by their partitions $[\lambda]$:

$$S \cong E_{l([\lambda])} \boxtimes C_{[\lambda]}(s) \oplus E_{l([\lambda'])} \boxtimes C_{[\lambda']}(s) \oplus \cdots. \tag{1.22}$$